

Toeplitz and slant Toeplitz operators on the polydisk

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Abstract

For $n \geq 1$, let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$, and let $\mathbb{T}^n$ be the n -torus. $L^2(\mathbb{T}^n)$ denotes the space of Lebesgue square integrable functions on $\mathbb{T}^n$. In this paper we define slant Toeplitz operators on $L^2(\mathbb{T}^n)$. Besides giving a necessary and sufficient condition for an operator on $L^2(\mathbb{T}^n)$ to be slant Toeplitz, we also establish several properties of slant Toeplitz operators.

Keywords Slant Toeplitz operator, Laurent matrix of level n , Toeplitz matrix of level n , Hyponormal operator

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. For $n \geq 1$, let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$, and let $\mathbb{T}^n$ be the n -torus which is the distinguished boundary of $\mathbb{D}^n$. Also let $d\sigma$ be the normalized Haar measure (or Lebesgue measure) on $\mathbb{T}^n$. In the sequel, z will always denote a vector $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$, and for $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $z^m := z_1^{m_1} \dots z_n^{m_n}$ and $|m| := m_1 + \dots + m_n$. Also for $\lambda \in \mathbb{Z}$ we denote the n -tuple $(\lambda, \dots, \lambda)$ as λ and $z^\lambda := z_1^\lambda \dots z_n^\lambda$. In particular, $z = z_1 \dots z_n$.

$L^2(\mathbb{T}^n)$ is the space of Lebesgue square integrable functions on $\mathbb{T}^n$ with respect to $d\sigma$. Thus,

$L^2(\mathbb{T}^n) = \{f : \mathbb{T}^n \rightarrow \mathbb{C} \mid f(z) = \sum_{m \in \mathbb{Z}^n} f_m z^m, \sum_{m \in \mathbb{Z}^n} |f_m|^2 < \infty\}$ and for $f, g \in L^2(\mathbb{T}^n)$, $\langle f, g \rangle := \int_{\mathbb{T}^n} f(z) \overline{g(z)} d\sigma(z)$. $L^\infty(\mathbb{T}^n)$ is the subspace of $L^2(\mathbb{T}^n)$, consisting of all $\varphi \in L^2(\mathbb{T}^n)$ having the following two properties:

1. $\varphi f \in L^2(\mathbb{T}^n) \forall f \in L^2(\mathbb{T}^n)$.
2. $\exists c > 0$ such that $\|\varphi f\| \leq c \|f\| \forall f \in L^2(\mathbb{T}^n)$.

For $\varphi \in L^\infty(\mathbb{T}^n)$, $\|\varphi\|_\infty$ is defined as $\|\varphi\|_\infty = \inf\{c > 0 \mid \|\varphi f\| \leq c \|f\| \forall f \in L^2(\mathbb{T}^n)\}$.

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The Hardy space $H^2(\mathbb{D}^n)$ consists of all holomorphic functions f on $\mathbb{D}^n$ such that $\sup_{0 < r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 d\sigma(z) < \infty$. For every function $f \in H^2(\mathbb{D}^n)$, the radial limit $\lim_{r \rightarrow 1^-} f(rz)$ exists for almost every $z \in \mathbb{T}^n$ [20]. If we denote this radial limit by $f(z)$, then the Hardy space $H^2(\mathbb{D}^n)$ can be isometrically identified with the closure of the polynomials in $L^2(\mathbb{T}^n)$. Thus we can now define P to be the orthogonal projection of $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$.

M. C. Ho [11] defined slant Toeplitz operators on $L^2(\mathbb{T})$ as those operators whose matrix representation with respect to an orthonormal basis can be obtained by eliminating every other row of a doubly infinite Toeplitz matrix. These types of operators appear frequently in wavelet analysis as their spectral properties have a connection with the smoothness of wavelets. The study of slant Toeplitz operators paved the way to the introduction of more new classes of operators over various function spaces, like k th order slant Toeplitz operators, essentially slant Toeplitz operators, weighted slant Toeplitz operators and so on. For relevant results on slant Toeplitz operators and their generalizations, we refer the reader to [1,3,5,10–13,17,18,23].

In this paper we define slant Toeplitz operators on $L^2(\mathbb{T}^n)$, for $n \geq 1$. Motivated by the matrix definition of slant Toeplitz operators on $L^2(\mathbb{T})$, as given by M. C. Ho [11], we define a slant Toeplitz matrix of level- n , and subsequently show that an operator on $L^2(\mathbb{T}^n)$ is a slant Toeplitz operator if and only if it can be represented as a slant Toeplitz matrix of level- n . For this we first define a Laurent matrix of level- n and establish its relation to a Laurent operator on $L^2(\mathbb{T}^n)$.

2. Laurent matrix of level n

Definition 2.1. For $\varphi \in L^\infty(\mathbb{T}^n)$, the Laurent operator M_φ is defined as the multiplication by φ . That is, $M_\varphi f = \varphi \cdot f \ \forall f \in L^2(\mathbb{T}^n)$.

Remark 2.2 (i). For $\varphi \in L^\infty(\mathbb{T}^n)$, we can easily show that $M_\varphi^* = M_{\bar{\varphi}}$.

(ii) Let i be an integer such that $1 \leq i \leq n$. If $\varphi(z) = z_i$ then $M_\varphi f = z_i f \ \forall f \in L^2(\mathbb{T}^n)$. In this case we denote M_φ as M_{z_i} .

Definition 2.3 (Laurent Matrix of Level n). Consider scalars $\{a_k\}_{k \in \mathbb{Z}^n}$. A matrix of the type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a_{(0,k_2,\dots,k_n)} & a_{(-1,k_2,\dots,k_n)} & a_{(-2,k_2,\dots,k_n)} & \cdots \\ \cdots & a_{(1,k_2,\dots,k_n)} & [a_{(0,k_2,\dots,k_n)}] & a_{(-1,k_2,\dots,k_n)} & \cdots \\ \cdots & a_{(2,k_2,\dots,k_n)} & a_{(1,k_2,\dots,k_n)} & a_{(0,k_2,\dots,k_n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be Laurent matrix of level 1 and is denoted as $L_{k_2,\dots,k_n}^{(1)}$.

A block matrix of the type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & L_{0,k_3,\dots,k_n}^{(1)} & L_{-1,k_3,\dots,k_n}^{(1)} & L_{-2,k_3,\dots,k_n}^{(1)} & \cdots \\ \cdots & L_{1,k_3,\dots,k_n}^{(1)} & [L_{0,k_3,\dots,k_n}^{(1)}] & L_{-1,k_3,\dots,k_n}^{(1)} & \cdots \\ \cdots & L_{2,k_3,\dots,k_n}^{(1)} & L_{1,k_3,\dots,k_n}^{(1)} & L_{0,k_3,\dots,k_n}^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be Laurent matrix of level 2 and is denoted by $L_{k_3,\dots,k_n}^{(2)}$.

A block matrix of the type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & L_{0,k_4,\dots,k_n}^{(2)} & L_{-1,k_4,\dots,k_n}^{(2)} & L_{-2,k_4,\dots,k_n}^{(2)} & \dots \\ \dots & L_{1,k_4,\dots,k_n}^{(2)} & [L_{0,k_4,\dots,k_n}^{(2)}] & L_{-1,k_4,\dots,k_n}^{(2)} & \dots \\ \dots & L_{2,k_4,\dots,k_n}^{(2)} & L_{1,k_4,\dots,k_n}^{(2)} & L_{0,k_4,\dots,k_n}^{(2)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be Laurent matrix of level 3 and is denoted as $L_{k_4,\dots,k_n}^{(3)}$. Continuing, we get a block matrix of type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & L_0^{(n-1)} & L_{-1}^{(n-1)} & L_{-2}^{(n-1)} & \dots \\ \dots & L_1^{(n-1)} & [L_0^{(n-1)}] & L_{-1}^{(n-1)} & \dots \\ \dots & L_2^{(n-1)} & L_1^{(n-1)} & L_0^{(n-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called Laurent matrix of level n denoted as $L^{(n)}$.

