

Note on a $W^{1,\infty}(L^2)$ -error estimate of a nonlinear finite volume scheme for a semi-linear heat equation

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Abstract

We consider, as discretization in space, the nonconforming mesh developed in SUSHI (Scheme Using Stabilization and Hybrid Interfaces) developed in Eymard et al. (2010) for a semi-linear heat equation. The time discretization is performed using a uniform mesh. We are concerned with a nonlinear scheme that has been studied in Bradji (2016) in the context of the general framework GDM (Gradient Discretization Method) (Droniou et al., 2018) which includes SUSHI. We provide sufficient conditions on the size of the spatial mesh and the time step which allow to prove a $W^{1,\infty}(L^2)$ -error estimate. This error estimate can be viewed as an improvement for the $W^{1,2}(L^2)$ -error estimate proved in Bradji (2016). The $W^{1,\infty}(L^2)$ -error estimate we want to prove in this note was stated without proof in Bradji (2016, Remark 7.2, Page 1302). Its proof is based on a comparison with an appropriately chosen auxiliary finite volume scheme along with the derivation of some new estimates on its solution.

Keywords Semilinear parabolic equations, SUSHI method, $W^{1,\infty}(L^2)$ -error estimate

Paper type Original Article

1. Problem to be solved and aim of this note

Let us consider the following semilinear parabolic problem:

$$u_t(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = f(u)(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T), \quad (1)$$

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where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$ with boundary denoted by $\partial\Omega$, $d \in \{2, 3\}$, $T > 0$, and f is a given function defined on $\mathbb{R}$ into $\mathbb{R}$. Nonlinear finite volume scheme

An initial condition is given by, for a given function u^0 defined on Ω

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (2)$$

For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T). \quad (3)$$

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Problem (1)–(3) arises for instance in combustion modeling, epidemic phenomena, and stochastic controls, see [15]. Finite volume methods have been widely used to approximate different types of partial differential equations which represent a large class of problems in application. Starting from the standard volume methods which use the so-called *admissible meshes* (cf. [10]), passing by the so-called SUSHI method (cf. [12]), and then a generalization to the framework of Gradient Schemes (cf. [9]), finite volume methods have then witnessed considerable developments in the last few decades. They can be applied in arbitrary geometries and are *locally conservative*, see [10] and the references therein. One of the principles of finite volume methods is the integration of the equation to be solved on the so called *control volumes*. We use then *numerical fluxes* to approximate, using the discrete unknowns, the continuous fluxes over the boundaries of the control volumes, which appear after the integration by parts. A widely used definition of admissible finite volume meshes for viscous conservation laws can be found in [10, Definition 9.1, Pages 762–763]. However, the control volumes in the sense of [10, Definition 9.1] are open polygonal convex sets. In addition to this, for admissibility, the mesh should satisfy an orthogonality condition, that is, there exists a family of points $(\mathbf{x}_K)_{K \in \mathcal{T}}$ such that for a given edge σ_{KL} , the line segment $\mathbf{x}_K \mathbf{x}_L$ is orthogonal to this edge. This condition allows to approximate the fluxes over a given edge using two point difference quotients. The construction of such meshes in general geometries is possible in many interesting cases but still linked to a number of challenging problems [13]. Consequently in many cases it is useful to drop the orthogonality condition and to assume general polyhedral control volumes, where the boundary of each control volume is a finite union of subsets of hyperplanes. Recently in [12], a large class of nonconforming meshes are used in the approximation of stationary anisotropic heterogeneous diffusion equations, and some error estimates have been provided. In addition to this, a discrete gradient is developed in [12] and it satisfies the properties of stability and consistency. The scheme which has been developed in [12] is called SUSHI.

In the recent work [4], the second author addressed an analysis for the convergence of the GDM for the problem (1)–(2). GDM is a generic framework has been developed recently in [9] which includes, in addition to SUSHI [12], several numerical methods:

- Conforming finite element methods, including mass lumped versions.
- The non-conforming finite element methods, again including mass lumped, versions.
- The mixed finite element methods and in particular the Raviart–Thomas ones.
- The symmetric interior penalty Galerkin version of the discontinuous Galerkin (DG) method.
- The multi-point flux approximation (MPFA) schemes and the discrete duality finite volume (DDFV) schemes on particular grids.
- The hybrid mimetic mixed (HMM) family of schemes, which includes the hybrid mimetic finite difference schemes, the SUSHI, and the mixed finite volume scheme.
- The nodal mimetic finite difference scheme.

In [4, Theorem 4.1, Pages 1290–1291] and [4, Corollary 6.2, Pages 1298–1299], the second author established some error estimates in discrete norms of $L^\infty(H^1)$, $W^{1,2}(L^2)$, and $L^\infty(L^2)$ in the context of the general framework GDM. Consequently, these results can be applied in the particular method of SUSHI. The aim of this note is to improve the $W^{1,2}(L^2)$ -error estimate, in the context of SUSHI method, in the sense that the estimate will be uniform in time, i.e. $W^{1,\infty}$ instead of $W^{1,2}$. This is strong since $W^{1,\infty}(L^2)$ -error estimate implies $W^{1,2}(L^2)$ -error estimate. We will address the extension to GDM in a future work. Such improvement, i.e. $W^{1,\infty}(L^2)$ -error estimate, exists in SUSHI for the linear case and it was treated in [3,5,6] but it does not exist for the nonlinear case. In fact, $L^\infty(H^1)$ and $W^{1,\infty}(L^2)$ -error estimates are useful since they allow to derive approximations for the gradient (spatial derivatives) and time derivative, respectively. The $W^{1,\infty}(L^2)$ -error estimate we want to prove was stated without proof in [4, Remark 7.2, Page 1302].

The proof of uniform-in-time convergence results has attracted considerable attention in the literature. Indeed, in [8], a numerical scheme is developed in the context of GDM for non-linear degenerate parabolic equations and a uniform-in-time strong-in-space convergence result is proved. In [1] (see also [2]), a numerical scheme given by a backward Euler in time and finite volume in space discretization is established for the drift–diffusion system and some uniform-in-time bounds for the approximate solutions are proved. In this work, we are concerned with the uniform-in-time convergence of the discrete time derivative of the approximate solutions towards the time derivative of the exact solution of the problem (1)–(3).

