

A Method of approximation for a zero of the sum of maximally monotone mappings in Hilbert spaces

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Abstract

Our purpose of this study is to construct an algorithm for finding a zero of the sum of two maximally monotone mappings in Hilbert spaces and discuss its convergence. The assumption that one of the mappings is α -inverse strongly monotone is dispensed with. In addition, we give some applications to the minimization problem. Our method of proof is of independent interest. Finally, a numerical example which supports our main result is presented. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

Keywords Firmly nonexpansive, Hilbert spaces, Maximally monotone mapping, Strong convergence, Zero points

Paper type Original Article

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $A : H \rightarrow 2^H$ be a nonlinear mapping. The domain, range, zero, and graph of A are respectively the sets $Dom(A) = \{x \in H : Ax \neq \emptyset\}$, $R(A) = \{Ax : x \in Dom(A)\}$, $Zero(A) = \{x \in H : 0 \in Ax\}$ and $Gph(A) = \{(x, y) \in H \times H : y \in Ax\}$. A mapping $A : H \rightarrow 2^H$ is called monotone if for any $x, y \in H$ and $u \in Ax, v \in Ay$ we have

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$$\langle u - v, x - y \rangle \geq 0. \quad (1.1) \text{ Approximation in Hilbert spaces}$$

A monotone mapping $A : H \rightarrow 2^H$ is called maximally monotone if $Gph(A)$ is not properly contained in the graph of any other monotone operator. The resolvent of A is given by $J_{\lambda A} = (I + \lambda A)^{-1}$, where I is the identity mapping on H and $\lambda > 0$.

Let $A, B : H \rightarrow 2^H$ be maximally monotone mappings. Consider the problem of finding $z \in H$ such that

$$0 \in Az + Bz. \quad (1.2)$$

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We denote the solution set of (1.2) by $(A + B)^{-1}(0)$. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. For solving problem (1.2), we remark that several authors have studied different iterative schemes (see, for example, [3,4,10,11,14,21] and the references therein).

In 1979, Passty [11] introduced *Forward-backward* splitting method which defines a sequence $\{x_n\}$ by

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, \quad n \geq 1, \quad (1.3)$$

where $\{r_n\}$ is a sequence of positive numbers, A and B are maximal monotone mappings with $Dom(A) \subset Dom(B)$, and A is single valued. This method is essentially a generalization of the classical gradient method for constrained convex optimization and monotone variational inequalities, and inherit restrictions similar to those methods such as A is single valued. In general, this method provides weak convergence even with the restriction that A is single-valued. It fails to provide weak convergence results to the zero of the sum of the general maximal monotone mappings.

In 1979, Lions and Mercier [7] introduced Peaceman–Rachford splitting method whose iteration method is given by

$$\begin{aligned} y_n &= (2(I + \lambda B)^{-1} - I)x_n, \\ z_n &= (2(I + \lambda A)^{-1} - I)y_n, \\ x_{n+1} &= (1 - \rho_n)x_n + \rho_n z_n, \quad n \geq 1, \end{aligned} \quad (1.4)$$

where $\lambda > 0$ is a fixed scalar, and $\{\rho_n\} \subset (0, 1]$ is a sequence of relaxation parameters. They proved weak convergence of this sequence to the solution of problem (1.2) under certain conditions.

In 2008, Eckstein and Svaiter [5] constructed new approach splitting algorithms which starts by reformulating (1.2) as the problem of locating a point in a certain *extended solution set* $S_e(A, B) \subset H \times H$ and proved weak convergence results provided that H has finite dimension or $A + B$ is maximal monotone mapping. The extended solution set for the problem (1.2), which is the subset of $H \times H$ is defined by:

$$S_e(A, B) = \{(z, w) \in H \times H : w \in B(z), -w \in A(z)\}. \quad (1.5)$$

We treat $H \times H$ as a Hilbert space by endowing it with the canonical inner product

$$\langle (x, x'), (y, y') \rangle = \langle x, y \rangle + \langle x', y' \rangle.$$

More recently, in order to solve problem (1.2), Svaiter [17] studied the following Algorithm: for any $\lambda > 0, x_0, b_0 \in H$,

Step 1. Compute, y_n, a_n such that $a_n \in A(y_n)$, $\lambda a_n + y_n = x_{n-1} - \lambda b_{n-1}$.

Step 2. Compute, x_n, b_n such that $b_n \in B(x_n)$, $\lambda b_n + x_n = y_n + \lambda b_{n-1}$, $n \geq 1$.

They proved that $\{(x_n, b_n)\}$ and $\{(y_n, -a_n)\}$ converge weakly to a point (z, w) in $S_e(A, B)$, where $z \in (A + B)^{-1}(0)$. We remark that the convergence is still weak convergence.

With regard to a strong convergence, several authors have studied different iterative schemes (see for example, [12,18–20,22] and the references therein) for a zero of the sum of monotone mappings A and B .