For example when $n = 4$ we consider the sequence $\{a_k\}_{k \in \mathbb{Z}^4}$ given by

$$a(k_1, k_2, k_3, k_4) = \begin{cases} 1, & \text{if } k_1 \text{ is even;} \\ 2, & \text{if } k_1 \text{ is odd, } k_2 \text{ is even;} \\ 3, & \text{if } k_1, k_2 \text{ are odd, } k_3 \text{ is even;} \\ 4, & \text{if } k_1, k_2, k_3 \text{ are odd, } k_4 \text{ is even;} \\ 5, & \text{if } k_1, k_2, k_3, k_4 \text{ are odd.} \end{cases}$$

Then

$$L_{k_2, k_3, k_4}^{(1)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & 1 & j & 1 & \dots \\ \dots & j & [1] & j & \dots \\ \dots & 1 & j & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\text{where } j = \begin{cases} 2, & \text{if } k_2 \text{ even;} \\ 3, & \text{if } k_2 \text{ odd, } k_3 \text{ even;} \\ 4, & \text{if } k_2, k_3 \text{ odd, } k_4 \text{ even;} \\ 5, & \text{if } k_2, k_3, k_4 \text{ odd.} \end{cases}$$

If we denote $L_{k_2, k_3, k_4}^{(1)}$ as A, B, C, D for j taking values 2, 3, 4, 5 respectively, then

$$L_{k_3, k_4}^{(2)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & A & B & A & \dots \\ \dots & B & [A] & B & \dots \\ \dots & A & B & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if k_3 is even;

$$L_{k_3, k_4}^{(2)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & A & C & A & \dots \\ \dots & C & [A] & C & \dots \\ \dots & A & C & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if k_3 is odd but k_4 is even; and

$$L_{k_3, k_4}^{(2)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & A & D & A & \dots \\ \dots & D & [A] & D & \dots \\ \dots & A & D & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if k_3 and k_4 are both odd. Let us denote these three matrices by E, F, G respectively. Then,

$$L_{k_4}^{(3)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & E & F & E & \dots \\ \dots & F & [E] & F & \dots \\ \dots & E & F & E & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if k_4 is even; and

$$L_{k_4}^{(3)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & E & G & E & \dots \\ \dots & G & [E] & G & \dots \\ \dots & E & G & E & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if k_4 is odd. If we denote these two matrices by I and J respectively, then

$$L^{(4)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & I & J & I & \dots \\ \dots & J & [I] & J & \dots \\ \dots & I & J & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Notation to be followed:

1. $\mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ denote the set of integers, set of non negative integers, and set of positive integers respectively.
2. For $k \in \mathbb{Z}^n$, let $e_k(z) := z^k$. Then $Q = \{e_k\}_{k \in \mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, and $\{e_k\}_{k \in \mathbb{Z}_+^n}$ is an orthonormal basis for $H^2(\mathbb{D}^n)$, as shown in [20].
3. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we say that k is even if each k_i is even. Otherwise k is said to be odd.

4. $S := \{(t_1, \dots, t_n) \in \mathbb{Z}^n : \text{each } t_i \text{ is either 0 or 1}\}$. If $o(S)$ denotes the order of the set S , then $o(S) = 2^n$.
5. For $j \in [1, n] \cap \mathbb{Z}$, let $\epsilon_j := (x_1, \dots, x_n)$ where $x_i = \delta_{ij}$.
6. For $j \in (1, n] \cap \mathbb{Z}$, and arbitrarily fixed $k_j, k_{j+1}, \dots, k_n \in \mathbb{Z}$, let $Q_{[k_j, \dots, k_n]} = \{e_{(k_1, \dots, k_n)} : k_i \in \mathbb{Z} \text{ for } 1 \leq i < j\}$. Then $Q_{[k_j, \dots, k_n]}$ is an orthonormal basis for $L^2(\mathbb{T}^{j-1})$.
7. For $1 < j \leq n$, and $k_i \in \mathbb{Z}$ with $j \leq i \leq n$, we often use the notation $\mathcal{H}_{j-1, (k_j, \dots, k_n)}$ for the space $L^2(\mathbb{T}^{j-1})$ to convey that the basis being considered is $Q_{[k_j, \dots, k_n]}$.

Theorem 2.4. For $i = 1, 2, \dots, n$, $M_{z_i}^* = M_{\bar{z}_i}$ and $M_{z_i}^* e_k = e_{k-\epsilon_i} \forall k \in \mathbb{Z}^n$.

Proof. Let $f \in L^2(\mathbb{T}^n)$ and $f(z) = \sum_{t \in \mathbb{Z}^n} a_t z^t$. Then for $i = 1, \dots, n$ and $k \in \mathbb{Z}^n$, we have,

$$\langle M_{z_i} f, e_k \rangle = \langle z_i f(z), z^k \rangle = \sum_{t \in \mathbb{Z}^n} a_t \langle z^{t+\epsilon_i}, z^k \rangle = a_{k-\epsilon_i} = \langle f, e_{k-\epsilon_i} \rangle.$$

Thus, $M_{z_i}^* e_k = e_{k-\epsilon_i} = \bar{z}_i e_k$ which implies that $M_{z_i}^* f(z) = \bar{z}_i f(z)$, so that $M_{z_i}^* = M_{\bar{z}_i}$. $\square$

Remark 2.5. (i) $M_{z_i} e_k = e_{k+\epsilon_i}$, for $1 \leq i \leq n$ and $k \in \mathbb{Z}^n$.

(ii) $M_{z_i} M_{z_i}^* = I = M_{z_i}^* M_{z_i} \forall 1 \leq i \leq n$.

Theorem 2.6. A bounded linear operator A on $L^2(\mathbb{T}^n)$ can be represented as a Laurent matrix of level- n if and only if $\langle A e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A e_k, e_t \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$.

Proof. Let us assume first that $\langle A e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A e_k, e_t \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$. Also let $\{a_{\eta, \xi}\}_{\eta, \xi \in \mathbb{Z}^n}$ be scalars such that

$$A e_\xi = \sum_{\eta \in \mathbb{Z}^n} a_{\eta, \xi} e_\eta \quad \forall \xi \in \mathbb{Z}^n \quad (2.1)$$

Step I: Let $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $t = (t_1, \dots, t_n) \in \mathbb{Z}^n$ be arbitrarily chosen. Keeping $(k_2, \dots, k_n)$ and $(t_2, \dots, t_n)$ fixed, if we vary k_1 and t_1 , then $\langle A e_{k+\epsilon_1}, e_{t+\epsilon_1} \rangle = \langle A e_k, e_t \rangle$ implies that $a_{t+\epsilon_1, k+\epsilon_1} = a_{t, k} \forall k_1, t_1 \in \mathbb{Z}$. This means that $A : \mathcal{H}_{1, [k_2, \dots, k_n]} \mapsto \mathcal{H}_{1, [t_2, \dots, t_n]}$ can be represented as a Laurent matrix of level-1, which we denote as $M_{(t_2, \dots, t_n)(k_2, \dots, k_n)}^1$.

Step II: Here we keep $(k_3, \dots, k_n)$ and $(t_3, \dots, t_n)$ fixed and vary k_2 and t_2 . Then $\langle A e_{k+\epsilon_2}, e_{t+\epsilon_2} \rangle = \langle A e_k, e_t \rangle$ implies that $M_{(t_2+1, t_3, \dots, t_n)(k_2+1, k_3, \dots, k_n)}^1 = M_{(t_2, \dots, t_n)(k_2, \dots, k_n)}^1$, and so $A : \mathcal{H}_{2, [k_3, \dots, k_n]} \mapsto \mathcal{H}_{2, [t_3, \dots, t_n]}$ can be represented as a Laurent matrix of level 2, say $M_{(t_3, \dots, t_n)(k_3, \dots, k_n)}^2$.

Step III: Keeping $(k_4, \dots, k_n)$ and $(t_4, \dots, t_n)$ fixed, and using the fact that $\langle A e_{k+\epsilon_3}, e_{t+\epsilon_3} \rangle = \langle A e_k, e_t \rangle$, we get $M_{(t_3+1, t_4, \dots, t_n)(k_3+1, k_4, \dots, k_n)}^2 = M_{(t_3, \dots, t_n)(k_3, \dots, k_n)}^2$ and so $A : \mathcal{H}_{3, [k_4, \dots, k_n]} \mapsto \mathcal{H}_{3, [t_4, \dots, t_n]}$ can be represented as a Laurent matrix of level 3. Continuing in the same way, after n steps we finally arrive at the fact that $A : L^2(\mathbb{T}^n) \mapsto L^2(\mathbb{T}^n)$ can be represented as a Laurent matrix of level- n .