The structure of this note is as follows. In Section 2, we describe the time and space discretizations. For the sake of completeness, we recall some convergence results concerning SUSHI applied to elliptic equations. We will also state in Section 2 some known results and the aim of this note. In Section 3, we state and prove our main result, that is Theorem 3.1. Section 4 is devoted to provide some numerical tests.

2. Meshes, schemes, and some preliminaries

We consider, as discretization in space, the mesh introduced in [12]. We recall its definition:

Definition 1 (*Space Discretization, cf. [12, Definition 2.1, Page 1012]*). Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. A discretization of Ω , denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

1. $\mathcal{M}$ is a finite family of non empty connected open disjoint subsets of Ω (the “control volumes”) such that $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K ; let $m(K) > 0$ denote the measure of K and h_K denote the diameter of K .
2. $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the “edges” of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non empty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$ -dimensional measure is strictly positive. We also assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote $\mathcal{M}_\sigma = \{K, \sigma \in \mathcal{E}_K\}$. We then assume that, for any $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, denoted by $\mathcal{E}_{\text{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, denoted by $\mathcal{E}_{\text{int}}$). For all $\sigma \in \mathcal{E}$, we denote by $\mathbf{x}_\sigma$ the barycenter of σ . For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K .
3. $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (\mathbf{x}_K)_{K \in \mathcal{M}}$, such that for all $K \in \mathcal{M}$, $\mathbf{x}_K \in K$ and K is assumed to be $\mathbf{x}_K$ -star-shaped, which means that for all $\mathbf{x} \in K$, the property $[\mathbf{x}_K, \mathbf{x}] \subset K$ holds. Denoting by $d_{K,\sigma}$ the Euclidean distance

between $\mathbf{x}_K$ and the hyperplane including σ , one assumes that $d_{K,\sigma} > 0$. We then denote by $\mathcal{D}_{K,\sigma}$ the cone with vertex $\mathbf{x}_K$ and basis σ .

The time discretization is performed with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^*$, and we shall denote $t_n = nk$, for $n \in \llbracket 0, N+1 \rrbracket$. We introduce the operator ∂^1 of the discrete temporal derivative $\partial^1 v^n = \frac{v^n - v^{n-1}}{k}$. We denote by ∂^2 the discrete second time derivative $\partial^2 v^{j+1} = \partial^1(\partial^1 v^{j+1}) = \frac{v^{j+1} - 2v^j + v^{j-1}}{k^2}$.

Throughout this paper, the letter C stands for a “generic” positive constant independent of the parameters of the space and time discretizations.

We define the discrete space $\mathcal{X}_{\mathcal{D}}$ as the set of all $v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}})$, where $v_K, v_\sigma \in \mathbb{R}$ and $\mathcal{X}_{\mathcal{D},0} \subset \mathcal{X}_{\mathcal{D}}$ is the set of all $v \in \mathcal{X}_{\mathcal{D}}$ such that $v_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$. Let $H_{\mathcal{M}}(\Omega) \subset L^2(\Omega)$ be the space of functions which are constant on each control volume K of the mesh $\mathcal{M}$. The space $H_{\mathcal{M}}(\Omega)$ is equipped with the norm $\|u\|_{1,2,\mathcal{M}}^2 = \sum_{\sigma \in \mathcal{E}} \frac{m(\sigma)}{d_\sigma} (u_L - u_K)^2$, where we have denoted $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\text{int}}$ and $u_L = 0$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$. For all $v \in \mathcal{X}_{\mathcal{D}}$, we denote by $\Pi_{\mathcal{M}} v \in H_{\mathcal{M}}(\Omega)$ the function defined by $\Pi_{\mathcal{M}} v(\mathbf{x}) = v_K$, for a.e. $\mathbf{x} \in K$, for all $K \in \mathcal{M}$. For all $\varphi \in \mathcal{C}(\overline{\Omega})$, we define $\mathcal{P}_{\mathcal{D}} \varphi = ((\varphi(\mathbf{x}_K))_{K \in \mathcal{M}}, (\varphi(\mathbf{x}_\sigma))_{\sigma \in \mathcal{E}}) \in \mathcal{X}_{\mathcal{D}}$. We denote by $\mathcal{P}_{\mathcal{M}} \varphi \in H_{\mathcal{M}}(\Omega)$ the function defined by $\mathcal{P}_{\mathcal{M}} \varphi(\mathbf{x}) = \varphi(\mathbf{x}_K)$, for a.e. $\mathbf{x} \in K$, for all $K \in \mathcal{M}$. In order to analyze the convergence, we need to consider the size $h_{\mathcal{D}}$ of the mesh $\mathcal{D}$ defined by $h_{\mathcal{D}} = \sup\{\text{diam}(K), K \in \mathcal{M}\}$ and the regularity of the mesh $\theta_{\mathcal{D}}$ given by

$$\theta_{\mathcal{D}} = \max \left(\max_{\sigma \in \mathcal{E}_{\text{int}}, K, L \in \mathcal{M}} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} \right).$$

The scheme we want to consider is based on the use of the discrete gradient given in [12]. For $u \in \mathcal{X}_{\mathcal{D}}$, we define, for all $K \in \mathcal{M}$, for a.e. $\mathbf{x} \in \mathcal{D}_{K,\sigma}$

$$\nabla_{\mathcal{D}} u(\mathbf{x}) = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}} (u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)) \right) \mathbf{n}_{K,\sigma}, \quad (4)$$

with

$$\nabla_K u = \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) (u_\sigma - u_K) \mathbf{n}_{K,\sigma}. \quad (5)$$

We introduce the bilinear form $\langle \cdot, \cdot \rangle_F$ defined on $\mathcal{X}_{\mathcal{D}} \times \mathcal{X}_{\mathcal{D}}$ by

$$\langle u, v \rangle_F = \int_{\Omega} \nabla_{\mathcal{D}} u(x) \cdot \nabla_{\mathcal{D}} v(x) dx, \quad \forall (u, v) \in \mathcal{X}_{\mathcal{D}} \times \mathcal{X}_{\mathcal{D}}. \quad (6)$$

