

The Q_0 -matrix completion problem

Q_0 -matrix
completion

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Abstract

A matrix is a Q_0 -matrix if for every $k \in \{1, 2, \dots, n\}$, the sum of all $k \times k$ principal minors is nonnegative. In this paper, we study some necessary and sufficient conditions for a digraph to have Q_0 -completion. Later on we discuss the relationship between Q and Q_0 -matrix completion problem. Finally, a classification of the digraphs of order up to four is done based on Q_0 -completion.

Keywords Partial matrix, Matrix completion, Q_0 -matrix, Q_0 -completion, Digraph

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1. Introduction

A *partial matrix* is a real square matrix with some specified entries while other entries are unspecified. By *completion* of a partial matrix, we have to choose specific values for the unspecified entries. The *matrix completion problem* studies those partial matrices which have desired type of completions.

A real $n \times n$ matrix A is a *P-matrix* (P_0 -matrix) if every principal minor of A is positive (nonnegative). A real $n \times n$ matrix $B = [b_{ij}]$ is a *Q-matrix* if for every $k \in \{1, 2, \dots, n\}$, $S_k(B) > 0$, where $S_k(B)$ is the sum of all $k \times k$ principal minors of B . The matrix B is Q_0 -matrix if for every $k \in \{1, 2, \dots, n\}$, $S_k(B) \geq 0$. Clearly a Q -matrix is a Q_0 -matrix but not conversely.

A *partial Q-matrix* C is a partial matrix in which $S_k(C) > 0$ for every $k \in \{1, 2, \dots, n\}$ for which all $k \times k$ principal submatrices are fully specified. Similarly a *partial Q_0 -matrix* C_1 is a partial matrix in which $S_k(C_1) \geq 0$ for every $k = 1, \dots, n$.

To make a *completion* of a partial matrix, a specific choice of value for the unspecified entries is chosen. Thus the main motive of *matrix completion problem* is to investigate the properties of partial matrices and find out those partial matrices which have a particular type of completions. In the last few years, research is done for different classes of matrices in the area of Matrix Completion Problems. Several researchers have developed many results of matrix completion problems for different classes of matrices including P and P_0 , Q -matrices (e.g., [2–5, 7, 8, 10, 14]). To see the details of the definition and properties of different classes of partial matrices (i.e. P , P_0 or Q -partial matrices) and results regarding matrix completion problems, we suggest [9].

From the beginning of matrix completion problems, we have found that graphs and digraphs are widely used in solving the matrix completion problems. A *digraph* D is a pair

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(V, A) , where V is a finite nonempty set of objects, called *vertices*, and A is a set of ordered pairs of vertices, called *arcs* or *directed edges*. We use V_D and A_D to denote the vertex set and the arc set of D respectively and we write frequently $v \in D$ (respectively, $(u, v) \in D$) to say that $v \in V(D)$ (respectively $(u, v) \in A(D)$). An arc $x = (u, u)$ in the arc set of a digraph D , which is called a loop at the vertex u is allowed in our given definition. Most of the graph-theoretic terms used in this article can be found in any standard book, for example [1,6]. However for the convenience of the readers' of this article, we request them to follow the introduction part of the article [11–13].

In this paper, we have studied the (combinatorial) Q_0 -matrix problem. In Section 2, we have defined the partial Q_0 -matrix and the Q_0 -matrix completion problem. We have discussed the relationship between digraphs and Q_0 -completion in Section 3. We have discussed some necessary and sufficient conditions for Q_0 -matrix completion problem in Section 4. In Section 5, we tried to find out the relationship between Q and Q_0 -matrix completion problem. Finally in Section 6, we have singled out of all digraphs of order up to 4 with Q_0 -matrix completion.

2. Partial Q_0 -matrices and their completion problem

A partial Q_0 -matrix $C = [c_{ij}]$ is a partial matrix in which $S_k(C) \geq 0$ for every $k = 1, \dots, n$ for which all $k \times k$ principal submatrices of C are fully specified. In Proposition 2.1, we characterize $C = [c_{ij}]$ as follows.