To prove the converse we assume that A can be represented as a Toeplitz matrix of level- n . Hence for $k = (k_1, \dots, k_n)$ and $t = (t_1, \dots, t_n)$ arbitrarily chosen, $A : \mathcal{H}_{j-1, [k_j, \dots, k_n]} \mapsto \mathcal{H}_{j-1, [t_j, \dots, t_n]}$ can be represented as a Toeplitz matrix of level $(j-1)$, denoted as $M_{(t_j, \dots, t_n)(k_j, \dots, k_n)}^{j-1}$, for each $j = 2, \dots, n$. This in turn implies that $\langle A e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A e_k, e_t \rangle$ for $j = 1, \dots, n-1$.

Finally to show that $\langle A e_{k+\epsilon_n}, e_{t+\epsilon_n} \rangle = \langle A e_k, e_t \rangle$ we consider the $(n-1)$ -torus $L^2(\mathbb{T}^{n-1})$. For each $k_n \in \mathbb{Z}$, $L^2(\mathbb{T}^{n-1})$ is an isomorphic copy of $\mathcal{H}_{n-1, (k_n)}$ and hence the n -torus $L^2(\mathbb{T}^n)$ can be

decomposed as $L^2(\mathbb{T}^n) = \bigoplus_{k_n \in \mathbb{Z}} \mathcal{H}_{n-1}, [k_n]$. Therefore, $A : L^2(\mathbb{T}^n) \mapsto L^2(\mathbb{T}^n)$ can be expressed as a matrix where the $(t_n, k_n)^{th}$ entry is M_{t_n, k_n}^{n-1} , and we also have $M_{t_n+1, k_n+1}^{n-1} = M_{t_n, k_n}^{n-1}$. This finally implies that $\langle Ae_{k+\epsilon_n}, e_{t+\epsilon_n} \rangle = \langle Ae_k, e_t \rangle$. $\square$

Theorem 2.7. *Let A be a bounded linear operator on $L^2(\mathbb{T}^n)$. Then the following conditions are equivalent:*

1. A can be represented as a Laurent matrix of level- n
2. A commutes with M_{z_j} for $1 \leq j \leq n$
3. $M_{z^k} A = AM_{z^k} \forall k \in \mathbb{Z}^n$

Proof. (1) $\Rightarrow$ (2)

Since A can be represented as a Laurent matrix of level- n , so by Theorem 2.6 we have, $\langle Ae_{k+\epsilon_j}, e_t \rangle = \langle Ae_k, e_{t-\epsilon_j} \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$. Applying Remark 2.5(i) and Theorem 2.4 to this relation we further get $\langle AM_{z_j} e_k, e_t \rangle = \langle M_{z_j} Ae_k, e_t \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$. Thus, $AM_{z_j} = M_{z_j} A \forall 1 \leq j \leq n$.

(2) $\Rightarrow$ (3)

Suppose $M_{z_i} A = AM_{z_i} \forall 1 \leq i \leq n$. Then for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we have $M_{z^k} A = M_{z_1}^{k_1} \dots M_{z_n}^{k_n} A = AM_{z_1}^{k_1} \dots M_{z_n}^{k_n} = AM_{z^k}$.

(3) $\Rightarrow$ (1)

As $M_{z^k} A = AM_{z^k} \forall k \in \mathbb{Z}^n$, so taking $k = \epsilon_i$ we get $M_{z_i} A = AM_{z_i}$ for each $i = 1, \dots, n$. Now for $k, t \in \mathbb{Z}^n$, we have $\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle AM_{z_j} e_k, M_{z_j} e_t \rangle = \langle M_{z_j} Ae_k, M_{z_j} e_t \rangle$, and using Remark 2.5(ii) we get $\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle Ae_k, e_t \rangle$ for all $k, t \in \mathbb{Z}^n$. Hence applying Theorem 2.6 we conclude that A can be represented as a Laurent matrix of level- n . $\square$

Theorem 2.8. *A bounded linear operator on $L^2(\mathbb{T}^n)$ is a Laurent operator if and only if it can be represented as a Laurent matrix of level n .*

Proof. For $\varphi \in L^\infty(\mathbb{T}^n)$, $M_\varphi M_{z_i} = M_{z_i} M_\varphi \forall 1 \leq i \leq n$, and so by Theorem 2.7, M_φ can be represented as a Laurent matrix of level n .

Conversely, let $\varphi = Ae_0$, so that $\varphi \in L^2(\mathbb{T}^n)$. We first show that φ is bounded.

We have $\|\varphi\| = \|Ae_0\| \leq \|A\| \|e_0\|$. Again, $\|\varphi^2\| = \|Ae_0 \cdot \varphi\| \leq \|A\|^2 \|e_0\|$. Repeating this argument inductively, we get $\|\varphi^m\| \leq \|A\|^m \|e_0\|$ for every positive integer m . This norm inequality can be expressed in the form $\int_{\mathbb{T}^n} \left| \frac{\varphi(z)}{\|A\|} \right|^{2m} d\sigma(z) \leq \|e_0\|^2 = 1$, which implies that $\left| \frac{\varphi(z)}{\|A\|} \right| \leq 1$ almost everywhere. Thus $|\varphi(z)| \leq \|A\|$ almost everywhere which shows that φ is bounded. Hence $\varphi \in L^\infty(\mathbb{T}^n)$ and $\|\varphi\|_\infty \leq \|A\|$.

Next we will show that $A = M_\varphi$. As A can be represented as a Laurent matrix of level n , so by Theorem 2.7, $AM_{z^k} = M_{z^k} A \forall k \in \mathbb{Z}^n$. Therefore,

$$\varphi(z)z^k = z^k \varphi(z) = M_{z^k} \varphi(z) = M_{z^k} Ae_0(z) = AM_{z^k} e_0(z) = Az^k.$$

Thus $Af = \varphi f$ for every polynomial f in $z_1, \dots, z_n$.

Let $f \in L^2(\mathbb{T}^n)$. Then there exist polynomials f_m in $z_1, \dots, z_n$ such that $f_m \rightarrow f$. This implies that $Af_m \rightarrow Af$ and $\varphi f_m \rightarrow \varphi f$, and since $Af_m = \varphi f_m \forall m$, so $Af = \varphi f$. Equivalently, $A = M_\varphi$. $\square$

The following corollary follows immediately from Theorems 2.6-2.8.

Corollary 2.9. Let A be a bounded linear operator on $L^2(\mathbb{T}^n)$. Then the following conditions are equivalent:

1. A is a Laurent operator
2. $\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle Ae_k, e_t \rangle \quad \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$
3. $M_{z_i} A = A M_{z_i} \quad \forall 1 \leq i \leq n$.
4. $M_{z^k} A = A M_{z^k} \quad \forall k \in \mathbb{Z}^n$.

The proof being obvious is omitted.

3. Toeplitz matrix of level n

Definition 3.1. For $\varphi \in L^\infty(\mathbb{T}^n)$, the Toeplitz operator T_φ is defined as the compression of M_φ to $H^2(\mathbb{D}^n)$. That is, $T_\varphi f = P(\varphi \cdot f) \quad \forall f \in H^2(\mathbb{D}^n)$, where P is the orthogonal projection of $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. Thus,

$$(T_\varphi f)(z) = P(\varphi(z) \cdot f(z)) \quad \forall f \in H^2(\mathbb{D}^n), z \in \mathbb{T}^n.$$

Toeplitz operators on $H^2(\mathbb{D}^n)$ have been earlier discussed by several authors. For the convenience of the reader we mention here a few significant references [4,6–8,14,16,19,21]. In the present work we give a matricial definition of Toeplitz operators on $H^2(\mathbb{D}^n)$. It may be mentioned that block Toeplitz operators with matrix symbol $F \in L^\infty(\mathbb{C}^{n \times n})$ was first discussed in [9]. Again in [19] we come across block matrix representation of Toeplitz and Hankel operators on $L^2(\mathbb{T}^n)$. In [15] the authors have used lexicographic ordering on $\mathbb{Z}_+^2$ for matrix representation of Toeplitz operators on $L^2(\mathbb{T}^n)$. These references motivated us to propose our definition of a Toeplitz matrix of level- n . However, our approach is significantly different from what is done in earlier works. We first propose the definition of a Toeplitz matrix of level n and then show that an operator on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if it can be represented as a Toeplitz matrix of level n .