Let us recall error estimates given in [12] of the finite volume approximate solution in the elliptic case. For this reason, we consider the following second order elliptic equation, for a given function F :

$$-\Delta U(\mathbf{x}) = F(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (7)$$

with the homogeneous Dirichlet boundary conditions, that is

$$U(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial \Omega. \quad (8)$$

Let us consider the finite volume scheme given in [12] and approximating the elliptic problem (7)–(8):

$$\langle U_D, v \rangle_F = (F, \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{D,0}. \quad (9)$$

The following theorem is the subject of [12, Theorem 4.8, Page 1033] and [12, (4.19), Page 1031] (see also [5,6] for some details):

Theorem 2.1 (Convergence Order in the Elliptic Case, cf. [12]). *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 1. Let $\nabla_{\mathcal{D}}$ be the discrete gradient defined by (4)–(5) and $\langle \cdot, \cdot \rangle_F$ be the bilinear form given by (6). Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \geq \theta_{\mathcal{D}}$, for some given $\theta > 0$. Then the discrete problem (9) has a unique solution. Assume in addition that the solution U of the elliptic problem (7)–(8) is satisfying $U \in \mathcal{C}^2(\overline{\Omega})$, then the following error estimates hold:*

- $L^2(\Omega)$ -estimate.

$$\|\Pi_{\mathcal{M}} U_D - \mathcal{P}_{\mathcal{M}} U\|_{L^2(\Omega)} \leq Ch_D \|U\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (10)$$

- $H^1(\Omega)$ -estimate.

$$\|\Pi_{\mathcal{M}} U_D - \mathcal{P}_{\mathcal{M}} U\|_{1,2,\mathcal{M}} \leq Ch_D \|U\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (11)$$

- Estimate in the gradient approximation .

$$\|\nabla_{\mathcal{D}} U_D - \nabla U\|_{L^2(\Omega)^d} \leq Ch_D \|U\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (12)$$

Using the fact that $\|U - \mathcal{P}_{\mathcal{M}} U\|_{L^\infty(\Omega)} \leq Ch_D \|U\|_{\mathcal{C}^1(\overline{\Omega})}$ (which stems from $|U(\mathbf{x}) - U(\mathbf{x}_K)| \leq Ch_D \|U\|_{\mathcal{C}^1(\overline{\Omega})}$, for all $\mathbf{x} \in \mathbf{K}$), estimate (10) together with the triangle inequality implies that

$$\|\Pi_{\mathcal{M}} U_D - U\|_{L^2(\Omega)} \leq Ch_D \|U\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (13)$$

As discretization for the problem (1)–(3), it is considered in [4] a non-linear scheme in the context of the general framework GDM (recall that GDM includes SUSHI method) which can be written in the particular case of SUSHI as: For any $n \in \llbracket 0, N \rrbracket$, find $u_D^{n+1} \in \mathcal{X}_{D,0}$ such that

$$(\partial^1 \Pi_{\mathcal{M}} u_D^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)} + \langle u_D^{n+1}, v \rangle_F = (f(\Pi_{\mathcal{M}} u_D^{n+1}), \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{D,0}, \quad (14)$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the $L^2(\Omega)$ -inner product and

$$\langle u_D^0, v \rangle_F = -(\Delta u^0, \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{D,0}. \quad (15)$$

Under assumptions that $u \in \mathcal{C}^2(\mathcal{C}^2)$, f is Lipschitz continuous, and $\theta \geq \theta_{\mathcal{D}}$ for some $\theta > 0$, it is shown in [4, Theorem 4.1, Pages 1290–1291] and [4, Corollary 6.2, Pages 1298–1299] that for sufficiently small k , there exists a unique solution $(u_D^n)_{n=0}^{N+1}$ for (14)–(15) and the following $W^{1,2}(L^2)$ -error estimate holds:

$$\left(\sum_{n=1}^{N+1} k \|u_t(t_n) - \partial^1 \Pi_{\mathcal{M}} u_D^n\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \leq C(k + h_D) \|u\|_{\mathcal{C}^2([0,T]; \mathcal{C}^2(\overline{\Omega}))}. \quad (16)$$

Our aim is to get a uniform version in time for (16), that is a $W^{1,\infty}(L^2)$ -error estimate. To analyze the convergence, we introduce the following auxiliary problem: For any $n \in \llbracket 0, N + 1 \rrbracket$, find $\overline{u}_D^n \in \mathcal{X}_{D,0}$ such that

$$\langle \overline{u}_D^n, v \rangle_F = (-\Delta u(t_n), \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{D,0}. \quad (17)$$

The well-posedness for scheme (17) is given in Theorem 2.1. Acting the discrete time derivatives ∂^j , for $j \in \{1, 2\}$, on the both sides of (17) yields, for all $n \in \llbracket j, N+1 \rrbracket$

$$\langle \partial^j \bar{u}_D^n, v \rangle_F = (-\Delta \partial^j u(t_n), \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{D,0}. \quad (18)$$

Therefore $\partial^j \bar{u}_D^n$ is satisfying the same scheme (9) with right hand side F given by $-\Delta \partial^j u(t_n)$. We are able then to apply error estimates (11) and (13) to get (see [4, (25), Page 1292], [4, Corollary 6.2, Pages 1298–1299], and also [5, (15), Page 1121])

$$\|\Pi_{\mathcal{M}} \partial^j \bar{u}_D^n - \partial^j \mathcal{P}_{\mathcal{M}} u(t_n)\|_{1,2,\mathcal{M}} \leq Ch_D \|\partial^j u(t_n)\|_{C^2(\bar{\Omega})} \leq Ch_D \|u\|_{C^j([0,T]; C^2(\bar{\Omega}))} \quad (19)$$

and

$$\|\Pi_{\mathcal{M}} \partial^j \bar{u}_D^n - \partial^j u(t_n)\|_{L^2(\Omega)} \leq Ch_D \|u\|_{C^2([0,T]; C^2(\bar{\Omega}))}. \quad (20)$$

In addition to (19)–(20), we will also need to use the following $L^\infty(L^2)$ -error estimate obtained in [4, (13), Page 1291]:

$$\max_{n=0}^{N+1} \|u(t_n) - \Pi_{\mathcal{M}} u_D^n\|_{L^2(\Omega)} \leq C(k + h_D) \|u\|_{C^2([0,T]; C^2(\bar{\Omega}))}. \quad (21)$$

The following assumptions are needed (see Remark 2.1):