Proposition 2.1. *Suppose $C = [c_{ij}]$ is a partial matrix. Then C is a partial Q_0 -matrix if and only if exactly one of the following holds:*

- (i) *At least one diagonal entry of C is not specified.*
- (ii) *All diagonal entries are specified so that $\text{Tr}(M) \geq 0$ and C has an off diagonal unspecified entry.*
- (iii) *All entries of C are specified and C is a Q_0 -matrix.*

A completion A of a partial Q_0 -matrix C is called a Q_0 -completion of C , if A is a Q_0 -matrix. If A is a Q_0 -matrix, then any matrix which is permutation similar to A is a Q_0 -matrix. As a consequence any digraph isomorphic to D which has Q_0 -completion also has Q_0 -completion.

Any partial Q_0 matrix C with all unspecified diagonal entries has Q_0 -completion. By choosing sufficiently large values for the unspecified diagonal entries, the desired Q_0 -completion of C is obtained. Now consider a partial Q_0 -matrix C with unspecified diagonal entries at (i, i) positions ($i = k + 1, \dots, n$). We may not get a Q_0 -completion of C in case $C[1, \dots, k]$ is fully specified. To see this, the partial matrix,

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & ? \end{bmatrix},$$

where ? denotes an unspecified entry, does not have Q_0 -completion. For any completion A of C , we have $S_2(A) \leq 0$. However, if $C[1, \dots, k]$ has an unspecified entry and has a Q -completion, then C has a Q_0 -completion. By choosing sufficiently large values for the unspecified diagonal entries, a Q_0 -completion of C can be obtained. We write our observations in the following results:

Theorem 2.2. *If a matrix C omits all diagonal entries, then C has Q_0 -completion.*

Theorem 2.3. Suppose C be a partial Q_0 -matrix in which the diagonal entry at $(r+1, r+1)$ position is unspecified. If the principal submatrix $C[1, \dots, r]$ of C is not fully specified and has Q -completion, then C has Q_0 -completion.

Corollary 2.4. Suppose C be a partial Q_0 -matrix in which the diagonal entries at (i, i) positions $(i = r+1, \dots, n)$ are unspecified. If the principal submatrix $C[1, \dots, r]$ of C is not fully specified and has Q -completion, then C has Q_0 -completion

The following example shows that the converse of [Corollary 2.4](#) is not true.

Example 2.5. Consider the partial matrix,

$$C = \begin{bmatrix} ? & c_{12} & ? & ? \\ c_{21} & d_2 & ? & c_{24} \\ c_{31} & c_{32} & ? & ? \\ c_{41} & ? & c_{43} & d_4 \end{bmatrix},$$

where ? denotes the unspecified entries. We show that C has Q_0 -completions, though there are occasions when $C[2, 4]$ does not have Q -completion. For $t > 0$, consider the completion $B(t)$ of C defined as follows:

$$B(t) = \begin{bmatrix} t & c_{12} & 0 & 0 \\ c_{21} & d_2 & t & c_{24} \\ c_{31} & c_{32} & t & t \\ c_{41} & t & c & d_4 \end{bmatrix}.$$

Then,

$$S_1(B(t)) = 2t + \sum d_i,$$

$$S_2(B(t)) = t^2 + f_1(t),$$

$$S_3(B(t)) = t^3 + f_2(t),$$

$$S_4(B(t)) = t^4 + f_3(t),$$

where $f_i(t)$ is a polynomial in t of degree at most $i, i = 1, 2, 3$. Consequently, $B(t)$ is a Q_0 -matrix for sufficiently large t , and therefore, C has Q_0 -completion. However, the partial Q -matrix

$$C[2, 4] = \begin{bmatrix} 0 & 0 \\ x_{42} & 1 \end{bmatrix},$$

with unspecified entry x_{42} , is the principal submatrix of C induced by its diagonal $\{2, 4\}$. That $C[2, 4]$ does not have Q -completion is evident, because $S_2(M[2, 4]) = 0$ for any completion of $C[2, 4]$.

Remark 2.6. We can see that $C[1, 2, \dots, r]$ in [Theorem 2.3](#) may not be a partial Q -matrix. If all the specified diagonal entries are zero, then [Theorem 2.3](#) does not hold automatically. Also [Example 2.5](#) shows that $C[2, 4]$ may not have Q_0 -completion. To see that consider a partial Q_0 -matrix

$$C[2, 4] = \begin{bmatrix} -1 & 0 \\ x_{42} & 1 \end{bmatrix},$$

with unspecified entry x_{42} . $C[2, 4]$ does not have Q_0 -completion since for any completion B_1 of $C[2, 4]$, we have $S_2(C[2, 4]) \leq 0$.