Definition 3.2 (Toeplitz Matrix of Level n). Consider a sequence of scalars $\{a_k\}_{k \in \mathbb{Z}^n}$. A matrix of the type

$$\begin{pmatrix} a_{(0,k_2,\dots,k_n)} & a_{(-1,k_2,\dots,k_n)} & a_{(-2,k_2,\dots,k_n)} & \dots \\ a_{(1,k_2,\dots,k_n)} & a_{(0,k_2,\dots,k_n)} & a_{(-1,k_2,\dots,k_n)} & \dots \\ a_{(2,k_2,\dots,k_n)} & a_{(1,k_2,\dots,k_n)} & a_{(0,k_2,\dots,k_n)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be a Toeplitz matrix of level 1, denoted as $T_{k_2,\dots,k_n}^{(1)}$.

A block matrix of the type

$$\begin{pmatrix} T_{0,k_3,\dots,k_n}^{(1)} & T_{-1,k_3,\dots,k_n}^{(1)} & T_{-2,k_3,\dots,k_n}^{(1)} & \dots \\ T_{1,k_3,\dots,k_n}^{(1)} & T_{0,k_3,\dots,k_n}^{(1)} & T_{-1,k_3,\dots,k_n}^{(1)} & \dots \\ T_{2,k_3,\dots,k_n}^{(1)} & T_{1,k_3,\dots,k_n}^{(1)} & T_{0,k_3,\dots,k_n}^{(1)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be Toeplitz matrix of level 2, denoted as $T_{k_3,\dots,k_n}^{(2)}$.

Continuing, we get a block matrix of type

$$\begin{pmatrix} T_0^{(n-1)} & T_{-1}^{(n-1)} & T_{-2}^{(n-1)} & \dots \\ T_1^{(n-1)} & T_0^{(n-1)} & T_{-1}^{(n-1)} & \dots \\ T_2^{(n-1)} & T_1^{(n-1)} & T_0^{(n-1)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called Toeplitz matrix of level n denoted as $T^{(n)}$.

Theorem 3.3. A bounded linear operator A on $H^2(\mathbb{D}^n)$ can be represented as a Toeplitz matrix of level n if and only if $\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle Ae_k, e_t \rangle \forall k, t \in \mathbb{Z}_+^n$ and $1 \leq j \leq n$.

Proof is similar to that of Theorem 2.6 and is therefore omitted.

Theorem 3.4. A necessary and sufficient condition that an operator on $H^2(\mathbb{D}^n)$ be a Toeplitz operator is that it can be represented as a Toeplitz matrix of level n .

Proof. Let $A = T_\varphi = PM_\varphi$ for $\varphi \in L^\infty(\mathbb{T}^n)$. Then for $k, t \in \mathbb{Z}_+^n$ and $1 \leq j \leq n$, we have

$$\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle PM_\varphi e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle M_\varphi e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle \quad (3.1)$$

As M_φ is a Laurent operator, so Corollary 2.9 implies that

$$\langle M_\varphi e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle M_\varphi e_k, e_t \rangle \quad (3.2)$$

Combining Eqs. (3.1) and (3.2) we get

$$\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle M_\varphi e_k, e_t \rangle = \langle PM_\varphi e_k, e_t \rangle = \langle Ae_k, e_t \rangle.$$

Thus by Theorem 3.3, A can be represented as a Toeplitz matrix of level n .

Conversely, suppose A is a bounded linear operator on $H^2(\mathbb{D}^n)$ which can be represented as a Toeplitz matrix of level n . Then by Theorem 3.3, we have

$$\langle Ae_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle Ae_k, e_t \rangle \forall k, t \in \mathbb{Z}_+^n, 1 \leq j \leq n \quad (3.3)$$

For $m \in \mathbb{Z}_+$, let $A_m : L^2(\mathbb{T}^n) \mapsto L^2(\mathbb{T}^n)$ be defined as $A_m = M_{z^m}^* A M_{z^m}$ where $z^m = z_1^m \dots z_n^m$. Then for $k = (k_1, \dots, k_n)$ and $t = (t_1, \dots, t_n)$ in $\mathbb{Z}_+^n$, and $m \in \mathbb{Z}_+$, repeated application of Eq. (3.3) gives,

$$\langle A_m e_k, e_t \rangle = \langle A e_{(k_1+m, \dots, k_n+m)}, e_{(t_1+m, \dots, t_n+m)} \rangle = \langle Ae_k, e_t \rangle \quad (3.4)$$

For $k, t \in \mathbb{Z}_+^n$, we choose $N \in \mathbb{Z}_+$, sufficiently large so that $k_i + N, t_i + N \in \mathbb{Z}_+ \forall 1 \leq i \leq n$, and let $\tilde{k} = (k_1 + N, \dots, k_n + N)$, $\tilde{t} = (t_1 + N, \dots, t_n + N)$. Then by Eq. (3.4), we have $\forall m \in \mathbb{Z}_+$,

$$\langle A_m e_{\tilde{k}}, e_{\tilde{t}} \rangle = \langle Ae_{\tilde{k}}, e_{\tilde{t}} \rangle \quad (3.5)$$

Also $e_{\tilde{t}} = M_{z^N} e_t$, $e_{\tilde{k}} = M_{z^N} e_k$ and $P e_{\tilde{k}} = e_{\tilde{k}}$. Using this in Eq. (3.5) we get

$$\langle M_{z^N}^* A_m P M_{z^N} e_k, e_t \rangle = \langle M_{z^N}^* A P M_{z^N} e_k, e_t \rangle \Rightarrow \langle A_{m+N} e_k, e_t \rangle = \langle A_N e_k, e_t \rangle$$

Thus $\forall k, t \in \mathbb{Z}_+^n, \exists N \in \mathbb{Z}_+$ such that

$$\langle A_{m+N} e_k, e_t \rangle = \langle A_N e_k, e_t \rangle \forall m \in \mathbb{Z}_+ \quad (3.6)$$

Therefore, if f and g are finite linear combinations of e_k for $k \in \mathbb{Z}^n$, then the sequence $\{\langle A_n f, g \rangle\}_{n \in \mathbb{Z}_+}$ is convergent. Also as $\|A_n\| \leq \|A\| \forall n \in \mathbb{Z}_+$, so the sequence $\{A_n\}_{n \in \mathbb{Z}_+}$ of operators on $L^2(\mathbb{T}^n)$ is weakly convergent to a bounded operator A_∞ on $L^2(\mathbb{T}^n)$.

Claim: A_∞ is a Laurent operator.

As in Eq. (3.6), for $k, t \in \mathbb{Z}^n$ and $N \in \mathbb{Z}_+$, we have

$$\langle A_{m+N} e_{k+\epsilon_i}, e_{t+\epsilon_i} \rangle = \langle A_N e_{k+\epsilon_i}, e_{t+\epsilon_i} \rangle \forall 1 \leq i \leq n \quad (3.7)$$

Taking limit as $m \rightarrow \infty$, in Eqs. (3.6) and (3.7) we get

$$\langle A_\infty e_k, e_t \rangle = \langle A_N e_k, e_t \rangle = \langle M_{z^N}^* A P M_{z^N} e_k, e_t \rangle = \langle A e_{\tilde{k}}, e_{\tilde{t}} \rangle, \text{ and}$$

$$\langle A_\infty e_{k+\epsilon_i}, e_{t+\epsilon_i} \rangle = \langle A_N e_{k+\epsilon_i}, e_{t+\epsilon_i} \rangle = \langle A e_{\tilde{k}+\epsilon_i}, e_{\tilde{t}+\epsilon_i} \rangle = \langle A e_{\tilde{k}}, e_{\tilde{t}} \rangle.$$

Thus $\langle A_\infty e_k, e_t \rangle = \langle A_\infty e_{k+\epsilon_i}, e_{t+\epsilon_i} \rangle \forall 1 \leq i \leq n, k, t \in \mathbb{Z}^n$, and hence Corollary 2.9 implies that A_∞ is a Laurent operator, and the claim is established.

From Eq. (3.4) we have $\langle A_m e_k, e_t \rangle = \langle A e_k, e_t \rangle \forall k, t \in \mathbb{Z}_+^n$, and $m \in \mathbb{Z}_+$. Thus for $f, g \in H^2(\mathbb{D}^n)$, we have $\langle A_m f, g \rangle = \langle A f, g \rangle$.