Assumption 1 (*Assumptions on $\mathcal{D}$ and k*). Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 1 and $k = T/(N+1)$ for a given $N \in \mathbb{N}^*$. We assume that for some given $\xi > 0$, the following assumption holds (see [7]):

$$\xi \leq \frac{m(K)}{h_D^d}, \quad \forall K \in \mathcal{M}. \quad (22)$$

In addition to this, we assume the following relation between h_D and k , for some given positive constant δ :

- When $d = 2$:

$$|\ln(h_D)|k \leq \delta. \quad (23)$$

- When $d = 3$:

$$h_D^{\frac{-1}{2}} k \leq \delta. \quad (24)$$

Assumption 2 (*Assumption on the Function f*). We assume that the function f is satisfying $f \in C^2(\mathbb{R})$ and for some positive constant κ

$$|f'(s)| + |f''(s)| \leq \kappa, \quad \forall s \in \mathbb{R}. \quad (25)$$

Remark 2.1 (*Comments on the Conditions (22), (23), and (24)*). Let us comment on the hypotheses of Assumption 1.

- The condition (22) is known in the literature, see for instance [7, Corollary 1, Page 777].
- If we assume that, for some $s > 0$, $k/h_D^s \leq \delta$, then (23) is satisfied. Indeed, since $\lim_{h_D \rightarrow 0} h_D^s |\ln(h_D)| = 0$, then for sufficiently small h_D , we have $|\ln(h_D)| \leq h_D^{-s}$. Consequently, $|\ln(h_D)|k \leq h_D^{-s}k$ for small h_D .

- The known condition $k/h_D^2 \leq 1/2$ for the convergence of the explicit scheme for the heat equation is sufficient to obtain (24) since $h_D^{-1} \leq h_D^{-2}$ for small h_D . Consequently, the conditions (23) and (24) are weaker than that known for the convergence of the explicit scheme for the heat equation.

From these facts, the hypotheses of [Assumption 1](#) are reasonable.

3. Statement of the main results

The main result of this paper is the following $W^{1,\infty}(L^2)$ -error estimate:

Theorem 3.1 (*$W^{1,\infty}(L^2)$ -Error Estimate*). Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d = 2$ or 3 , and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the solution of (1)–(2) is satisfying $u \in C^3([0, T]; C^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^*$, and denote by $t_n = nk$, for $n \in \llbracket 0, N+1 \rrbracket$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of [Definition 1](#). Let $\nabla_{\mathcal{D}}$ be the discrete gradient defined by (4)–(5) and $\langle \cdot, \cdot \rangle_F$ be the bilinear form given by (6). Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \geq \theta_{\mathcal{D}}$, for some given $\theta > 0$. There exists, according to [4], $0 < k_0 < T$ such that for all $k \leq k_0$, there exists a unique solution $(u_{\mathcal{D}}^n)_{n=0}^{N+1}$ for the scheme (14)–(15). Then, under [Assumptions 1](#) and [2](#), and for sufficiently small $h_{\mathcal{D}}$ and k , the following $W^{1,\infty}(0, T; L^2(\Omega))$ -error estimate holds, for all $n \in \llbracket 0, N+1 \rrbracket$:

$$\|u_t(t_n) - \partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^n\|_{L^2(\Omega)} \leq C(k + h_{\mathcal{D}}) \|u\|_{C^3([0, T]; C^2(\overline{\Omega}))}. \quad (26)$$

Remark 3.1 (*Importance of the Uniform-in-time Estimate (26) and Similar Results in Literature*). The error estimate (26) of [Theorem 3.1](#) provides optimal convergence order for the error $e_{\mathcal{D}}^n = u_t(t_n) - \partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^n$ between a discrete time derivative of the approximate solution and the time derivative u_t of the exact solution u in the uniform norm $L^\infty(L^2)$ -norm. This is stronger than that of (16) since (26) implies (16).

On the other hand, estimate (26) means that the discrete time derivative $\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^n$ of the approximate solution approximates the time derivative u_t . This is important in some applications, e.g. u_t represents the velocity.

As stated in [Section 1](#), we find in the literature several works which aim to prove error estimates in the uniform norm with respect to time, see for instance [1,2,8].

As mentioned also in [Section 1](#), we proved in [3,5,6], $W^{1,\infty}(L^2)$ -error estimates for the finite volume methods for linear case of heat equation $u_t - \Delta u = f$. However, $W^{1,\infty}(L^2)$ -error estimate has not been proved for the non-linear case $u_t - \Delta u = f(u)$. It is worth mentioning that the $W^{1,\infty}(L^2)$ -error estimates proved for the linear case of the heat equation $u_t - \Delta u = f$ do not require any relation between $h_{\mathcal{D}}$ and k .

The proof of [Theorem 3.1](#) is not straightforward and it is based on the L^∞ -estimates of [Lemma 3.1](#) as well as the technical [Lemma 3.2](#) given.

Lemma 3.1 (*Some L^∞ -estimates*). Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d = 2$ or 3 , and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the solution of (1)–(2) is satisfying $u \in C^3([0, T]; C^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^*$, and denote $t_n = nk$, for $n \in \llbracket 0, N+1 \rrbracket$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of [Definition 1](#). Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \geq \theta_{\mathcal{D}}$. There exists a unique solution $(u_{\mathcal{D}}^n)_{n=0}^{N+1}$ for (14)–(15) when k is sufficiently small. For all $n \in \llbracket 0, N+1 \rrbracket$, let $\bar{u}_{\mathcal{D}}^n$ be the solution of (17) and $\eta_{\mathcal{D}}^n$ be defined as $\eta_{\mathcal{D}}^n = u_{\mathcal{D}}^n - \bar{u}_{\mathcal{D}}^n$. Then, for $h_{\mathcal{D}} \leq \frac{1}{e}$ (where e is the Euler's number, i.e. $\ln(e) = 1$) if $d = 2$ and for arbitrary $h_{\mathcal{D}}$ if $d = 3$

$$\|\partial^1 \Pi_{\mathcal{M}} \bar{u}_{\mathcal{D}}^n\|_{L^\infty(\Omega)} \leq C \|u\|_{C^2([0, T]; C^2(\overline{\Omega}))}, \quad \forall n \in \llbracket 1, N+1 \rrbracket. \quad (27)$$

If we assume in addition that [Assumptions 2 and 1](#) are satisfied, then for $h_D \leq \frac{1}{e}$ if $d = 2$ and for arbitrary h_D if $d = 3$, the following estimate holds: Nonlinear finite volume scheme

$$\|\Pi_{\mathcal{M}} \eta_D^n\|_{L^\infty(\Omega)} \leq C \|u\|_{C^2([0,T];C^2(\overline{\Omega}))}, \quad \forall n \in \llbracket 1, N+1 \rrbracket. \quad (28)$$

Proof. We will prove [Lemma 3.1](#) item by item.

