

Nonlinear Jordan centralizer of strictly upper triangular matrices

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Jordan
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Abstract

Let \mathcal{F} be a field of zero characteristic, let $N_n(\mathcal{F})$ denote the algebra of $n \times n$ strictly upper triangular matrices with entries in \mathcal{F} , and let $f : N_n(\mathcal{F}) \rightarrow N_n(\mathcal{F})$ be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$, that is, a map satisfying that $f(XY + YX) = Xf(Y) + f(Y)X$, for all $X, Y \in N_n(\mathcal{F})$. We prove that $f(X) = \lambda X + \eta(X)$ where $\lambda \in \mathcal{F}$ and η is a map from $N_n(\mathcal{F})$ into its center $\mathcal{Z}(N_n(\mathcal{F}))$ satisfying that $\eta(XY + YX) = 0$ for every X, Y in $N_n(\mathcal{F})$.

Keywords Jordan centralizer, Strictly upper triangular matrices, Commuting map

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1. Introduction

Consider a ring R . An additive mapping $T : R \rightarrow R$ is called a left (respectively right) centralizer if $T(ab) = T(a)b$ (respectively $T(ab) = aT(b)$) for all $a, b \in R$. The map T is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [11] Zalar proved the following interesting result: if R is a 2-torsion free semiprime ring and T is an additive mapping such that $T(a^2) = T(a)a$ (or $T(a^2) = aT(a)$), then T is a centralizer. Vukman [10] considered additive maps satisfying similar conditions, namely $2T(a^2) = T(a)a + aT(a)$ for any $a \in R$, and showed that if R is a 2-torsion free semiprime ring then T is also a centralizer. Since then, the centralizers have been intensively investigated by many mathematicians (see, e.g., [2–5,7]).

Let R be a ring. An additive map $f : R \rightarrow R$, is called a Jordan centralizer of R if

$$\forall x, y \in R \quad f(xy + yx) = xf(y) + f(y)x. \quad (1)$$

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Recently, Ghomanjani and Bahmani [8] dealt with the structure of Lie centralizers of trivial extension algebras, whereas Fošner and Jing [6] studied Lie centralizers of triangular rings.

The inspiration of this paper comes from the articles [1,4,6] in which the authors deal with the Lie centralizer maps of triangular algebras and rings. In this note we will consider nonlinear Jordan centralizers on strictly upper triangular matrices over a field of zero characteristic.

Throughout this article, \mathcal{F} is a field of zero characteristic. Let $M_n(\mathcal{F})$ and $N_n(\mathcal{F})$ denote the algebra of all $n \times n$ matrices and the algebra of all $n \times n$ strictly upper triangular matrices over \mathcal{F} , respectively. We use $diag(a_1, a_2, \dots, a_n)$ to represent a diagonal matrix with diagonal (a_1, a_2, \dots, a_n) where $a_i \in \mathcal{F}$. The set of all $n \times n$ diagonal matrices over \mathcal{F} is denoted by $D_n(\mathcal{F})$. Let I_n be the identity in $M_n(\mathcal{F})$, $J = \sum_{i=1}^{n-1} E_{i,i+1}$ and $\{E_{ij} : 1 \leq i, j \leq n\}$ the canonical basis of $M_n(\mathcal{F})$, where E_{ij} is the matrix with 1 in the (i, j) position and zeros elsewhere. By $C_{N_n(\mathcal{F})}(X)$ we will denote the centralizer of the element X in the ring $N_n(\mathcal{F})$.

The notation $f : N_n(\mathcal{F}) \rightarrow N_n(\mathcal{F})$ means a nonlinear map satisfying $\forall X, Y \in N_n(\mathcal{F}) : f(XY + YX) = Xf(Y) + f(Y)X$.

Notice that it is easy to check that the $\mathcal{Z}(N_n(\mathcal{F})) = \mathcal{F}E_{1n}$.

The main result in this paper is the following:

Theorem 1. *Let \mathcal{F} be a field of zero characteristic. If $f : N_n(\mathcal{F}) \rightarrow N_n(\mathcal{F})$ is a nonlinear Jordan centralizer then there exists $\lambda \in \mathcal{F}$ and a map $\eta : N_n(\mathcal{F}) \rightarrow \mathcal{Z}(N_n(\mathcal{F}))$ satisfying $\eta(XY + YX) = 0$ for every X, Y in $N_n(\mathcal{F})$ such that $f(X) = \lambda X + \eta(X)$ for all X in $N_n(\mathcal{F})$.*

2. Proof of the main result

Let us start with some basic properties of Lie centralizers.

Lemma 2. *Let f be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$. Then*

- (1) $f(0) = 0$,
- (2) For every $X, Y \in N_n(\mathcal{F})$, we have $f(XY + YX) = Yf(X) + f(X)Y$.