Therefore $\langle P A_\infty f, g \rangle = \langle A_\infty f, g \rangle = \lim_{m \rightarrow \infty} \langle A_m f, g \rangle = \langle A f, g \rangle \forall f, g \in H^2(\mathbb{D}^n)$. This implies that $P A_\infty|_{H^2(\mathbb{D}^n)} = A$, where A_∞ is a Laurent operator. Thus, A is a Toeplitz operator. $\square$

Definition 3.5. For $1 \leq j \leq n$, we define $U_{z_j} : H^2(\mathbb{D}^n) \mapsto H^2(\mathbb{D}^n)$ as $U_{z_j} f(z) = z_j f(z) \forall f \in H^2(\mathbb{D}^n), z \in \mathbb{T}^n$.

Remark 3.6. $U_{z_j} e_k = e_{k+\epsilon_j} \forall k \in \mathbb{Z}_+^n, 1 \leq j \leq n$.

Theorem 3.7. For $1 \leq j \leq n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we have

$$U_{z_j}^* e_k = \begin{cases} e_{k-\epsilon_j} & \text{if } k_j \neq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Proof. Let $f \in H^2(\mathbb{D}^n)$ where $f(z) = \sum_{m \in \mathbb{Z}_+^n} f_m z^m$.

Then for $1 \leq j \leq n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} \langle U_{z_j} f, e_k \rangle &= \left\langle \sum_{m \in \mathbb{Z}_+^n} f_m z^{m+\epsilon_j}, z^k \right\rangle = \sum_{m \in \mathbb{Z}_+^n} f_m \langle z^{m+\epsilon_j}, z^k \rangle \\ &= \begin{cases} f_{k-\epsilon_j} & \text{if } k_j \neq 0 \\ 0 & \text{if } k_j = 0 \end{cases} = \begin{cases} \langle f, e_{k-\epsilon_j} \rangle & \text{if } k_j \neq 0 \\ 0 & \text{if } k_j = 0 \end{cases} \end{aligned}$$

$$\text{Thus, } U_{z_j}^* e_k = \begin{cases} e_{k-\epsilon_j} & \text{if } k_j \neq 0 \\ 0 & \text{Otherwise} \end{cases} \quad \square$$

Remark 3.8. For $1 \leq j \leq n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we have

$$U_{z_j}^* U_{z_j} e_k = e_k \text{ and } U_{z_j} U_{z_j}^* e_k = \begin{cases} e_k & \text{if } k_j \neq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Theorem 3.9. Let A be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then A is a Toeplitz operator if and only if $U_{z_j}^* A U_{z_j} = A \quad \forall 1 \leq j \leq n$.

Proof. If A is a Toeplitz operator on $H^2(\mathbb{D}^n)$, then, by Theorem 3.4, A can be represented as a Toeplitz matrix of level- n . This implies, by Theorem 3.3, that $\langle A e_{k+\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A e_k, e_t \rangle \quad \forall k, t \in \mathbb{Z}_+^n, 1 \leq j \leq n$. Applying Remark 3.6 to this relation we further deduce that $\langle U_{z_j}^* A U_{z_j} e_k, e_t \rangle = \langle A e_k, e_t \rangle \quad \forall k, t \in \mathbb{Z}_+^n, 1 \leq j \leq n$, which in turn implies that $U_{z_j}^* A U_{z_j} = A \quad \forall 1 \leq j \leq n$.

Similarly, assuming $U_{z_j}^* A U_{z_j} = A \quad \forall 1 \leq j \leq n$ and applying Theorems 3.3, 3.4 and Remark 3.6 it can be shown that A is a Toeplitz operator on $H^2(\mathbb{D}^n)$. $\square$

4. Slant Toeplitz operator on $L^2(\mathbb{T}^n)$

Let us consider a sequence of scalars $\{a_k\}_{k \in \mathbb{Z}^n}$. A matrix of the type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a_{(-1, k_2, \dots, k_n)} & a_{(-2, k_2, \dots, k_n)} & a_{(-3, k_2, \dots, k_n)} & \cdots \\ \cdots & a_{(1, k_2, \dots, k_n)} & [a_{(0, k_2, \dots, k_n)}] & a_{(-1, k_2, \dots, k_n)} & \cdots \\ \cdots & a_{(3, k_2, \dots, k_n)} & a_{(2, k_2, \dots, k_n)} & a_{(1, k_2, \dots, k_n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is said to be slant Toeplitz matrix of level-1, denoted as $A_{k_2, \dots, k_n}^{(1)}$. Observe that it is the matrix obtained by eliminating all odd rows of the Laurent matrix of level-1 namely $L_{k_2, \dots, k_n}^{(1)}$.

Similarly if we eliminate all odd rows of the block Laurent matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & A_{0, k_3, \dots, k_n}^{(1)} & A_{-1, k_3, \dots, k_n}^{(1)} & A_{-2, k_3, \dots, k_n}^{(1)} & \cdots \\ \cdots & A_{1, k_3, \dots, k_n}^{(1)} & [A_{0, k_3, \dots, k_n}^{(1)}] & A_{-1, k_3, \dots, k_n}^{(1)} & \cdots \\ \cdots & A_{2, k_3, \dots, k_n}^{(1)} & A_{1, k_3, \dots, k_n}^{(1)} & A_{0, k_3, \dots, k_n}^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we get the slant Toeplitz matrix of level-2, denoted by $A_{k_3, \dots, k_n}^{(2)}$ and given by

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & A_{-1, k_3, \dots, k_n}^{(1)} & A_{-2, k_3, \dots, k_n}^{(1)} & A_{-3, k_3, \dots, k_n}^{(1)} & \cdots \\ \cdots & A_{1, k_3, \dots, k_n}^{(1)} & [A_{0, k_3, \dots, k_n}^{(1)}] & A_{-1, k_3, \dots, k_n}^{(1)} & \cdots \\ \cdots & A_{3, k_3, \dots, k_n}^{(1)} & A_{2, k_3, \dots, k_n}^{(1)} & A_{1, k_3, \dots, k_n}^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Continuing in this way, we eliminate the odd rows of the block Laurent matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & A_0^{(n-1)} & A_{-1}^{(n-1)} & A_{-2}^{(n-1)} & \dots \\ \dots & A_1^{(n-1)} & [A_0^{(n-1)}] & A_{-1}^{(n-1)} & \dots \\ \dots & A_2^{(n-1)} & A_1^{(n-1)} & A_0^{(n-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

to get the slant Toeplitz matrix of level- n , which is

$$A^{(n)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & A_{-1}^{(n-1)} & A_{-2}^{(n-1)} & A_{-3}^{(n-1)} & \dots \\ \dots & A_1^{(n-1)} & [A_0^{(n-1)}] & A_{-1}^{(n-1)} & \dots \\ \dots & A_3^{(n-1)} & A_2^{(n-1)} & A_1^{(n-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 4.1. *If T is a bounded linear operator on $L^2(\mathbb{T}^n)$, then T can be represented as a slant Toeplitz matrix of level- n if and only if $\langle T e_{k+2\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle T e_k, e_t \rangle \forall k, t \in \mathbb{Z}^n$, and $1 \leq j \leq n$.*

The proof follows as in [Theorem 2.6](#), and is therefore omitted.

Theorem 4.2. *A bounded linear operator T on $L^2(\mathbb{T}^n)$ can be represented as a slant Toeplitz matrix of level- n if and only if $M_{z_j} T = T M_{z_j}^2 \forall 1 \leq j \leq n$.*

Proof. Let j be an integer such that $1 \leq j \leq n$. Then,

$$M_{z_j} T = T M_{z_j}^2 \forall 1 \leq j \leq n$$

if and only if $\langle M_{z_j} T e_k, e_t \rangle = \langle T M_{z_j}^2 e_k, e_t \rangle, \forall k, t \in \mathbb{Z}^n$ and $\forall 1 \leq j \leq n$

if and only if $\langle T e_k, e_{t-\epsilon_j} \rangle = \langle T e_{k+2\epsilon_j}, e_t \rangle, \forall k, t \in \mathbb{Z}^n$ and $\forall 1 \leq j \leq n$

if and only if $\langle T e_k, e_t \rangle = \langle T e_{k+2\epsilon_j}, e_{t+\epsilon_j} \rangle, \forall k, t \in \mathbb{Z}^n$ and $\forall 1 \leq j \leq n$

The result now follows from [Theorem 4.1](#). $\square$

Definition 4.3. $W : L^2(\mathbb{T}^n) \mapsto L^2(\mathbb{T}^n)$ is defined to be the linear operator such that for

$$\text{each } k \in \mathbb{Z}^n, W e_k = \begin{cases} e_{\frac{k}{2}}, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

So for $f(z) = \sum_{j \in \mathbb{Z}^n} a_j z^j \in L^2(\mathbb{T}^n)$, we have $W f(z) = \sum_{j \in \mathbb{Z}^n} a_{2j} z^j$.