In 2012, Takahashi et al. [20] proved some strong convergence theorems of the Halpern-type iteration in a Hilbert space H , which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_n B}(x_n - r_n A x_n)), \quad \forall n \geq 1, \quad (1.6)$$

where $u \in H$ is a fixed point and A is an α -inverse strongly monotone mapping on H and B is a maximal monotone mapping on H . Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a zero point of $A + B$. A monotone mapping $A : H \rightarrow H$ is said to be α -inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H. \quad (1.7)$$

Several researchers have also studied and obtained similar results in Hilbert and Banach spaces more general than Hilbert spaces (see, e.g. [8,13,16]). However, we observe that the strong convergence results available for the zero of the sum of monotone mappings A and B , are either one of the mappings is α -inverse strongly monotone or single-valued.

Question 1. A natural question arises whether we can obtain strong convergence result for approximating a zero of the sum of two maximally monotone mappings?

Motivated and inspired by the above results, our purpose in this paper is to construct an algorithm for finding a zero of the sum of maximally monotone mappings via the extended solution set $S_e(A, B) \subset H \times H$ and discuss its strong convergence. The assumption that one of the mappings is α -inverse strongly monotone is dispensed with. Our results provide an affirmative answer to our concern. Our method of proof is of independent interest. Our results improve, extend, and generalize many results in the literature.

2. Preliminaries

Let H be a real Hilbert space, C be a nonempty closed convex subset of H . The set of fixed point of the mapping $T : C \rightarrow C$ is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. The following lemmas shall be used in the sequel.

Definition 2.1. Let H be a real Hilbert space, C be a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be Lipschitz if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \quad (2.1)$$

T is said to be *nonexpansive* mapping if $L = 1$ and it is called *contraction* if $L \in [0, 1)$.

We remark that every contraction mapping is nonexpansive and every nonexpansive mapping is Lipschitz mapping.

Definition 2.2. Let H be a real Hilbert space, C be a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C. \quad (2.2)$$

Lemma 2.3 ([5]). *Finding a point in $S_e(A, B)$ is equivalent to solving (1.2) in the sense that*

$$z \in (A + B)^{-1}(0) \Leftrightarrow \exists w \in H : (z, w) \in S_e(A, B). \quad (2.3)$$

Lemma 2.4 ([5]). Given any two maximal monotone mappings $A, B : H \rightarrow 2^H$, the corresponding extended solution set $S_e(A, B)$ is closed and convex in $H \times H$. Approximation
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Lemma 2.5 ([5]). Given any two points $(x, b) \in \text{Gph}(B)$ and $(y, a) \in \text{Gph}(A)$, define the function $\varphi : H \times H \rightarrow \mathbb{R}$ via

$$\varphi(z, w) = \langle z - x, b - w \rangle + \langle z - y, a + w \rangle.$$

Then, for any $(z, w) \in S_e(A, B)$ we have $\varphi(z, w) \leq 0$, that is

$$S_e(A, B) \subseteq \{(z, w) \in H \times H \mid \varphi(z, w) \leq 0\}.$$

Additionally, φ is both continuous and affine, $\nabla \varphi = (a + b, x - y)$, and

$$\nabla \varphi = 0 \Leftrightarrow (x, b) \in S_e(A, B), x = y, a = -b.$$

Furthermore, $\nabla \varphi = 0$ implies $\varphi(z, w) = 0$ for all $z, w \in H$.

The function φ in Lemma 2.5 is called decomposable separators.

Lemma 2.6 ([23]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n \delta_n| < \infty$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 ([2]). Let $x, y \in H$. If H is a real Hilbert space, then the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.8 ([6] Demiclosedness Principle). Let H be a Hilbert space, C be a closed and convex subset of H , and $T : C \rightarrow C$ be a nonexpansive operator with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converge to $x \in C$ and $\{(I - T)x_n\}$ converge strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.

Lemma 2.9 ([24]). Let C be a closed and convex subset of a real Hilbert space H , given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if the following inequality holds:

$$\langle x - y, z - y \rangle \leq 0$$

for all $z \in C$.

3. Main results

In this section, we introduce our Algorithm for finding a point which is a solution of the sum of maximally monotone mappings as a problem of locating a point in the extended solution set $S_e(A, B)$ in Hilbert spaces and discuss its convergence.

Let H be a real Hilbert space. Hereafter, let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and let $\{\beta_n\}$ be a decreasing sequence in $(0, 1]$ such that $\beta_0 = 1$, and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

We now propose the following algorithm which uses basically Algorithm 2 of [5].

Algorithm 3.1.

Step 0: Select initial guess $p_0 = (z_0, w_0) \in H \times H$.

Step 1: Given $(z_n, w_n) \in H \times H$, for $n \geq 0$, compute:

$$x_n = (I + \lambda_n B)^{-1}(z_n + \lambda_n w_n),$$

$$y_n = (I + \mu_n A)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n),$$

where $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\theta_n \in \mathbb{R}$.

Step 2: Compute $a_n = \frac{1}{\mu_n}((1 - \theta_n)z_n + \theta_n x_n - y_n) - w_n$ and $b_n = w_n + \frac{1}{\lambda_n}(z_n - x_n)$.