1. Proof of estimate (27). Let us consider a (generic) function $u_{\mathcal{M}} \in H_{\mathcal{M}}(\Omega)$. We use the discrete inequality of [[7](#), Lemma 1, Page 774] to get, for all $q \in [1, \infty)$ when $d = 2$ and for all $q \in [1, 6]$ when $d = 3$

$$\|u_{\mathcal{M}}\|_{L^q(\Omega)} \leq Cq \|u_{\mathcal{M}}\|_{1,2,\mathcal{M}}, \quad (29)$$

where C in the previous estimate is independent of q . Using now the reasoning of [[7](#), Proof of Corollary 1, Page 777] yields $\|u_{\mathcal{M}}\|_{L^\infty(\Omega)} \leq Ch_D^{\frac{d}{2}} \|u_{\mathcal{M}}\|_{L^q(\Omega)}$. Which, using (29), implies that

$$\|u_{\mathcal{M}}\|_{L^\infty(\Omega)} \leq Cqh_D^{\frac{d}{2}} \|u_{\mathcal{M}}\|_{1,2,\mathcal{M}}, \quad \forall u_{\mathcal{M}} \in H_{\mathcal{M}}(\Omega). \quad (30)$$

Taking $u_{\mathcal{M}} = \partial^1(\mathcal{P}_{\mathcal{M}}u(t_n) - \Pi_{\mathcal{M}}\bar{u}_D^n)$ in (30) and using (19) with $j = 1$ yield

$$\|\partial^1(\mathcal{P}_{\mathcal{M}}u(t_n) - \Pi_{\mathcal{M}}\bar{u}_D^n)\|_{L^\infty(\Omega)} \leq Cqh_D^{1-\frac{d}{q}} \|u\|_{C^1([0,T];C^2(\overline{\Omega}))}. \quad (31)$$

We have, for all $n \in \llbracket 1, N+1 \rrbracket$

$$\begin{aligned} \|\partial^1 \mathcal{P}_{\mathcal{M}}u(t_n)\|_{L^\infty(\Omega)} &= \|\mathcal{P}_{\mathcal{M}}(\partial^1 u(t_n))\|_{L^\infty(\Omega)} \leq \|\partial^1 u(t_n)\|_{L^\infty(\Omega)} \\ &= \left\| \frac{1}{k} \int_{t_{n-1}}^{t_n} u_t(t) dt \right\|_{L^\infty(\Omega)} \\ &\leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|u_t(t)\|_{L^\infty(\Omega)} dt \\ &\leq \|u\|_{C^1([0,T];C(\overline{\Omega}))}. \end{aligned} \quad (32)$$

Gathering this with (31) and the triangle inequality implies that

$$\begin{aligned} \|\Pi_{\mathcal{M}} \partial^1 \bar{u}_D^n\|_{L^\infty(\Omega)} &\leq \|\partial^1(\mathcal{P}_{\mathcal{M}}u(t_n) - \Pi_{\mathcal{M}}\bar{u}_D^n)\|_{L^\infty(\Omega)} + \|\partial^1 \mathcal{P}_{\mathcal{M}}u(t_n)\|_{L^\infty(\Omega)} \\ &\leq Cqh_D^{1-\frac{d}{q}} \|u\|_{C^1([0,T];C^2(\overline{\Omega}))} + \|\partial^1 \mathcal{P}_{\mathcal{M}}u(t_n)\|_{L^\infty(\Omega)} \\ &\leq Cqh_D^{1-\frac{d}{q}} \|u\|_{C^1([0,T];C^2(\overline{\Omega}))} + \|u\|_{C^1([0,T];C(\overline{\Omega}))}. \end{aligned} \quad (33)$$

Let us discuss the two cases of $d = 2$ or 3 :

1.1. The case $d = 2$. A study for the function $q \mapsto qh_D^{1-\frac{d}{q}}$ on $q \in [1, \infty)$ shows that it takes its minimum when $q = -2\ln(h_D)$. Since q should satisfy $q \geq 1$, then h_D should verify $|\ln(h_D)| \geq \frac{1}{2}$. This can be satisfied when for instance $h_D \leq \frac{1}{e}$. Taking $q = -2\ln(h_D)$ in (33) gives

$$\|\partial^1 \Pi_{\mathcal{M}} \bar{u}_D^n\|_{L^\infty(\Omega)} \leq Ch_D |\ln(h_D)| \|u\|_{C^1([0,T];C^2(\overline{\Omega}))} + \|u\|_{C^1([0,T];C(\overline{\Omega}))}. \quad (34)$$

We remark that the function $h_D \mapsto h_D |\ln(h_D)|$ defined on $(0, 1/e)$ is increasing. We deduce that $h_D |\ln(h_D)| \leq 1/e$ for all $h_D \leq 1/e$. This with (34) implies the desired estimate (27) when $d = 2$.

1.2. The case $d = 3$. Taking $q = 6$ in (33) yields $\|\partial^1(\mathcal{P}_M u(t_n) - \Pi_M \bar{u}_D^n)\|_{L^\infty(\Omega)}$ is order $\sqrt{h_D}$. This with the fact that $h_D < \text{diam}(\Omega)$ yields the desired estimate (27) when $d = 3$.