Remark 4.4.

(i) For $k, t \in \mathbb{Z}^n$

$$\langle W e_k, e_t \rangle = \begin{cases} 1 & \text{if } 2t = k, \\ 0 & \text{Otherwise.} \end{cases}$$

and so $\|W\| = 1$

- (ii) $W^* e_t = e_{2t} \forall t \in \mathbb{Z}^n$, because $\langle W e_k, e_t \rangle = \langle e_k, e_{2t} \rangle \forall k, t \in \mathbb{Z}^n$. So for $f(z) = \sum_{p \in \mathbb{Z}^n} a_p z^p$ $\in L^2(\mathbb{T}^n)$, we have $W^* f(z) = \sum_{p \in \mathbb{Z}^n} a_p z^{2p} = f(z^2)$.

Definition 4.5. For $\varphi \in L^\infty(\mathbb{T}^n)$, we define the slant Toeplitz operator A_φ as $W M_\varphi$. Hence, if φ is the constant function 1 then $W = A_1$.

Theorem 4.6. $\|A_\varphi\| \leq \|\varphi\|_\infty$.

Proof. $\|A_\varphi\| = \|W M_\varphi\| \leq \|W\| \|M_\varphi\| = \|M_\varphi\| = \|\varphi\|_\infty$. $\square$

Theorem 4.7. If A_φ is a slant Toeplitz operator on $L^2(\mathbb{T}^n)$, then

$$\langle A_\varphi e_{k+2\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A_\varphi e_k, e_t \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n.$$

Proof. Since $A_\varphi = W M_\varphi$, the result follows immediately from [Corollary 2.9](#) and [Remark 4.4\(ii\)](#). $\square$

Theorem 4.8. A bounded linear operator A on $L^2(\mathbb{T}^n)$ is a slant Toeplitz operator if and only if A can be represented as a slant Toeplitz matrix of level- n .

If A is a slant Toeplitz operator then by [Theorems 4.1](#) and [4.7](#) it follows immediately that A can be represented as a slant Toeplitz matrix of level- n . However, for the converse implication we defer the proof to [Section 6](#) as we first have to establish a few more results needed thereof.

5. Properties of W and W^*

Definition 5.1. Let P_e be the projection of $L^2(\mathbb{T}^n)$ onto the closed span of $\{e_{2k}\}_{k \in \mathbb{Z}^n}$ in $L^2(\mathbb{T}^n)$. Thus for $j \in \mathbb{Z}^n$ we have $P_e e_j = \begin{cases} e_j, & \text{if } j \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$

Or equivalently we have $P_e(\sum_{k \in \mathbb{Z}^n} a_k z^k) = \sum_{k \in \mathbb{Z}^n} a_{2k} z^{2k}$.

We make the following observations:

- (i) $W W^* = I$, and so W is a co-isometry on $L^2(\mathbb{T}^n)$.
- (ii) $W^* W = P_e$, and so W is an isometry on $P_e(L^2(\mathbb{T}^n))$.

In the following few results we refer to the subset S of $\mathbb{Z}^n$, the definition for which was already given in [Section 2](#).

Lemma 5.2. Let $S = \{(t_1, \dots, t_n) \in \mathbb{Z}^n \mid \text{each } t_i \text{ is either 0 or 1}\}$. Then for $k \in \mathbb{Z}^n$, k odd, there exist unique $\tilde{k} \in \mathbb{Z}^n$ and $0 \neq t \in S$ such that $k = 2\tilde{k} + t$.

Proof. Let $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. For each $1 \leq i \leq n$ there exists $\tilde{k}_i \in \mathbb{Z}$ such that

$$k_i = \begin{cases} 2\tilde{k}_i & \text{if } k_i \text{ is even,} \\ 2\tilde{k}_i + 1 & \text{if } k_i \text{ is odd.} \end{cases}$$

Let $\tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_n)$ and $t = (t_1, \dots, t_n)$ where

$$t_i = \begin{cases} 0 & \text{if } k_i \text{ is even,} \\ 1 & \text{if } k_i \text{ is odd.} \end{cases}$$

Then $k = 2\tilde{k} + t$ where $\tilde{k} \in \mathbb{Z}^n$, $t \in S$. Also k is odd implies $t_i = 1$ for at least one i , so that t is non-zero.

Uniqueness Suppose $\exists k' = (k'_1, \dots, k'_n) \in \mathbb{Z}^n$ and $t' = (t'_1, \dots, t'_n) \in S$ such that $k = 2k' + t'$.

Let if possible $t' \neq t$. Without loss of generality we suppose $t'_1 \neq t_1$, so that either $t'_1 = 1$ and $t_1 = 0$, or $t'_1 = 0$ and $t_1 = 1$.

Now $t'_1 = 1$ implies $k_1 = 2k'_1 + t'_1 = 2k'_1 + 1$, which is odd,

and $t_1 = 0$ implies $k_1 = 2k_1 + t_1 = 2k_1$, which is even. Thus we get a contradiction.

Similarly, $t'_1 = 0$ and $t_1 = 1$ give us a contradiction.

Thus $t' = t$ implies $2k' = 2\bar{k}$ or $k' = \bar{k}$. $\square$

Remark 5.3. Every non zero entry in S is odd. Also $\epsilon_j \in S$ for $j = 1, 2, \dots, n$.

Lemma 5.4. For $t, k \in S$ with $t \neq 0, k \neq 0$, we have $t + k$ is even if and only if $t = k$.

Proof. Suppose $t = k$. Then $t + k = 2t$, which is even.

Conversely, suppose $t \neq k$. Then without loss of generality we assume that $t_1 \neq k_1$, where $t = (t_1, \dots, t_n), k = (k_1, \dots, k_n)$. Thus, either $t_1 = 0$ and $k_1 = 1$; or $t_1 = 1$ and $k_1 = 0$. In any case,

$t + k = (t_1 + k_1, \dots, t_n + k_n)$, where $t_1 + k_1 = 1$, and so $t + k$ is odd.

Thus $t \neq k$ implies that $t + k$ odd, or equivalently, $t + k$ even implies that $t = k$. $\square$

Theorem 5.5. $WM_{z^{2k}}W^* = M_{z^k}$ if $k \in \mathbb{Z}^n$, and $WM_{z^t}W^* = 0$ if $t \in \mathbb{Z}^n$ is odd.

Proof. For $k, p \in \mathbb{Z}^n$, $WM_{z^{2k}}W^*e_p = WM_{z^{2k}}e_{2p} = We_{2(p+k)} = e_{(p+k)} = M_{z^k}e_p$. Now if $t \in \mathbb{Z}^n$ is odd, then by Lemma 5.2, $2p + t$ is odd, and so we have $WM_{z^t}W^*e_p = WM_{z^t}e_{2p} = We_{(2p+t)} = 0$. $\square$

Corollary 5.6. $WM_{z_j}W^* = 0 \forall 1 \leq j \leq n$.

Proof. This follows from Theorem 5.5. Taking $t = \epsilon_j$ we get $z^t = z_j$ where t is odd. Thus $WM_{z_j}W^* = 0$. $\square$

Corollary 5.7. $WM_zW^* = 0$

Proof. Let $t = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n$. Then t is odd and $z^t = z_1 \dots z_n = z$. So $WM_zW^* = 0$. $\square$

Theorem 5.8. Let $f, g \in L^2(\mathbb{T}^n)$ such that $fg \in L^2(\mathbb{T}^n)$. Then

$$(i) W^*(fg) = (W^*f)(W^*g),$$

$$(ii) W((W^*f)(W^*g)) = fg.$$

Proof. (i) $W^*(e_k e_t) = W^*e_{(k+t)} = e_{2(k+t)} = e_{2k}e_{2t} = (W^*e_k)(W^*e_t)$ for $k, t \in \mathbb{Z}^n$. Thus for $f, g \in L^2(\mathbb{T}^n)$, $W^*(fg) = (W^*f)(W^*g)$, provided $fg \in L^2(\mathbb{T}^n)$.