Step 3: Define $\varphi_i : H \times H \rightarrow \mathbb{R}$ to be

$$\varphi_i(z, w) := \langle z - x_i, b_i - w \rangle + \langle z - y_i, a_i + w \rangle,$$

and compute

$$T_i p_n = \begin{cases} (z_n, w_n) & \text{if } \varphi_i(z_n, w_n) \leq 0, \\ (z_n, w_n) - \left(\frac{\varphi_i(z_n, w_n)}{\|\nabla \varphi_i\|^2} \right) \nabla \varphi_i & \text{if } \varphi_i(z_n, w_n) > 0, \end{cases} \quad (3.1)$$

where $T_i p_n$ is the metric projection of $p_n = (z_n, w_n)$ onto the set

$$H_i = \{(z, w) \in H \times H : \varphi_i(z, w) \leq 0\}, \text{ for each } i = 1, 2, \dots, n.$$

Step 4: Compute

$$\begin{cases} q_n = \beta_n p_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n, \\ p_{n+1} = \alpha_n f(p_n) + (1 - \alpha_n) q_n, \quad \forall n \geq 0, \end{cases} \quad (3.2)$$

where $f : H \times H \rightarrow H \times H$ is a contraction mapping with constant α . Set $n := n + 1$ and go to **Step 1**.

Remark 3.2. By maximality of B the resolvent mapping $(I + \lambda_n B)^{-1}$ is single-valued and well-defined (see, e.g., [9]). Hence

$$(x_n, b_n) = \left((I + \lambda_n B)^{-1}(z_n + \lambda_n w_n), w_n + \frac{1}{\lambda_n}(z_n - x_n) \right),$$

exists and is unique. Similarly by maximality of A the resolvent mapping $(I + \mu_n A)^{-1}$ is single-valued and well-defined, thus

$$(y_n, a_n) = \left((I + \mu_n A)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \frac{1}{\mu_n}((1 - \theta_n)z_n + \theta_n x_n - y_n) - w_n \right),$$

exists and is unique. By rearranging equations in Step 1, one has the following equations:

$$\begin{cases} x_n + \lambda_n b_n = z_n + \lambda_n w_n; \\ y_n + \mu_n a_n = (1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n; \end{cases} \quad (3.3)$$

where $a_n \in A(y_n)$ and $b_n \in B(x_n)$. Hence decomposable separators function φ in [Lemma 2.5](#) is well defined. Approximation
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We proceed with the following lemmas which will be used to prove our main theorem.

Lemma 3.3. *The sequence $\{p_n\}$ generated by [Algorithm 3.1](#) is bounded.*

Proof. Let $p^* = (z^*, w^*) \in S_e(A, B)$, then by [Lemma 2.5](#) we have $p^* \in H_i$ for all $i \geq 0$ and hence $S_e(A, B) \subset \cap_{i=1}^{\infty} F(T_i) = \mathcal{F}$. Now, from [\(3.2\)](#) and the fact that T_i is firmly nonexpansive we get

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$$\begin{aligned} \|q_n - p^*\| &= \left\| \beta_n p_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n - p^* \right\| \\ &= \left\| \beta_n p_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n - \sum_{i=1}^n (\beta_{i-1} - \beta_i) p^* + \sum_{i=1}^n (\beta_{i-1} - \beta_i) p^* - p^* \right\| \\ &= \left\| \beta_n (p_n - p^*) + \sum_{i=1}^n (\beta_{i-1} - \beta_i) (T_i p_n - p^*) \right\| \\ &\leq \beta_n \|p_n - p^*\| + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|p_n - p^*\| = \|p_n - p^*\|, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|p_{n+1} - p^*\| &= \|\alpha_n f(p_n) + (1 - \alpha_n) q_n - p^*\| \\ &= \|\alpha_n (f(p_n) - p^*) + (1 - \alpha_n) (q_n - p^*)\| \\ &\leq \alpha_n \|f(p_n) - p^*\| + (1 - \alpha_n) \|q_n - p^*\| \\ &\leq \alpha_n \|f(p_n) - f(p^*)\| + \alpha_n \|f(p^*) - p^*\| + (1 - \alpha_n) \|q_n - p^*\| \\ &\leq \alpha_n \|f(p_n) - f(p^*)\| + \alpha_n \|f(p^*) - p^*\| + (1 - \alpha_n) \|p_n - p^*\| \\ &\leq \alpha \alpha_n \|p_n - p^*\| + \alpha_n \|f(p^*) - p^*\| + (1 - \alpha_n) \|p_n - p^*\| \\ &= (1 - \alpha_n(1 - \alpha)) \|p_n - p^*\| + \alpha_n \|f(p^*) - p^*\| \\ &\leq \max \left\{ \|p_n - p^*\|, \frac{\|f(p^*) - p^*\|}{1 - \alpha} \right\}, \end{aligned}$$

and hence by induction we have

$$\|p_n - p^*\| \leq \max \left\{ \|p_0 - p^*\|, \frac{\|f(p^*) - p^*\|}{1 - \alpha} \right\},$$

which implies that $\{p_n\}$ is bounded, so are $\{T_i p_n\}$, $\{q_n\}$ and $\{f(p_n)\}$. $\square$

Lemma 3.4. *The sequence $\{p_n\}$ generated by [Algorithm 3.1](#) converges strongly to $p^* = (z^*, w^*)$ in $\mathcal{F} = \cap_{i=1}^{\infty} F(T_i)$.*

Proof. We divide the proof into the following steps.