2. Proof of estimate (28). Using [4, (39), Page 1293] and [4, Corollary 6.2, Pages 1298–1299] implies that $\|\Pi_M \eta_D^n\|_{1,2,\mathcal{M}}$ is of order $k + h_D$. This implies, when taking $u_M = \Pi_M \eta_D^n$ in (30), for all $q \in [1, \infty)$ when $d = 2$ and for all $q \in [1, 6]$ when $d = 3$

$$\|\Pi_M \eta_D^n\|_{L^\infty(\Omega)} \leq C q h_D^{\frac{q}{2}} (k + h_D) \|u\|_{C^2([0,T];C^2(\bar{\Omega}))} \quad (35)$$

Let us discuss the two cases for $d = 2$ or 3 as in the previous items:

2.1. The case $d = 2$. Taking $q = -2 \ln(h_D)$ (recall that the hypothesis $h_D \leq 1/e$ of Lemma 3.1 implies that $\ln(h_D) \leq -1$ which gives $q = -2 \ln(h_D) \geq 2 > 1$) in (35) gives

$$\|\Pi_M \eta_D^n\|_{L^\infty(\Omega)} \leq C |\ln(h_D)| (k + h_D) \|u\|_{C^2([0,T];C^2(\bar{\Omega}))}. \quad (36)$$

This implies that, thanks to hypothesis (23) and the fact that $h_D |\ln(h_D)| \leq 1/e$ for all $h_D \leq 1/e$, the desired estimate (28) when $d = 2$.

2.2. The case $d = 3$. Taking $q = 6$ in (35) and using (24) to get (28). $\square$

The following technical lemma will be useful to prove Theorem 3.1, see Remark 3.2.

Lemma 3.2 (Technical Lemma). Under Assumption 2, we consider two real sequences $(\alpha^n)_n$ and $(\beta^n)_n$. Then, the following estimate holds:

$$|\partial^1 f(\alpha^{n+1}) - \partial^1 f(\beta^{n+1})| \leq \kappa |\partial^1(\alpha^{n+1} - \beta^{n+1})| + \kappa (|\partial^1 \alpha^{n+1}| + |\partial^1 \beta^{n+1}|) |\alpha^n - \beta^n|.$$

Proof. We will use mainly the following Taylor expansion, for all $\psi \in C^1[0, T]$

$$\psi(r) - \psi(s) = (r - s) \int_0^1 \psi'(r + t(s - r)) dt. \quad (37)$$

Using the Taylor expansion (37) to get

$$\partial^1 f(\alpha^{n+1}) - \partial^1 f(\beta^{n+1}) = \partial^1 \left((\alpha^{n+1} - \beta^{n+1}) \int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \right) \quad (38)$$

For two sequences $(L^n)_n$ and $(M^n)_n$, we also need to use the following rule of a discrete version for the derivative (time derivative) of the product of two functions

$$\partial^1(L^{n+1}M^{n+1}) = L^{n+1}\partial^1M^{n+1} + M^n\partial^1L^{n+1}. \quad (39)$$

Let us set

$$L^{n+1} = \int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \quad \text{and} \quad M^{n+1} = \alpha^{n+1} - \beta^{n+1}$$

Applying the rule (39) and using (38) leads to

$$\begin{aligned} \partial^1 f(\alpha^{n+1}) - \partial^1 f(\beta^{n+1}) &= \left(\int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \right) \partial^1(\alpha^{n+1} - \beta^{n+1}) \\ &\quad + (\alpha^n - \beta^n) \partial^1 \left(\int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \right). \end{aligned} \quad (40)$$

But using the Taylor expansion (37) yields

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$$\partial^1 \left(\int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \right) = \int_0^1 \left(\gamma^n(t) \left(\int_0^1 f''(G^n(s)) ds \right) \right) dt, \quad (41)$$

where

$$\gamma^n(t) = (1-t)\partial^1 \alpha^{n+1} + t\partial^1 \beta^{n+1}, \quad (42)$$

and

$$G^n(s) = \alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1}) + s(\alpha^n + t(\beta^n - \alpha^n) - \alpha^{n+1} - t(\beta^{n+1} - \alpha^{n+1})). \quad (43)$$

The expression (41) with Assumption 2 yields

$$\left| \partial^1 \left(\int_0^1 f'(\alpha^{n+1} + t(\beta^{n+1} - \alpha^{n+1})) dt \right) \right| \leq \kappa(|\partial^1 \alpha^{n+1}| + |\partial^1 \beta^{n+1}|). \quad (44)$$

This with (40), and Assumption 2 imply the desired estimate of Lemma 3.2. $\square$

Remark 3.2 (Comments on Lemma 3.2). Lemma 3.2 can be viewed as a discrete version for the following estimate, for any sufficiently smooth functions ϕ and ψ :

$$|(\phi(t))_t - (\psi(t))_t| \leq \kappa|\phi_t(t) - \psi_t(t)| + \kappa|\psi_t(t)| |\phi(t) - \psi(t)|.$$

This estimate stems from the following identity together with Assumption 2:

$$\begin{aligned} (f(\phi(t)))_t - (f(\psi(t)))_t &= \phi_t(t)f_t(\phi(t)) - \psi_t(t)f_t(\psi(t)) \\ &= (\phi_t(t) - \psi_t(t))f_t(\phi(t)) + \psi_t(t)f_t(\phi(t)) - \psi_t(t)f_t(\psi(t)) \\ &= (\phi_t(t) - \psi_t(t))f_t(\phi(t)) + \psi_t(t)(f_t(\phi(t)) - f_t(\psi(t))). \end{aligned}$$

Proof of Theorem 3.1 Writing scheme (17) in the level $n+1$ and subtracting the result from (14) to get, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$(\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)} + \langle \eta_{\mathcal{D}}^{n+1}, v \rangle_F = (\mathcal{S}^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad (45)$$

where $\eta_{\mathcal{D}}^n = u_{\mathcal{D}}^n - \bar{u}_{\mathcal{D}}^n$ and $\mathcal{S}^{n+1} = \mathbb{T}^1 + \mathbb{T}^2$ with

$$\mathbb{T}^1 = f(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}) - f(u(t_{n+1})) \quad \text{and} \quad \mathbb{T}^2 = u_t(t_{n+1}) - \partial^1 \Pi_{\mathcal{M}} \bar{u}_{\mathcal{D}}^{n+1}. \quad (46)$$

We act the discrete time derivative ∂^1 on the both sides of Eq. (45) to get, for all $n \in \llbracket 1, N \rrbracket$

$$(\partial^2 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)} + \langle \partial^1 \eta_{\mathcal{D}}^{n+1}, v \rangle_F = (\partial^1 \mathcal{S}^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)}. \quad (47)$$