3. Digraphs and Q_0 -completions

An $n \times n$ partial matrix C specifies a digraph $D = (\{1, 2, \dots, n\}, A_D)$ if for $1 \leq i, j \leq n$, $(i, j) \in A_D$ if and only if the (i, j) th entry of C is specified. As an example, we can see that the partial Q_0 -matrix C in [Example 2.5](#) specifies the digraph D in [Figure 1](#).

Theorem 3.1. Suppose C is a partial matrix specifying the digraph D . If the partial submatrix of C induced by every strongly connected induced subdigraph of D has Q_0 -completion, then C has Q_0 -completion.

Proof. First we consider the case when D has two strong components say, D_1 and D_2 . Later on the general result will automatically follow from induction. If required, by a relabelling of the vertices of D , we have

$$C = \begin{bmatrix} C_{11} & C_{12} \\ X & C_{22} \end{bmatrix},$$

where C_{ii} is a partial Q_0 -matrix specifying D_i , $i = 1, 2$, and X contains all unspecified entries. Now, we have C_{ii} has a Q_0 -completion B_{ii} . Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in X as well as all unspecified entries in C_{12} as 0. Then, for $2 \leq k \leq |D|$ we have,

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11})S_{k-r}(B_{22}) \geq 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii} . Thus C can be completed to a Q_0 -matrix ■.

The proof of the following result is similar.

Theorem 3.2. Suppose C is a partial matrix specifying the digraph D . If the partial submatrix of C induced by each component of D has a Q_0 -completion, then C has a Q_0 -completion.

The converse of [Theorem 3.1](#) is not true. For example, every partial Q_0 -matrix specifying the digraph D in [Figure 1](#) has Q_0 -completion, although the strong component D_1 induced by vertices $\{1, 2\}$ does not have Q_0 -completion (see [Example 3.3](#)).

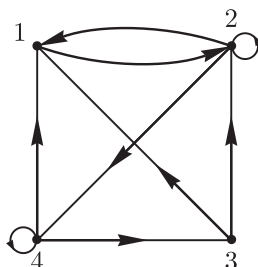


Figure 1.
The digraph D .

Example 3.3. Consider the digraph D in Figure 1. We show that D has Q_0 -completion, but the strong component D_1 induced by vertices $\{1, 2\}$ does not have Q_0 -completion. Let $C = [c_{ij}]$ be a partial Q_0 -matrix specifying D . Then for $t > 0$, C can be completed to a Q_0 -matrix $B(t)$ (see Example 2.5) but the principal submatrix induced by the digraph D_1 i.e. $C[1, 2]$ does not have Q_0 -completion. To see that $C[1, 2]$ does not have Q_0 -completion, consider the partial Q_0 -matrix

$$C[1, 2] = \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix},$$

with unspecified entry x . Then for any Q_0 -completion B of $C[1, 2]$, we have $S_2(B) \leq 0$ and hence $C[1, 2]$ does not have Q_0 -completion.

The property of having Q_0 -completion is not hereditary which can be also seen from Example 2.5.

4. The Q_0 -completion problem

A digraph D has Q_0 -completion, if every partial Q_0 -matrix specifying D can be completed to a Q_0 -matrix. The main motive of Q_0 -matrix completion problem is to study and classify all digraphs D based on Q_0 -completion.

4.1 Sufficient conditions for Q_0 -matrix completion

Theorem 4.1. If a digraph $D \neq K_n$ of order n has Q_0 -completion, then any spanning subdigraph of D has Q_0 -completion.

Proof. Suppose H be a spanning subdigraph of D and C_H be a partial Q_0 -matrix specifying the digraph H . Consider a partial matrix C_D obtained from C_H by specifying the entries corresponding to $(i, j) \in C_D \setminus C_H$ as 0. Since $D \neq K_n$, by Proposition 2.1, C_D is a partial Q_0 -matrix specifying D . Let B be a Q_0 -completion of C_D . Clearly, B is a Q_0 -completion of C_H . ■

Theorem 4.2. Suppose $D \neq K_n$ be a digraph such that $\bar{D}$ is stratified. If it is possible to sign the arcs of $\bar{D}$ so that the sign of every cycle in D is of positive sign, then D has Q_0 -completion.