Step 1. We prove that $\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0$.

From (3.2), we have

$$\begin{aligned} \|q_n - q_{n-1}\| &= \left\| \beta_n p_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n - \beta_{n-1} p_{n-1} \right. \\ &\quad \left. - \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) T_i p_{n-1} \right\| \\ &\leq \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i p_n - T_i p_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i p_{n-1}\| \\ &\leq \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &\quad + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i p_{n-1}\| \\ &= \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| (\|p_{n-1}\| + \|T_i p_{n-1}\|), \end{aligned} \quad (3.5)$$

hence Eqs. (3.2) and (3.5) yield

$$\begin{aligned} \|p_{n+1} - p_n\| &= \|\alpha_n f(p_n) + (1 - \alpha_n) q_n - \alpha_{n-1} f(p_{n-1}) - (1 - \alpha_{n-1}) q_{n-1}\| \\ &\leq \alpha_n \|f(p_n) - f(p_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(p_{n-1})\| \\ &\quad + (1 - \alpha_n) \|q_n - q_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|q_{n-1}\| \\ &\leq \alpha \alpha_n \|p_n - p_{n-1}\| + (1 - \alpha_n) \|q_n - q_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|f(p_{n-1})\| + \|q_{n-1}\|) \\ &\leq \alpha \alpha_n \|p_n - p_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(p_{n-1})\| + \|q_{n-1}\|) \\ &\quad + (1 - \alpha_n) (\|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| (\|p_{n-1}\| + \|T_i p_{n-1}\|)) \\ &= (1 - \alpha_n (1 - \alpha)) \|p_n - p_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(p_{n-1})\| + \|q_{n-1}\|) \\ &\quad + (1 - \alpha_n) |\beta_n - \beta_{n-1}| (\|p_{n-1}\| + \|T_i p_{n-1}\|). \end{aligned}$$

Thus, from Lemma 2.6 and conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, we obtain that

$$\|p_{n+1} - p_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.6)$$

Step 2. We prove that $\|T_i p_n - p_n\| \rightarrow 0$ as $n \rightarrow \infty$. Take $p^* = P_{\mathcal{F}}(f(p^*))$. Then, we have

$$\begin{aligned} \|p_n - p^*\|^2 &\geq \|T_i p_n - T_i p^*\|^2 \\ &= \|T_i p_n - p_n + p_n - T_i p^*\|^2 \\ &= \|T_i p_n - p_n\|^2 + \|p_n - p^*\|^2 + 2\langle T_i p_n - p_n, p_n - p^* \rangle, \end{aligned} \quad (3.7)$$

which implies that

$$\frac{1}{2} \|T_i p_n - p_n\|^2 \leq \langle p_n - T_i p_n, p_n - p^* \rangle. \quad (3.8)$$

Thus, from (3.2) and (3.8), we obtain

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i p_n - p_n\|^2 &\leq \sum_{i=1}^n (\beta_{i-1} - \beta_i) \langle p_n - T_i p_n, p_n - p^* \rangle \\
 &= \left\langle (1 - \beta_n) p_n - \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n, p_n - p^* \right\rangle \\
 &= \langle (1 - \beta_n) p_n - q_n + \beta_n p_n, p_n - p^* \rangle \\
 &= \langle p_n - q_n, p_n - p^* \rangle \\
 &= \langle p_n - p_{n+1}, p_n - p^* \rangle \\
 &\quad + \langle p_{n+1} - q_n, p_n - p^* \rangle.
 \end{aligned} \tag{3.9}$$

From (3.2) and (3.9), we also have

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i p_n - p_n\|^2 &\leq \langle p_n - p_{n+1}, p_n - p^* \rangle \\
 &\quad + \langle \alpha_n f(p_n) + (1 - \alpha_n) q_n - q_n, p_n - p^* \rangle \\
 &\leq \|p_n - p_{n+1}\| \|p_n - p^*\| \\
 &\quad + \langle \alpha_n f(p_n) - \alpha_n q_n, p_n - p^* \rangle \\
 &\leq \|p_n - p_{n+1}\| \|p_n - p^*\| \\
 &\quad + \alpha_n \|f(p_n) - q_n\| \|p_n - p^*\|.
 \end{aligned} \tag{3.10}$$

Now, from Lemma 3.3, (3.6), (3.10) and the condition of $\{\alpha_n\}$, we obtain that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|T_i p_n - p_n\|^2 = 0. \tag{3.11}$$

Since $\{\beta_n\}$ is strictly decreasing, for every $i \in \mathbb{N}$, equality (3.11) yields

$$\lim_{n \rightarrow \infty} \|T_i p_n - p_n\| = 0, \tag{3.12}$$

and

$$\lim_{n \rightarrow \infty} \|T_n p_n - p_n\| = 0. \tag{3.13}$$

Step 3. We prove that $\limsup_{n \rightarrow \infty} \langle f(p^*) - p^*, p_n - p^* \rangle \leq 0$.