Taking $v = \partial^1 \eta_{\mathcal{D}}^{n+1}$ in (47), using the fact that $\partial^2 \eta_{\mathcal{D}}^{n+1} = (\partial^1 \eta_{\mathcal{D}}^{n+1} - \partial^1 \eta_{\mathcal{D}}^n)/k$, and summing the result over $n \in \llbracket 1, J-1 \rrbracket$ where $J \in \llbracket 2, N+1 \rrbracket$ yield that

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^J\|_{L^2(\Omega)} \leq \sum_{n=1}^{J-1} k \|\partial^1 \mathcal{S}^{n+1}\|_{L^2(\Omega)} + \|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^1\|_{L^2(\Omega)}. \quad (48)$$

Let us now estimate the terms on the right hand side of (48). We begin to estimate $\|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^1\|_{L^2(\Omega)}$. Taking $n=0$ and $v = \partial^1 \eta_{\mathcal{D}}^1$ in Eq. (45) leads to

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_D^1\|_{L^2(\Omega)}^2 + \langle \eta_D^1, \partial^1 \eta_D^1 \rangle_F = (\mathcal{S}^1, \Pi_{\mathcal{M}} \partial^1 \eta_D^1)_{L^2(\Omega)}. \quad (49)$$

Since $\eta_D^0 = 0$, then $\partial^1 \eta_D^1 = (\eta_D^1 - \eta_D^0)/k = \eta_D^1/k$. Inserting this in (49) and using the Cauchy Schwarz inequality to obtain $\|\partial^1 \Pi_{\mathcal{M}} \eta_D^1\|_{L^2(\Omega)} \leq \|\mathcal{S}^1\|_{L^2(\Omega)}$. Gathering this with (48) gives

$$\|\partial^1 \Pi_{\mathcal{D}\mathcal{M}} \eta_D^J\|_{L^2(\Omega)} \leq \sum_{n=1}^{J-1} k \|\partial^1 \mathcal{S}^{n+1}\|_{L^2(\Omega)} + \|\mathcal{S}^1\|_{L^2(\Omega)}. \quad (50)$$

Using the triangle inequality together with Assumption 2 and estimates (21) and (20) (when $j = 1$), we get $\|\mathcal{S}^{n+1}\|_{L^2(\Omega)} \leq C(k + h_D) \|u\|_{C^2([0,T]; C^2(\bar{\Omega}))}$. This (when $n = 0$) with (50) leads to

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_D^J\|_{L^2(\Omega)} \leq \sum_{n=1}^{J-1} k \|\partial^1 \mathcal{S}^{n+1}\|_{L^2(\Omega)} + C(k + h_D) \|u\|_{C^2([0,T]; C^2(\bar{\Omega}))}. \quad (51)$$

Using the triangle inequality together with a Taylor expansion and estimates (20) (when $j = 2$) to get

$$\|\partial^1 \mathbb{T}^2\|_{L^2(\Omega)} \leq C(k + h_D) \|u\|_{C^3([0,T]; C^2(\bar{\Omega}))}. \quad (52)$$

Let us apply Lemma 3.2 to get a convenient estimate for $\|\partial^1 \mathbb{T}^1\|_{L^2(\Omega)}$. Using the definition of $\mathbb{T}^1$ in (46) and the triangle inequality to get

$$\|\partial^1 \mathbb{T}^1\|_{L^2(\Omega)} \leq \|\partial^1 \mathbb{T}^3\|_{L^2(\Omega)} + \|\partial^1 \mathbb{T}^4\|_{L^2(\Omega)} + \|\partial^1 \mathbb{T}^5\|_{L^2(\Omega)}, \quad (53)$$

where

$$\mathbb{T}^3 = f(\Pi_{\mathcal{M}} u_D^{n+1}) - f(\Pi_{\mathcal{M}} \bar{u}_D^{n+1}), \quad \mathbb{T}^4 = f(\Pi_{\mathcal{M}} \bar{u}_D^{n+1}) - f(\mathcal{P}_{\mathcal{M}} u(t_{n+1}))$$

and

$$\mathbb{T}^5 = f(\mathcal{P}_{\mathcal{M}} u(t_{n+1})) - f(u(t_{n+1})).$$

Applying Lemma 3.2 and using estimates (20)–(21) and (27)–(28), we get

$$\sum_{i=3}^5 \|\partial^1 \mathbb{T}^i\|_{L^2(\Omega)} \leq C \|\partial^1 \Pi_{\mathcal{M}} \eta_D^{n+1}\|_{L^2(\Omega)} + C(k + h_D) \|u\|_{C^2([0,T]; C^2(\bar{\Omega}))}. \quad (54)$$

Gathering (51)–(54) yields

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_D^J\|_{L^2(\Omega)} \leq C \sum_{n=1}^{J-1} k \|\partial^1 \Pi_{\mathcal{M}} \eta_D^{n+1}\|_{L^2(\Omega)} + C(k + h_D) \|u\|_{C^3([0,T]; C^2(\bar{\Omega}))}.$$

Choosing k such that $\frac{1}{2} \leq 1 - Ck$ gives

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_D^J\|_{L^2(\Omega)} \leq C \sum_{n=2}^{J-1} k \|\partial^1 \Pi_{\mathcal{M}} \eta_D^n\|_{L^2(\Omega)} + C(k + h_D) \|u\|_{C^3([0,T]; C^2(\bar{\Omega}))}.$$

This with a discrete Gronwall's lemma imply that $\|\partial^1 \Pi_{\mathcal{M}} \eta_D^J\|_{L^2(\Omega)}$ is of order $k + h_D$. Gathering this with estimate (20) (when $j = 1$), the fact that $\partial^1 u(t_n) - u_t u(t_n)$ is of order k , and

the triangle inequality yields the desired estimate (26). This completes the proof of [Theorem 3.1](#). $\square$ Nonlinear finite volume scheme

4. Some computational results

This section is devoted to give some computational results supporting the theoretical results included in [Theorem 3.1](#). We consider $\Omega = (0, 1)^2$ meshed with the rectangular meshes described as in [10, Pages 756–758], with uniform meshes with mesh size h , that is a mesh $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ given by

- $\mathcal{M}$ is a set of rectangles $\mathcal{M} = \{K_{ij} = [(i-1)h, ih] \times [(j-1)h, jh], (i, j) \in \llbracket 1, N \rrbracket \times \llbracket 1, N \rrbracket\}$, where $N \in \mathbb{N} \setminus \{0\}$ is given and $h = 1/N$.
- $\mathcal{E}$ is the set of the edges of the elements K_{ij} of $\mathcal{M}$.
- The family $\mathcal{P}$ is the set of points $((i - \frac{1}{2})h, (j - \frac{1}{2})h)$, where $(i, j) \in \llbracket 1, N \rrbracket \times \llbracket 1, N \rrbracket$.