(ii) As $WW^* = I$ so the result follows immediately from (i). $\square$

Theorem 5.9. For $\varphi \in L^\infty(\mathbb{T}^n)$, $WA_\varphi^* = M_\psi$ where $\psi = W\bar{\varphi}$.

Proof. Let $\varphi(z) = \sum_{t \in \mathbb{Z}^n} a_t z^t$, so that $\bar{\varphi}(z) = \sum_{t \in \mathbb{Z}^n} \bar{a}_t z^{-t}$. Then $\psi(z) = W\bar{\varphi}(z) = \sum_{t \in \mathbb{Z}^n} \bar{a}_t Wz^{-t} = \sum_{t \in \mathbb{Z}^n} \bar{a}_{2t} z^{-t}$ and $M_\psi = \sum_{t \in \mathbb{Z}^n} \bar{a}_{2t} M_{z^{-t}}$.

Again, $A_\varphi = WM_\varphi$ implies $A_\varphi^* = M_\varphi^* W^* = M_{\bar{\varphi}} W^* = \sum_{t \in \mathbb{Z}^n} \bar{a}_t M_{z^{-t}} W^*$.

Hence by Theorem 5.5,

$$WA_\varphi^* = \sum_{t \in \mathbb{Z}^n} \bar{a}_t WM_{z^{-t}} W^* = \sum_{t \in \mathbb{Z}^n} \bar{a}_{2t} M_{z^{-t}} = M_\psi. \quad \square$$

Corollary 5.10. If $\varphi \in L^\infty(\mathbb{T}^n)$, then $W\varphi \in L^\infty(\mathbb{T}^n)$.

Proof. Let $\varphi \in L^\infty(\mathbb{T}^n)$. Then by Theorem 5.9, $M_{W\varphi} = WA_\varphi^*$, which is bounded on $L^2(\mathbb{T}^n)$. Therefore $W\varphi \in L^\infty(\mathbb{T}^n)$. $\square$

Lemma 5.11. Let $f \in L^2(\mathbb{T}^n)$ and $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. If for $t \in S$, we define $f_t(z) = \sum_{k \in \mathbb{Z}^n} a_{2k+t} z^k$, then $f(z) = \sum_{t \in S} z^t f_t(z^2)$. Also $f_0(z^2) = P_e f(z)$ and $f_0(z) = W f(z)$.

Proof. Let $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. If $g(z) := \sum_{k \in \mathbb{Z}^n, k \text{ even}} a_k z^k$ and $h(z) := \sum_{k \in \mathbb{Z}^n, k \text{ odd}} a_k z^k$, then $f(z) = g(z) + h(z)$. For $t \in S$ define $f_t(z) = \sum_{k \in \mathbb{Z}^n} a_{2k+t} z^k$. As $f \in L^2(\mathbb{T}^n)$, so $f_t \in L^2(\mathbb{T}^n) \forall t \in S$. We have

$$g(z) = \sum_{k \in \mathbb{Z}^n, k \text{ even}} a_k z^k = \sum_{k \in \mathbb{Z}^n} a_{2k} z^{2k} = f_0(z^2).$$

Again, by Lemma 5.2,

$$h(z) = \sum_{0 \neq t \in S} \sum_{k \in \mathbb{Z}^n} a_{2k+t} z^{2k+t} = \sum_{0 \neq t \in S} z^t \left(\sum_{k \in \mathbb{Z}^n} a_{2k+t} z^{2k} \right) = \sum_{0 \neq t \in S} z^t f_t(z^2). \text{ Thus we have,}$$

$$f(z) = f_0(z^2) + \sum_{0 \neq t \in S} z^t f_t(z^2) = \sum_{t \in S} z^t f_t(z^2).$$

Moreover, by Definition 4.3, we have $W g(z) = \sum_{k \in \mathbb{Z}^n} a_{2k} z^k = f_0(z)$, and $W h(z) = 0$, so that $W f(z) = f_0(z)$.

Again, by Definition 5.1, we have $P_e g(z) = g(z)$ and $P_e h(z) = 0$ so that $P_e f(z) = P_e g(z) = g(z) = f_0(z^2)$. $\square$

Theorem 5.12. Let $f, g \in L^2(\mathbb{T}^n)$ such that one of f and g is in $L^\infty(\mathbb{T}^n)$. Then $W(fg) = (Wf)(Wg) + \sum_{0 \neq t \in S} z^t (W \overline{z}^t f)(W \overline{z}^t g)$.

Proof. By Lemma 5.11, we have, $f(z) = f_0(z^2) + \sum_{0 \neq t \in S} z^t f_t(z^2)$ and $g(z) = g_0(z^2) + \sum_{0 \neq t \in S} z^t g_t(z^2)$, where $f_0(z^2) = P_e f(z)$, and $g_0(z^2) = P_e g(z)$. Hence,

$$\begin{aligned} fg &= f_0(z^2)g_0(z^2) + f_0(z^2) \left(\sum_{0 \neq t \in S} z^t g_t(z^2) \right) + g_0(z^2) \left(\sum_{0 \neq t \in S} z^t f_t(z^2) \right) \\ &\quad + \left(\sum_{0 \neq t \in S} z^t f_t(z^2) \right) \left(\sum_{0 \neq t \in S} z^t g_t(z^2) \right) \end{aligned} \quad (5.1)$$

Applying Remark 4.4(ii), Theorem 5.8(ii) and Lemma 5.11 in that order, we get,

$$W(f_0(z^2)g_0(z^2)) = W(W^* f_0(z) \cdot W^* g_0(z)) = f_0(z)g_0(z) = (Wf(z))(Wg(z)) \quad (5.2)$$

As $f_0(z^2)(\sum_{0 \neq t \in S} z^t g_t(z^2))$ and $g_0(z^2)(\sum_{0 \neq t \in S} z^t f_t(z^2))$ are expressions that involve only odd powers of z , so by Definition 4.3,

$$W \left[f_0(z^2) \sum_{t \in S, t \neq 0} z^t g_t(z^2) \right] = 0 \text{ and } W \left[g_0(z^2) \sum_{t \in S, t \neq 0} z^t f_t(z^2) \right] = 0. \quad (5.3)$$

Again, $(\sum_{0 \neq t \in S} z^t g_t(z^2))(\sum_{0 \neq t \in S} z^t f_t(z^2)) = \sum_{0 \neq t \in S} \sum_{0 \neq t \in S} z^{t+k} g_t(z^2) f_k(z^2)$, where $t+k$ is even iff $t=k$, by Lemma 5.4. Therefore by Definition 4.3,

$$\begin{aligned} W \left(\sum_{0 \neq t \in S} z^t g_t(z^2) \right) \left(\sum_{0 \neq t \in S} z^t f_t(z^2) \right) &= \sum_{0 \neq t \in S} W(z^{2t}) W(g_t(z^2)) W(f_t(z^2)) \\ &= \sum_{0 \neq t \in S} z^t W(g_t(z^2)) W(f_t(z^2)). \end{aligned} \quad (5.4)$$

As $g(z) = \sum_{k \in S} z^k g_k(z^2)$, so for any $0 \neq t \in S$, we have

$z^t g_t(z^2) = g(z) - \sum_{k \in S, k \neq t} z^k g_k(z^2) \Rightarrow g_t(z^2) = \bar{z}^t g(z) - \sum_{k \in S, k \neq t} z^{k-t} g_k(z^2)$. Therefore,

$$W(g_t(z^2)) = W(\bar{z}^t g(z)) \quad (5.5)$$

Similarly,

$$W(f_t(z^2)) = W(\bar{z}^t f(z)). \quad (5.6)$$

So, Eqs. (5.1)–(5.6) together imply,

$$W(fg) = (Wf)(Wg) + \sum_{r \in S, t \neq 0} z^t (W\bar{z}^t f)(W\bar{z}^t g). \quad \square$$

Remark 5.13. The above relation is correct provided the expression on the RHS is in $L^2(\mathbb{T}^n)$ i.e. $(Wf)(Wg)$ and $(W\bar{z}^t f)(W\bar{z}^t g)$ should be $L^2(\mathbb{T}^n)$ functions. This is guaranteed by Corollary 5.10.