Since $\{p_n\}$ is bounded, there exist $u \in H \times H$ and a subsequence $\{p_{n_k}\}$ of p_n such that

$$\limsup_{n \rightarrow \infty} \langle f(p^*) - p^*, p_n - p^* \rangle = \lim_{k \rightarrow \infty} \langle f(p^*) - p^*, p_{n_k} - p^* \rangle,$$

and $p_{n_k} \rightharpoonup u$. Now, using (3.12) and Lemma 2.8, we obtain that $u \in F(T_i)$ for each $i \geq 1$ and hence $u \in \mathcal{F}$. Since $p^* = P_{\mathcal{F}}(f(p^*))$, by Lemma 2.9, we obtain

$$\begin{aligned}\limsup_{n \rightarrow \infty} \langle f(p^*) - p^*, p_n - p^* \rangle &= \lim_{k \rightarrow \infty} \langle f(p^*) - p^*, p_{n_k} - p^* \rangle \\ &= \langle f(p^*) - p^*, u - p^* \rangle \leq 0.\end{aligned}\quad (3.14)$$

Step 4. Finally, we prove that $p_n \rightarrow p^*$ as $n \rightarrow \infty$. From Lemma 2.7, (3.2) and (3.4), we have

$$\begin{aligned}\|p_{n+1} - p^*\|^2 &= \|\alpha_n f(p_n) + (1 - \alpha_n)q_n - p^*\|^2 \\ &\leq \|\alpha_n(f(p_n) - f(p^*)) + (1 - \alpha_n)(q_n - p^*)\|^2 \\ &\quad + 2\alpha_n \langle f(p^*) - p^*, p_{n+1} - p^* \rangle \\ &\leq (\alpha_n \|p_n - p^*\| + (1 - \alpha_n) \|p_n - p^*\|)^2 \\ &\quad + 2\alpha_n \langle f(p^*) - p^*, p_{n+1} - p^* \rangle \\ &= (1 - (1 - \alpha)\alpha_n)^2 \|p_n - p^*\|^2 + 2\alpha_n \langle f(p^*) - p^*, p_{n+1} - p^* \rangle \\ &\leq (1 - (1 - \alpha)\alpha_n) \|p_n - p^*\|^2 + 2\alpha_n \langle f(p^*) - p^*, p_{n+1} - p^* \rangle \\ &\leq (1 - (1 - \alpha)\alpha_n) \|p_n - p^*\|^2 + 2\alpha_n \langle f(p^*) - p^*, p_n - p^* \rangle \\ &\quad + 2\alpha_n \|f(p^*) - p^*\| \|p_{n+1} - p_n\|.\end{aligned}\quad (3.15)$$

Hence, from (3.14), (3.15) and Lemma 2.6, we get that $p_n \rightarrow p^*$ as $n \rightarrow \infty$. $\square$

Following the method of proof of [5] we obtain the following lemma.

Lemma 3.5. *If in Algorithm 3.1, the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0, \quad (3.16)$$

then, there exists some scalar $\eta > 0$ such that $\varphi_n(p_n) \geq \eta \|\nabla \varphi_n\|^2$ for all $n \geq 0$.

Next, we state and prove our main theorem.

Theorem 3.6. *Let H be a real Hilbert space and let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. If the sequences $\{\lambda_n\}, \{\mu_n\} \subset [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0,$$

then, the sequence $\{p_n\}$ generated by Algorithm 3.1 converges strongly to an element $p^ = (z^*, w^*)$ of $S_e(A, B)$, where $z^* \in \Omega$.*

Proof. By Lemma 3.5, there exists $\eta > 0$ such that

$$\varphi_n(p_n) \geq \eta \|\nabla \varphi_n\|^2. \quad (3.17)$$

This implies that $\varphi_n(p_n)$ is always nonnegative, so from (3.1), we have

$$\|T_n p_n - p_n\| = \left\| - \left(\frac{\varphi_n(z_n, w_n)}{\|\nabla \varphi_n\|^2} \right) \nabla \varphi_n \right\| = \frac{\varphi_n(z_n, w_n)}{\|\nabla \varphi_n\|}, \quad (3.18)$$

for all n for which $\nabla \varphi_n \neq 0$. Thus, dividing both sides of the inequality (3.17) by $\|\nabla \varphi_n\|$, we obtain

$$\|T_n p_n - p_n\| \geq \eta \|\nabla \varphi_n\|, \quad (3.19) \quad \text{Approximation in Hilbert spaces}$$

which is also true for n with $\nabla \varphi_n = 0$. Now, (3.13) and (3.19) imply that $\nabla \varphi_n \rightarrow 0$ as $n \rightarrow \infty$. Now, from the expression for $\nabla \varphi_n = (a_n + b_n, x_n - y_n)$, we have

$$x_n - y_n \rightarrow 0 \text{ and } a_n + b_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.20)$$