Proof. Suppose C be a partial Q_0 -matrix specifying the digraph D . For any $t > 0$, consider a completion B of C by choosing the unspecified entry $x_{ij} = \text{sgn}(i, j)t$ (using the sign of the arc in $\bar{D}$). Then for each $k = 2, 3, \dots, n$, we have,

$$S_k(B) = c_k t^k + r_k(t) \quad (1)$$

where c_k is the number of permutation subdigraphs of order k in D and $r_k(t)$ is a polynomial of degree less than k . If D contains all loops, then the trace of any partial Q_0 -matrix specifying D is nonnegative; if D omits a loop, then $S_1(B) = c_1 t + r_0$, where c_1 is the number of loops in D and $r_0 \in \mathbb{R}$. Now by choosing t sufficiently large results, B becomes a Q_0 -matrix. ■

Example 4.3. Consider the complement $\bar{D}$ in Figure 2 of the digraph D in Figure 1. It can be easily seen that the digraph $\bar{D}$ is stratified. Also it is possible to sign the arcs of $\bar{D}$ with positive sign, thus by Theorem 4.2 the digraph D has Q_0 -completion.

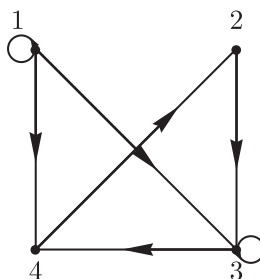
Corollary 4.4. If D is a digraph and D has a stratified spanning subdigraph that has a signing in which the sign of every cycle is +, then D has Q_0 -completion.

4.2 Necessary conditions for Q_0 -matrix completion

In this section we provide some necessary conditions for a digraph to have Q_0 -completion.

Theorem 4.5. Suppose D be a digraph of order n which includes all loops. If $\bar{D}$ has no 2-cycle, then D does not have Q_0 -completion.

Figure 2.
The digraph $\bar{D}$.



Proof. Let D be a digraph of order n which includes all loops. Suppose $C = [c_{ij}]$ be a partial Q_0 -matrix specifying the digraph D which is defined as follows:

$$c_{ij} = \begin{cases} 1, & \text{if } (i, j) = (1, 1), (i, j) \in D \\ -1, & \text{if } (i, j) = (2, 2), (i, j) \in D \\ 0, & \text{for all } (i, j) \in D \setminus \{(1, 1), (2, 2)\}. \end{cases}$$

It is clear that C is a partial Q_0 -matrix specifying D . Now $\bar{D}$ does not contain a 2-cycle, then for any completion B of C , we have $S_2(B) \leq 0$. Thus D does not have Q_0 -completion. ■

Example 4.6. Consider the digraph D_2 In Figure 3. Suppose

$$C = \begin{bmatrix} 1 & 0 \\ ? & -1 \end{bmatrix},$$

be a partial Q_0 -matrix specifying the digraph D_2 . Then for any completion B of C , we have $S_2(B) \leq 0$ by Theorem 4.5. Hence, C cannot be completed to a Q_0 -matrix.

Corollary 4.7. If a digraph D of order n includes all loops and has Q_0 -completion, then $\bar{D}$ must not be a tournament or subdigraph of a tournament.

Proof. If $\bar{D}$ is a tournament or a subdigraph of a tournament, then it does not contain a 2-cycle. Hence, the result follows. ■

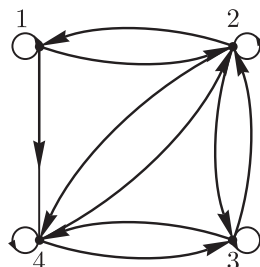
The converse of Theorem 4.5 is not true which follows from Example 4.8.

Example 4.8. Consider the digraph D_3 in Figure 4. The complement of the digraph D_3 i.e. $\bar{D}_3$ contains a 2-cycle. But D_3 does not have Q_0 -completion. Consider a partial Q_0 -matrix

Figure 3.
The digraph D_2 .



Figure 4.
The digraph D_3 .



$$C = \begin{bmatrix} 0 & 1 & x_{13} & 0 \\ 1 & 0 & 0 & 0 \\ x_{31} & 0 & 1 & 0 \\ x_{41} & 0 & 0 & 0 \end{bmatrix},$$

specifying the digraph D_3 . Then for any completion B of C , we have $S_3(B) \leq 0$. Hence, C cannot be completed to a Q_0 -matrix.