Proof. To prove (1) it suffices to notice that

$$f(0) = 0f(0) + f(0)0 = 0.$$

(2) Observe that if $f(XY + YX) = Yf(X) + f(X)Y$, Interchanging X and Y in the above identity, we have $f(XY + YX) = Yf(X) + f(X)Y$. ■

Lemma 3. *Let f be a nonlinear Jordan centralizer of $N_n(\mathcal{F})$. Then*

- (1) $f(\sum_{i=1}^{n-1} a_i E_{i,i+1}) = \sum_{i=1}^{n-1} b_i E_{i,i+1}$,
- (2) There exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$.

Proof. Let $D = \sum_{i=1}^n \alpha_i E_{i,i} \in D_n(\mathcal{F})$. As \mathcal{F} is infinite, we can find a set $\{\alpha_i \in \mathcal{F} / 1 \leq i \leq n\}$ whose elements satisfy conditions: $\alpha_i + \alpha_{i+1} = 1$ for $1 \leq i \leq n-1$ and $\alpha_i + \alpha_j \neq 1$ for $j \neq i+1$.

(1) Consider $A \in M_n(\mathcal{F})$. It is well known that $DA + AD = A$ if and only if $A = \sum_{i=1}^n \alpha_i E_{i,i+1}$.

Hence, if $A = \sum_{i=1}^{n-1} a_i E_{i,i+1}$, we have $A = DA + AD$. Thus $f(A) = f(DA + AD) = Df(A) + f(A)D$. Therefore $f(A) = \sum_{i=1}^{n-1} b_i E_{i,i+1}$.

(2) As in (1), let $N = \sum_{i=1}^{n-1} (-1)^i E_{i,i+1} \in N_n(\mathcal{F})$, consider $A = \sum_{i=1}^{n-1} a_i E_{i,i+1}$. for some $a_i \in \mathcal{F}$. Then $NA + AN = 0$ if and only if $A = aJ$ for some $a \in \mathcal{F}$.

Indeed, $f(J) = \sum_{i=1}^{n-1} a_i E_{i,i+1}$. by (1). Thus, $0 = f(0) = f(NA + AN) = Nf(A) + f(A)N$. Hence, there exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$. ■

We will need the following lemma.

Lemma 4 (Lemma 2.1, [9]). *Suppose that \mathcal{F} is an arbitrary field. If $G, H \in UT_n(\mathcal{F})$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, then G and H are conjugated in $UT_n(\mathcal{F})$.*

Here $UT_n(\mathcal{F})$ is the multiplicative group of $n \times n$ upper triangular matrices with only 1's in the main diagonal. From the lemma above we obtain the following corollary.

Corollary 5. *Let \mathcal{F} be a field. For every $A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, there exists $B \in T_n(\mathcal{F})$ such that $B^{-1}AB = J$ and $T_n(\mathcal{F})$ is the ring of upper triangular matrices.*

Proof. Let A be a matrix in $N_n(\mathcal{F})$ of the mentioned form. Then $I_n + A$ is a unitriangular matrix. Let us notice first that there exists $B_1 \in D_n(\mathcal{F})$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. We can construct $B_1 \in D_n(\mathcal{F})$ recursively by:

$$(B_1)_{11} = 1, (B_1)_{i+1,i+1} = (B_1)_i \cdot (A_{i,i+1})^{-1} \text{ for } i \geq 1.$$

Consider the matrix $I_n + B_1^{-1}AB \in UT_n(\mathcal{F})$. The unitriangular matrices $I_n + J$ and $I_n + B_1^{-1}AB$ fulfill the condition in Lemma 4. Hence, there exists $B_2 \in UT_n(\mathcal{F})$ such that $I_n + J = B_2^{-1}(I_n + B_1^{-1}AB_1)B_2$. Then $J = B_2^{-1}(B_1^{-1}AB_1)B_2$. Taking $B = B_1B_2 \in T_n(\mathcal{F})$, we get $J = B^{-1}AB$ as wanted. ■

Lemma 6. *Let $A = \sum_{i < j} a_{ij} E_{ij}$ be a matrix in $N_n(\mathcal{F})$ with $a_{i,i+1} \neq 0$ for every $i = 1, \dots, n-1$. Then there exists $\lambda_A \in \mathcal{F}$ such that $f(A) = \lambda_A A$.*

Proof. Since $A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $T \in T_n(\mathcal{F})$ such that $TAT^{-1} = J$ by the previous corollary. Define $h : N_n(\mathcal{F}) \rightarrow N_n(\mathcal{F})$ by $h(X) = Tf(T^{-1}XT)T^{-1}$. Then h is a nonlinear Jordan centralizer map. Indeed, $\forall X, Y \in N_n(\mathcal{F})$, we have:

$$\begin{aligned} h(XY + YX) &= Tf(T^{-1}(XY + YX)T)T^{-1} \\ &= Tf(T^{-1}(XY + YX)T)T^{-1} \\ &= Tf(T^{-1}XTT^{-1}YT + T^{-1}YT T^{-1}XT)T^{-1} \\ &= Tf((T^{-1}XT)(T^{-1}YT) + (T^{-1}YT)(T^{-1}XT))T^{-1} \\ &= T[(T^{-1}XT)f(T^{-1}YT) + f(T^{-1}YT)(T^{-1}XT)]T^{-1} \\ &= XTf(T^{-1}YT)T^{-1} + Tf(T^{-1}YT)T^{-1}X \\ &= Xh(Y) + h(Y)X \end{aligned}$$