Moreover, subtracting y_n from both sides of the first equation and adding $\mu_n b_n - \theta_n y_n$ to both sides of the second equation in (3.3) and rearranging we obtain

$$\begin{cases} (z_n - y_n) + \lambda_n(w_n - b_n) = (x_n - y_n), \\ (\theta_n - 1)(z_n - y_n) + \mu_n(w_n - b_n) = \theta_n(x_n - y_n) - \mu_n(a_n + b_n), \quad n \geq 0. \end{cases}$$

Thus, solving for $(w_n - b_n)$ from the above equations we obtain that

$$(\mu_n + (1 - \theta_n)\lambda_n)(w_n - b_n) = (x_n - y_n) - \mu_n(a_n + b_n). \quad (3.21)$$

Therefore, since $(\mu_n + (1 - \theta_n)\lambda_n) > 0$, from (3.20) and (3.21) we have $w_n - b_n \rightarrow 0$ and hence this with (3.20) yields $z_n - y_n \rightarrow 0$, as $n \rightarrow \infty$. As a consequence, we obtain

$$z_n - x_n \rightarrow 0, \quad z_n - y_n \rightarrow 0, \quad w_n - b_n \rightarrow 0, \quad w_n + a_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.22)$$

In addition, from Lemma 3.4 we know that the sequence $\{p_n\} = \{(z_n, w_n)\}$ converges strongly to a point $p^* = (z^*, w^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$. Now, we show that $p^* \in S_e(A, B)$. But $z_n - x_n \rightarrow 0$ implies that $x_n \rightarrow z^*$. Furthermore, since $w_n - b_n \rightarrow 0$, we have $b_n \rightarrow w^*$. As a consequence, since the $Gph(B)$ is closed and $(x_n, b_n) \in Gph(B)$ for all n , we conclude that $(z^*, w^*) \in Gph(B)$. Similarly, $(z^*, -w^*) \in Gph(A)$, so $(z^*, w^*) \in S_e(A, B)$. Therefore, by Lemma 2.3 we obtain $z^* \in (A + B)^{-1}(0)$. The proof is complete. $\square$

If, in Algorithm 3.1, we replace the contraction mapping f by a constant $p = (z, w) \in H \times H$, then we get the following corollary for the approximation of a zero of the sum of maximally monotone mappings.

Corollary 3.7. *Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$ and $\{\theta_n\} \subset \mathbb{R}$ satisfy the following:*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0.$$

For arbitrary $(z_0, w_0) \in H \times H$ define an iterative algorithm by

$$\begin{cases} q_n = \beta_n p_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) T_i p_n, \\ p_{n+1} = \alpha_n p + (1 - \alpha_n) q_n, \quad \forall n \geq 0, \end{cases} \quad (3.23)$$

where $p = (z, w) \in H \times H$ and $T_i p_n$ is as in (3.1). Then $\{p_n\}$ converges strongly to an element $p^* = (z^*, w^*)$ of $S_e(A, B)$, where $z^* \in \Omega$.

Remark 3.8. We observe that Algorithm 3.1 is equivalent to the following scheme:

$$\begin{cases} p_0 = (z_0, w_0) \in H \times H \text{ chosen arbitrarily,} \\ x_n = (I + \lambda_n B)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n A)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ (z_{n+1}, w_{n+1}) = \alpha_n f(z_n, w_n) + (1 - \alpha_n)(c_n, d_n), \end{cases} \quad (3.24)$$

where

$$c_n = \begin{cases} \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) z_n & \text{if } \delta_i \leq 0, \\ \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)(z_n - \delta_i(a_i + b_i)) & \text{if } \delta_i > 0, \end{cases} \quad (3.25)$$

$$d_n = \begin{cases} \beta_n w_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) w_n & \text{if } \delta_i \leq 0, \\ \beta_n w_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i)(w_n - \delta_i(x_i - y_i)) & \text{if } \delta_i > 0, \end{cases} \quad (3.26)$$

and $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$, $\theta_n \in \mathbb{R}$, with

$$\delta_i = \frac{\langle z_n - x_i, b_i - w_n \rangle + \langle z_n - y_i, a_i + w_n \rangle}{\|a_i + b_i\|^2 + \|x_i - y_i\|^2},$$

for $b_i \in B(x_i)$ and $a_i \in A(y_i)$.

If, in (3.24) we assume $\theta_n = 0$ for all $n \geq 0$ then we get the following corollary for approximating a zero of the sum of maximally monotone mappings.