For the sake of simplicity, we will consider the discrete gradient described in [11, (211)–(212), Page 333] (which is also a particular case of the discrete gradient introduced recently in [12]). The exact solution is given by

$$u(x, y, t) = \sin(\pi x) \sin(\pi y) (1 - \exp(-t)), \quad (x, y, t) \in (0, 1)^2 \times (0, 1).$$

By this way $u^0 \equiv 0$ and $f(x, y, t, s) = (2\pi^2 - 1)s + \sin(\pi x) \sin(\pi y)$, for all $(x, y, t) \in (0, 1)^2 \times (0, 1) \times \mathbb{R}$.

By some computations, we find that the discrete gradient is given by, for $K_{ij} = [(i-1)h, ih] \times [(j-1)h, jh]$

$$\nabla_{\mathcal{D}} u_{\mathcal{D}}|_{K_{ij}} = \left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}, \frac{u_{i,j+1} - u_{i,j-1}}{2h} \right), K_{ij} \cap \partial \Omega = \emptyset,$$

and with slightly modification when $K_{ij} \cap \partial \Omega \neq \emptyset$.

We presented the $W^{1,\infty}(L^2)$ -error estimate (26) for the finite volume scheme (14)–(15) which is a fixed point problem. Consequently, to deal with the non linear scheme (14)–(15), we use instead linearized schemes. In [4], we performed simulations using a linearized Crank–Nicolson scheme. These simulations show that the convergence order is $k^2 + h$ in not only $W^{1,2}(L^2)$ -norm but in $W^{1,\infty}(L^2)$ -norm. In addition to these simulations of [4] which show that the convergence order of a linearized scheme for (14)–(15) is at least $k + h$ in $W^{1,\infty}(L^2)$ -norm which support the theoretical results of [Theorem 3.1](#), we consider other simulations on the following linearized scheme for (14):

$$(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v)_{L^2(\Omega)} + \langle u_{\mathcal{D}}^{n+1}, v \rangle_F = (f(\Pi_{\mathcal{M}} u_{\mathcal{D}}^n), \Pi_{\mathcal{M}} v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \quad (55)$$

We will show numerically that the convergence order of (55) is $k + h$ in $W^{1,\infty}(L^2)$ -norm which supports the theoretical results of [Theorem 3.1](#). However, in all these tests, we have not needed to use [Assumption 1](#).

To check the numerical order, we assume an expression for the error in a given norm denoted by $|\cdot|$ of the form $|e_{h,k}| = C \cdot h^r + D \cdot k^p$, where C and D are two constants independent of h and k . This yields the following values for r and p

$$p = \log_2 \left(\frac{|e_{h,k}| - |e_{h,\frac{k}{2}}|}{|e_{h,\frac{k}{2}}| - |e_{h,\frac{k}{4}}|} \right) \quad \text{and} \quad r = \log_2 \left(\frac{|e_{h,k}| - |e_{\frac{h}{2},k}|}{|e_{\frac{h}{2},k}| - |e_{\frac{h}{4},k}|} \right). \quad (56)$$

A. First test: Convergence order with respect to the mesh size k of the time discretization
The results of the following table are obtained using Scilab programs with $h = 1/25$. The notation “–” in the table corresponds to values of k for which the numerical order cannot be computed using the second formula in (56).

Table 1 shows that the numerical order in time is one. This confirms the results of Theorem 3.1.

B. Second test: Convergence order with respect to the mesh size of the space discretization
The results of the following table are obtained using Scilab programs with $k = 0.001$.

Table 2 shows that the numerical order in space is not only one as stated in Theorem 3.1 but it is two, i.e. space discretization is performed using uniform squares. This confirms the observation stated in [12, Lines 31–33, Page 1022] where the numerical tests show a better rate of convergence of the gradient in the case of uniform squares.

5. Conclusion and perspectives

We considered a nonlinear scheme studied recently in [4] in the gradient schemes framework for the semilinear heat equation. We were interested to improve an error estimate proved in [4] but in the particular case of SUSHI. The discretization in space is performed using “SUSHI” method whereas the time discretization is uniform. We provided a sufficient condition on the function f (Assumption 2) as well as some relations (Assumption 1) between the size of the spatial mesh and the time step which allow to prove a $W^{1,\infty}(L^2)$ –error estimate. Assumption 2 on f is reasonable from the point of view that it is known in the literature of numerical methods for semilinear parabolic equations, see [14, Beginning of Section 3, Page 338]. Assumption 1 contains three conditions, namely (22), (23), and (24). The condition (22) is well known in the literature of finite volume methods, cf. [7], whereas the conditions (23) and (24) are similar to those known by CFL conditions or the one assumed to get the convergence of explicit schemes for the heat equation $u_t - u_{xx} = 0$, that is $k/h^2 < 1/2$. The $W^{1,\infty}(L^2)$ –error

Table 1.

k	Error	$ \text{Error} _{W^{1,\infty}(L^2)}$	Order
1/250	0.0341637		–
1/500	0.0181005		–
1/1000	0.0095616		0.9116372
1/2000	0.0051566		0.9549082

Table 2.

h	Error	$ \text{Error} _{W^{1,\infty}(L^2)}$	Order
1/10	0.0778465		–
1/20	0.0757006		–
1/40	0.0751631		1.9972462
1/80	0.0750287		1.9997316

estimate proved in this note can be viewed as an improvement for the $W^{1,\infty}(L^2)$ –error estimate presented in [4]. In fact $W^{1,2}(L^2)$ –error estimate is a uniform-in-time version for the $W^{1,2}(L^2)$ –error estimate. Uniform convergence-in-time is an interesting task which has been studied for instance in [8]. We aim in a future paper to extend the obtained results to GDM.

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Further reading

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