5. Comparison between Q -completion and Q_0 -completion

Although every Q -matrix is a Q_0 -matrix, but the completion problem of these two classes is different. We list these observations in the following result.

Theorem 5.1. *If a digraph D has Q_0 -completion, then it must also have Q -completion.*

Proof. Suppose D be a digraph that has Q_0 -completion and M be a partial Q -matrix specifying the digraph D . Then, the sums of all fully specified principal minor of same order of M are positive. Since the determinant and each principal minor of a matrix are a continuous function of its entries, there is $\epsilon > 0$ such that the partial matrix M_0 obtained from M by decreasing the specified diagonal entries by ϵ is a partial Q -matrix. Since a partial Q -matrix is a partial Q_0 -matrix, M_0 is a partial Q_0 -matrix specifying D . Consequently, M_0 has a Q_0 -completion B_0 . We now have a Q -completion of M , namely, $B = B_0 + \epsilon I$, where I is the identity matrix. ■

The following equivalent corollary is immediate.

Corollary 5.2. *Any digraph which does not have Q -completion does not have Q_0 -completion.*

But the converse of Theorem 5.1 is not completely true which can be seen in the following two cases.

Case 1. Suppose D includes all loops. In this case D has Q -completion but does not have Q_0 -completion.

Example 5.3. Consider the symmetric 4-cycle C_4 (Figure 5) which includes all loops. Now C_4 has Q -completion (see Example 2.2 of [4]). To see that C_4 does not have Q_0 -completion, consider the partial Q_0 -matrix

$$M = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 1 & 0 & x_{24} \\ x_{31} & 0 & 0 & 1 \\ 0 & x_{42} & 1 & -1 \end{bmatrix}$$

specifying C_4 . For a completion B of M , the 3×3 principal minor is given by

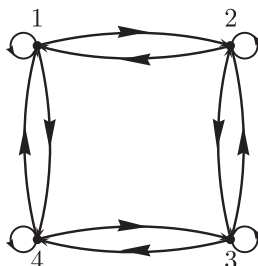


Figure 5.
The digraph D .

$$B(1, 2, 3) = -x_{13}x_{31}$$

$$B(1, 2, 4) = 0$$

$$B(1, 3, 4) = x_{13}x_{31}$$

$$B(2, 3, 4) = -1$$

Then we have $S_3(B) = -1 \leq 0$ and M cannot be completed to a Q_0 -matrix.

Case 2. Suppose D omits at least a loop. Then we have the following theorem:

Theorem 5.4. *Suppose D be a digraph such that D omits at least a loop. If D has Q -completion, then D must have Q_0 -completion.*

Proof. Suppose M be a partial Q_0 -matrix specifying D . Since D omits at least a loop, thus at least a diagonal entry of M is unspecified. Thus M is a partial Q -matrix. Since D has Q -completion, M can be completed to a Q -matrix B . Consequently, B is a Q_0 -matrix. ■

6. Q_0 -Completion of digraphs of small order

With the help of the results obtained in the previous sections, we will sort out all digraphs of order ≤ 4 which have loops at all its vertices and have Q_0 -completion. In this regard we will take the help of the nomenclature of the digraphs as per their order in [6, Appendix, p. 233]. Here, $D_p(q, n)$ denotes the n th member digraph with loop at each p vertices and it has q (non-loop) arcs.

As we know that any matrix under permutation similarity to a Q_0 -matrix is also a Q_0 -matrix, if a digraph D has Q_0 -completion, then any isomorphic digraph of D has Q_0 -completion, that is, any digraph obtained by labelling the unlabelled digraph associated to D has Q_0 -completion.

Clearly, any digraph of order 1 (with or without a loop) has Q_0 -completion. There are only two non-isomorphic digraphs of order 2 with loops say, $D_2(0, 1)$ and $D_2(2, 1)$ have Q_0 -completion.

The rest of the section is broken up into a series of lemmas.