Corollary 3.9. Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$. Let $f : H \times H \rightarrow H \times H$ be a contraction mapping with constant α . For arbitrary $(z_0, w_0) \in H \times H$ define an iterative algorithm by

$$\begin{cases} x_n = (I + \lambda_n B)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n A)^{-1}(z_n - \mu_n w_n), \\ (z_{n+1}, w_{n+1}) = \alpha_n f(z_n, w_n) + (1 - \alpha_n)(c_n, d_n), \end{cases} \quad (3.27)$$

where $\{c_n\}$ and $\{d_n\}$ are as in (3.25) and (3.26), respectively. Then $\{p_n\}$ converges strongly to an element $p^* = (z^*, w^*)$ of $S_c(A, B)$, where $z^* \in \Omega$.

If, in (3.24) we assume $f = p = (z, w) \in H \times H$ and $\theta_n = 0$ for all $n \geq 0$, then we get the following corollary for approximating a zero of the sum of maximally monotone mappings.

Corollary 3.10. Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$. For arbitrary $(z_0, w_0) \in H \times H$ define an iterative algorithm by

$$\begin{cases} x_n = (I + \lambda_n B)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n A)^{-1}(z_n - \mu_n w_n), \\ (z_{n+1}, w_{n+1}) = \alpha_n (z, w) + (1 - \alpha_n)(c_n, d_n), \end{cases} \quad (3.28)$$

where $\{c_n\}$ and $\{d_n\}$ are as in (3.25) and (3.26), respectively. Then $\{p_n\}$ converges strongly to an element $p^* = (z^*, w^*)$ of $S_e(A, B)$, where $z^* \in \Omega$.

We note that if in Corollary 3.10 we assume that $p = (0, 0)$ we get the following theorem for approximating the minimum-norm point of the extended solution set of the sum of maximally monotone mappings.

Theorem 3.11. *Let H be a real Hilbert space. Let $A, B : H \rightarrow 2^H$ be maximal monotone mappings satisfying $\Omega := \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$. Let $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$. For arbitrary $(z_0, w_0) \in H \times H$ define an iterative algorithm by*

$$\begin{cases} x_n = (I + \lambda_n B)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n A)^{-1}(z_n - \mu_n w_n), \\ (z_{n+1}, w_{n+1}) = (1 - \alpha_n)(c_n, d_n), \end{cases} \quad (3.29)$$

where $\{c_n\}$ and $\{d_n\}$ are as in (3.25) and (3.26), respectively. Then $\{p_n\}$ converges strongly to the minimum-norm point $p^* = (z^*, w^*)$ of $S_e(A, B)$.

Proof. We note that since $p^* = (z^*, w^*) = P_{\mathcal{F}}(0)$, where $p^* \in S_e(A, B) \subset \mathcal{F}$ we obtain that p^* is the minimum-norm point of $S_e(A, B)$. $\square$

4. Application

4.1 Application to convex minimization problem

In this section, we apply Theorem 3.6 to study the convex minimization problem. Let $h : H \rightarrow \mathbb{R}$ be a convex smooth function and $g : H \rightarrow \mathbb{R}$ be a convex, lower semicontinuous functions. We consider the problem of finding $x^* \in H$ such that

$$h(x^*) + g(x^*) = \min_{x \in H} \{h(x) + g(x)\}, \quad (4.1)$$

This problem is equivalent, by Fermats rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \quad (4.2)$$

where ∇h is a gradient of h and ∂g is a subdifferential of g . Note that ∇h and ∂g are maximally monotone mappings (see, for example, [1,15]). Thus, the following result can be obtained from Theorem 3.6.

Theorem 4.1. *Let H be a real Hilbert space. Let $h : H \rightarrow \mathbb{R}$ be a convex smooth function and $g : H \rightarrow \mathbb{R}$ be a convex, lower semicontinuous functions such that $\Omega = \min_{x \in H} \{h(x) + g(x)\} \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be real sequences such that $\lambda_n, \mu_n \in [\underline{\lambda}, \bar{\lambda}]$ for some $\bar{\lambda} > \underline{\lambda} > 0$, and $\{\theta_n\} \subset \mathbb{R}$, satisfy the condition*

$$\inf_{n \geq 0} \left\{ \frac{\mu_n}{\lambda_n} - \left(\frac{\theta_n}{2} \right)^2 \right\} > 0.$$

Let $f : H \times H \rightarrow H \times H$ be a contraction mapping with constant α . For arbitrary $(z_0, w_0) \in H \times H$ define an iterative algorithm by

$$\begin{cases} p_0 = (z_0, w_0) \in H \times H \text{ chosen arbitrarily,} \\ x_n = (I + \lambda_n \partial g)^{-1}(z_n + \lambda_n w_n), \\ y_n = (I + \mu_n \nabla h)^{-1}((1 - \theta_n)z_n + \theta_n x_n - \mu_n w_n), \\ (z_{n+1}, w_{n+1}) = \alpha_n f(z_n, w_n) + (1 - \alpha_n)(c_n, d_n), \end{cases} \quad (4.3)$$

where

$$c_n = \begin{cases} \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) z_n & \text{if } \delta_i \leq 0, \\ \beta_n z_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) (z_n - \delta_i (a_i + b_i)) & \text{if } \delta_i > 0, \end{cases} \quad (4.4)$$