Hence, $h(J) = \lambda_A J$ by lemma 2.2. Then

$$Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}.$$

Multiplying the left and right sides by T^{-1} and T respectively yields $f(A) = \lambda_A A$. ■

Now we wish to extend Lemma 2.3 to all elements of $N_n(\mathcal{F})$. In order to do this, let us introduce the following set:

$$S = \{B = (b_{ij}) \in N_n(\mathcal{F}) : b_{i,i+1} \neq 0 \forall i = 1, \dots, n-1\}.$$

This set has an important property that is established below.

Lemma 7. Let \mathcal{F} be a field. Every element of $N_n(\mathcal{F})$ can be written as a sum of at most two elements of \mathcal{S} .

Proof. If $a_{i,i+1} \neq 0$ for all $i = 1, \dots, n-1$, then A belongs to \mathcal{S} , so there is nothing to prove. If A is not in \mathcal{S} , then we can define B_1 and B_2 as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i + 1 \\ a_{ij} & \text{if } j > i + 1, \end{cases} \quad (B_2)_{ij} = \begin{cases} b_i & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where b_i is an element in \mathcal{F} different from $a_{i,i+1}$. It is easy to see that B_1, B_2 are in \mathcal{S} , and $A = B_1 + B_2$, so we wanted. ■

Lemma 8. Let \mathcal{F} be a field. For arbitrary elements A, B of $N_n(\mathcal{F})$, there exists $\lambda_{A,B} \in \mathcal{F}$ such that

$$f(A + B) = f(A) + f(B) + \lambda_{A,B} E_{1n}.$$

Proof. For any A, B, X of $N_n(\mathcal{F})$, we have

$$\begin{aligned} f((A + B)X + X(A + B)) &= Xf(A + B) + f(A + B)X \\ &= Xf(A + B) + f(A + B)X \\ &= Af(X) + f(X)A + Bf(X) + f(X)B \\ &= f(AX + XA) + f(BX + XB) \\ &= Xf(A) + f(A)X + Xf(B) + f(B)X \end{aligned}$$

hence

$$X(f(A) + f(B) - f(A + B)) = (f(A + B) - f(B) - f(A))X$$

which implies that $(f(A + B) - f(A) - f(B))^2 \in \mathcal{Z}(N_n(\mathcal{F}))$. Thus, there exists $\lambda_{A,B} \in \mathcal{F}$ such that $f(A + B) = f(A) + f(B) + \lambda_{A,B} E_{1n}$. ■

Now we can prove the main theorem.

Proof of Theorem 1. For every $X \in N_n(\mathcal{F})$ there exists a $A, B \in \mathcal{S}$ such that $X = A + B$.

First take $A, B \in \mathcal{S}$ such that $AB + BA \neq 0$. Then, by Lemma 2.3, $f(A) = \lambda_A A$, $f(B) = \lambda_B B$ for some $\lambda_A, \lambda_B \in \mathcal{F}$. Since f is nonlinear Jordan centralizer map, the following holds:

$$f(AB + BA) = Af(B) + f(B)A = Bf(A) + f(A)B$$

we must have $\lambda_A = \lambda_B$.

Consider now A and B from \mathcal{S} such that $AB + BA = 0$. Then there exists $C \in \mathcal{S}$ such that the pairs C and A, C and B, C are $AC + CA \neq 0$ and $BC + CB \neq 0$, so we have $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$.

Thus, there exists $\lambda \in \mathcal{F}, \eta : N_n(\mathcal{F}) \rightarrow \mathcal{Z}(N_n(\mathcal{F}))$ nonlinear Jordan centralizer map such that $f(X) = \lambda X + \eta(X)$ for all $X \in N_n(\mathcal{F})$.

we have

$$\begin{aligned} f(XY + YX) &= \lambda(XY + YX) + \eta(XY + YX) \\ &= Xf(Y) + f(Y)X \\ &= X(\lambda Y + \eta(Y)) + (\lambda Y + \eta(Y))X \\ &= \lambda(XY + YX) + X\eta(Y) + \eta(Y)X \end{aligned}$$

we obtain that $\eta(XY + YX) = X\eta(Y) + \eta(Y)X$ for all $X, Y \in N_n(\mathcal{F})$.

Now we use Lemma 2.5 we get $f(X) = \lambda X + \eta(X)$ for all $X \in N_n(\mathcal{F})$, where $\eta : N_n(\mathcal{F}) \rightarrow \mathcal{Z}(N_n(\mathcal{F}))$ is a nonlinear Jordan centralizer map and $\eta(X) = 0$ for all $X \in \mathcal{S}$. \square

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