Lemma 6.1. *For $1 \leq p \leq 4$, the digraphs $D_p(q, n)$ which are listed below do not have Q_0 -completion.*

$$\begin{array}{lll} p = 2; & q = 2; & n = 1 \\ p = 3; & q = 3; & n = 2, 3 \\ & q = 4; & n = 2, 3, 4 \\ & q = 5; & n = 1 \\ p = 4; & q = 6; & n = 45 - 48 \\ & q = 7; & n = 29 - 38 \\ & q = 8; & n = 16 - 27 \\ & q = 9; & n = 4 - 13 \\ & q = 10; & n = 1 - 5 \\ & q = 11; & n = 1. \end{array}$$

Proof. Each of the digraphs listed above satisfies [Theorem 4.5](#) and hence the result follows. ■

Lemma 6.2. *The digraphs $D_4(7, 2)$ and $D_4(8, 2)$ do not have Q_0 -completion.*

Proof. In [Example 5.3](#), it is seen that the digraph $D_4(8, 2)$ (i.e. C_4) does not have Q_0 -completion. Suppose

$$M = \begin{bmatrix} 1 & 0 & x_{13} & 1 \\ 0 & 0 & x_{23} & x_{24} \\ x_{31} & 0 & -1 & 0 \\ 1 & x_{42} & 0 & 0 \end{bmatrix},$$

be a partial Q_0 -matrix specifying the digraph $D_4(7, 2)$. Now for any Q_0 -completion B of M , we have $S_3(B) = -1$. Hence $D_4(7, 2)$ does not have Q_0 -completion. ■

Lemma 6.3. For $1 \leq p \leq 4$, the digraphs $D_p(q, n)$ which are listed below do not have Q_0 -completion.

- $p = 3;$ $q = 2;$ $n = 1, 3, 4$
- $q = 3;$ $n = 1, 4$
- $q = 4;$ $n = 1$
- $p = 4;$ $q = 3;$ $n = 8, 11$
- $q = 4;$ $n = 10, 12, 14, 15, 21, 27$
- $q = 5;$ $n = 4-6, 11, 14-17, 19, 21-24, 26, 28, 29, 31, 34, 36, 37$
- $q = 6;$ $n = 1, 2, 9-13, 15-23, 26, 27, 29, 30, 32-41, 43, 44$
- $q = 7;$ $n = 1, 3-28$
- $q = 8;$ $n = 1, 3-15$
- $q = 9;$ $n = 1-3.$

Proof. Each of the digraphs does not have Q -completion, thus by [Corollary 5.2](#) the above digraphs do not have Q_0 -completion. ■

Theorem 6.4. For $1 \leq p \leq 4$, the digraphs $D_p(q, n)$ which are listed below have Q_0 -completion.

- $p = 2;$ $q = 0, 2;$ $n = 1$
- $p = 3;$ $q = 0, 1;$ $n = 1$
- $q = 2;$ $n = 2$
- $q = 6;$ $n = 1$
- $p = 4;$ $q = 0, 1;$ $n = 1$
- $q = 2;$ $n = 1-5$
- $q = 3;$ $n = 1-7, 9, 10, 12, 13$
- $q = 4;$ $n = 1, 3-9, 11, 13, 16, 17, 19, 20, 22-26$
- $q = 5;$ $n = 1, 3, 7, 8, 10, 12, 13, 18, 20, 25, 27, 30, 32, 33, 35, 38$
- $q = 6;$ $n = 7, 24, 25, 28, 31, 42$
- $q = 12;$ $n = 1.$

Proof. The complement $\overline{D_p(q, n)}$ of each of the digraphs $D_p(q, n)$ is stratified and it is possible to sign the arcs of the $\overline{D_p(q, n)}$ with positive sign, thus by [Theorem 4.2](#), each of the digraphs listed above has Q_0 -completion. ■

Remark 6.5. In this paper, the Q_0 -matrix completion is discussed. Some necessary and sufficient conditions for a digraph to have Q_0 -completion are discussed. Although these conditions helped us to single out the digraphs of order at most 4 as to Q_0 -completion, the problem is far from being completely solved. A complete characterization for a digraph to have Q_0 -completion is still unresolved. Out of 218 digraphs of order 4, only 11 digraphs are still not singled out to have Q_0 -completion or not. Since the Q_0 -completion problem is not still

fully solved, thus the following digraphs $D_p(q, n)$, $1 \leq p \leq 4$ are not classified according to the Q_0 -completion.

$$\begin{array}{lll} p = 4; & q = 4; & n = 2, 18 \\ & q = 5; & n = 2, 9 \\ & q = 6; & n = 3-6, 8, 14. \end{array}$$

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Further reading

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