$$d_n = \begin{cases} \beta_n w_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) w_n & \text{if } \delta_i \leq 0, \\ \beta_n w_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) (w_n - \delta_i (x_i - y_i)) & \text{if } \delta_i > 0, \end{cases} \quad (4.5)$$

with

$$\delta_i = \frac{\langle z_n - x_i, b_i - w_n \rangle + \langle z_n - y_i, a_i + w_n \rangle}{\|a_i + b_i\|^2 + \|x_i - y_i\|^2},$$

for $b_i \in \partial g(x_i)$ and $a_i \in \nabla h(y_i)$. Then $\{p_n\}$ converges strongly to an element $p^* = (z^*, w^*)$ of $S_e(\nabla h, \partial g)$, where $z^* \in \Omega$

Proof. Let $A = \nabla h$ and $B = \partial g$ in Theorem 4.1. Thus, Theorem 3.6 provides the conclusion of the theorem. $\square$

5. Numerical example

In this section, we present some numerical experiment result to explain the conclusion of our result. The following numerical example verifies the conclusion of Corollary 3.9.

Example 5.1. Let $H = l_2$, where l_2 is the space of sequences. Let $A, B : l_2 \rightarrow l_2$ be defined by $Ax = 2x + (1, 2, 3, 0, 0, \dots)$ and $Bx = x + (1, 1, 1, 0, 0, \dots)$, where $x = (x_1, x_2, \dots) \in l_2$. We see that A and B are maximally monotone with $R(I + \gamma A) = l_2 = R(I + \gamma B)$ for all $\gamma > 0$. Now by direct calculation we get that

n	z_n	$\ z_{n+1} - z_n\ _{l_2}$
1	(1.0000, 4.0000, 5.0000, 0, 0, ...)	
2	(0.5000, 2.4998, 3.0831, 0, 0, ...)	2.4850
3	(0.1319, 1.3956, 1.6734, 0, 0, ...)	1.8269
4	(-0.0320, 0.9039, 1.0465, 0, 0, ...)	0.8134
5	(-0.1089, 0.6730, 0.7528, 0, 0, ...)	0.3814
10	(-0.3328, 0.0014, -0.0986, 0, 0, ...)	0.1263
100	(-0.6237, -0.8714, -1.1904, 0, 0, ...)	0.0022
200	(-0.6478, -0.9436, -1.2774, 0, 0, ...)	0.0010
300	(-0.6563, -0.9694, -1.3077, 0, 0, ...)	2.8284e-04
400	(-0.6607, -0.9825, -1.3228, 0, 0, ...)	1.4142e-04
500	(-0.6633, -0.9904, -1.3318, 0, 0, ...)	1.0000e-04
	$\vdots$	$\vdots$
	$\downarrow$	$\downarrow$
	$(-\frac{2}{3}, -1, -\frac{4}{3}, 0, 0, \dots)$	0

Table 1.
Convergence of the
sequence z_n .

$$x_n = \frac{z_n + \lambda_n w_n - \lambda_n(1, 1, 1, 0, 0, \dots)}{1 + \lambda_n}, \quad y_n = \frac{z_n - \mu_n w_n - \mu_n(1, 2, 3, 0, 0, \dots)}{1 + 2\mu_n},$$

$$a_n = \frac{2z_n - 2\mu_n w_n + (1, 2, 3, 0, 0, \dots)}{1 + 2\mu_n} \text{ and } b_n = \frac{z_n + \lambda_n w_n + (1, 1, 1, 0, 0, \dots)}{1 + \lambda_n}.$$

Thus, if we assume $\lambda_n = \mu_n = 1$, $\alpha_n = \frac{1}{1000(n+10)}$, $\beta_n = \frac{1}{n}$, for all $n \geq 1$, and $f(x) = \frac{x}{1000}$ with initial point $z_1 = (1, 4, 5, 0, 0, \dots)$ and $w_1 = (0, -1, -2, 0, 0, \dots)$ then the iteration Scheme (3.27) provides the following numerical experiment result using MATLAB (see, Table 1). From this we obtain that the Algorithm z_n converges strongly to $z^* = (-\frac{2}{3}, -1, -\frac{4}{3}, 0, 0, \dots) \in (A+B)^{-1}(0)$.

6. Conclusion

In this paper, we have constructed and studied splitting algorithms which starts by reformulating (1.2) as the problem of locating a point in a certain *extended solution set* $S_e(A, B) \subset H \times H$ which converges strongly to a zero of the sum of two maximally monotone mappings in Hilbert spaces. The assumption that one of the mappings is single-valued, α -inverse strongly monotone or α -strongly monotone is dispensed with. In addition, we applied our main results to study the convex minimization problem. Finally, we provided a numerical example to support our results. Our results extend the results of [5] in the sense that our theorems provide strong convergence in arbitrary Hilbert spaces. In particular, Theorem 3.6 extends Proposition 3 of Eckstein [5] from weak to strong convergence. Moreover, our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

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