6. Properties of A_φ and A_φ^*

Lemma 6.1. For a bounded linear operator A on $L^2(\mathbb{T}^n)$, we have $M_{z_j} A = A M_{z_j}^2 \forall 1 \leq j \leq n$ if and only if $M_{z^k} A = A M_{z^{2k}} \forall k \in \mathbb{Z}^n$.

Proof. Suppose $M_{z^k} A = A M_{z^{2k}} \forall k \in \mathbb{Z}^n$. In this expression if we put $k = \epsilon_j$ for any integer j with $1 \leq j \leq n$, we get $M_{z_j} A = A M_{z_j}^2$. Conversely, if $M_{z_j} A = A M_{z_j}^2 \forall 1 \leq j \leq n$, then for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we have $M_{z^k} A = M_{z_1}^{k_1} \dots M_{z_n}^{k_n} A = A M_{z_1}^{2k_1} \dots M_{z_n}^{2k_n} = A M_{z^{2k}}$. $\square$

Theorem 6.2. A bounded linear operator A on $L^2(\mathbb{T}^n)$ is a slant Toeplitz operator if and only if $M_{z^k} A = A M_{z^{2k}} \forall k \in \mathbb{Z}^n$. In such a case we have $A = A_\varphi$ with $\varphi(z) = \sum_{t \in S} \bar{z}^t (Ae_t)(z^2)$.

Proof. If A is a slant Toeplitz operator, then the result follows from Theorems 4.8, 4.2 and Lemma 6.1.

Conversely, let A be a bounded linear operator on $L^2(\mathbb{T}^n)$ and suppose $M_{z^k} A = A M_{z^{2k}} \forall k \in \mathbb{Z}^n$.

We prove the result in the following steps:

- (i) For $t \in S$, define $\varphi_t(z) = \bar{z}^t (Ae_t)(z^2)$. We will show that $\varphi_t \in L^\infty(\mathbb{T}^n)$.
- (ii) Define $\varphi(z) = \sum_{t \in S} \varphi_t(z)$. We will show that $A = A_\varphi$.

Proof of step (i): Let $h \in L^2(\mathbb{T}^n)$ with $h(z) = \sum_{k \in \mathbb{Z}^n} \delta_k z^k$. If $\xi(z) := h(z^2)$, then $\xi \in L^2(\mathbb{T}^n)$, $\xi(z) = \sum_{k \in \mathbb{Z}^n} \delta_k z^{2k}$ and $\|\xi\| = \|h\|$. For $t \in S$ we have

$$\begin{aligned} A(z^t \xi(z)) &= A\left(z^t \sum_{k \in \mathbb{Z}^n} \delta_k z^{2k}\right) \\ &= \sum_{k \in \mathbb{Z}^n} \delta_k A M_{z^{2k}} e_t(z) = \sum_{k \in \mathbb{Z}^n} \delta_k M_{z^k} A e_t(z) \\ &= \left(\sum_{k \in \mathbb{Z}^n} \delta_k z^k\right) A e_t(z) = h(z) \cdot A e_t(z) = (h \cdot A e_t)(z) \end{aligned}$$

$$\text{Thus, } A M_{z^t} \xi = h \cdot A e_t \text{ and } A(z^t h(z^2)) = (h \cdot A e_t)(z) \forall t \in S \quad (6.1)$$

Also, $\|M_{Ae_t} \cdot h\| = \|Ae_t \cdot h\| = \|A M_{z^t} \xi\| \leq \|A\| \|M_{z^t}\|^t \|\xi\| = \|A\| \|h\|$

Therefore M_{Ae_t} is bounded and so $Ae_t \in L^\infty(\mathbb{T}^n) \forall t \in S$.

This implies that $\varphi_t \in L^\infty(\mathbb{T}^n) \forall t \in S$.

Proof of step (ii): As $\varphi_t \in L^\infty(\mathbb{T}^n) \forall t \in S$, so $\varphi \in L^\infty(\mathbb{T}^n)$. Let $F \in L^2(\mathbb{T}^n)$. So by Lemma 5.11, $F(z) = F_0(z^2) + \sum_{0 \neq t \in S} z^t F_t(z^2)$, where $WF(z) = F_0(z)$. Again applying Theorem 5.12 we get

$$\begin{aligned} A_\varphi F(z) &= W M_\varphi F(z) = W(\varphi(z)F(z)) \\ &= (W\varphi(z))(WF(z)) + \sum_{0 \neq t \in S} z^t (W\bar{z}^t \varphi(z))(W\bar{z}^t F(z)) \end{aligned}$$

By Definition 4.3, we have $W\varphi_t(z) = 0$ for $0 \neq t \in S$, and so

$$W\varphi(z) = W\varphi_0(z) = W(Ae_0(z^2)) = Ae_0(z).$$

Also, by applying Lemma 5.4 we get

$$\begin{aligned} (W\bar{z}^t \varphi(z))(W\bar{z}^t F(z)) &= \left(\sum_{k \in S} W\bar{z}^{k+t} (Ae_k)(z^2) \right) \left(\sum_{k \in S} W\bar{z}^{k-t} F_k(z^2) \right) \\ &= (W\bar{z}^{2t} Ae_t(z^2))(WF_t(z^2)) = \bar{z}^t Ae_t(z)F_t(z). \end{aligned}$$

Therefore, $A_\varphi F(z) = Ae_0(z)F_0(z) + \sum_{0 \neq t \in S} |z|^{2t} Ae_t(z)F_t(z)$

$$\begin{aligned} &= (F_0 \cdot Ae_0)(z) + \sum_{0 \neq t \in S} (F_t \cdot Ae_t)(z) \\ &= A(F_0(z^2)) + \sum_{0 \neq t \in S} A(z^t F_t(z^2)), \text{ by Eq. (6.1)} \\ &= A[F_0(z^2) + \sum_{0 \neq t \in S} z^t F_t(z^2)] = AF(z) \end{aligned}$$

Thus, $A_\varphi F = AF \forall F \in L^2(\mathbb{T}^n)$ which implies that $A_\varphi = A$. $\square$

We are now in a position to complete the proof of Theorem 4.8, stated in Section 4.

Proof of Theorem 4.8. Suppose A is a bounded linear operator on $L^2(\mathbb{T}^n)$ such that A can be represented as a slant Toeplitz matrix of level- n . Then by Theorem 4.2 we have $M_{z_j} A = A M_{z_j^2} \forall 1 \leq j \leq n$. This together with Lemma 6.1 implies that $M_{z^k} A = A M_{z^{2k}} \forall k \in \mathbb{Z}^n$. Hence, applying Theorem 6.2, we conclude that A is a slant Toeplitz operator.

Theorem 6.3. $A_\varphi A_\varphi^* = M_\psi$ where $\psi = W(|\varphi|^2)$.

Proof. $A_\varphi A_\varphi^* = W M_\varphi M_\varphi^* W^* = W M_{|\varphi|^2} W^* = W(W M_{|\varphi|^2})^* = W A_{|\varphi|^2}^* = M_\psi$, where $\psi = W(|\varphi|^2)$, by Theorem 5.9. $\square$

Corollary 6.4. $\|A_\varphi\|^2 = \|\psi\|_\infty$, where $\psi = W(|\varphi|^2)$.

Proof. $\|A_\varphi\|^2 = \|A_\varphi A_\varphi^*\| = \|M_\psi\| = \|\psi\|_\infty$. $\square$

Corollary 6.5. A_φ^* is an isometry iff $W|\varphi|^2 = 1$.

Theorem 6.6. The map $\tau : L^\infty(\mathbb{T}^n) \mapsto \mathcal{B}(L^2(\mathbb{T}^n))$ defined as $\tau(\varphi) = A_\varphi$ is linear and injective. Here $\mathcal{B}(L^2(\mathbb{T}^n))$ denotes the algebra of all bounded linear operators on $L^2(\mathbb{T}^n)$.

Proof. For $\varphi, \psi \in L^\infty(\mathbb{T}^n)$ and $\lambda \in \mathbb{C}$,

$\tau(\lambda\varphi + \psi) = A_{\lambda\varphi + \psi} = W M_{\lambda\varphi + \psi} = \lambda(W M_\varphi) + (W M_\psi) = \lambda A_\varphi + A_\psi = \lambda\tau(\varphi) + \tau(\psi)$. Therefore τ is linear.

Next to show τ is injective: Suppose $\tau(\varphi) = 0$. This implies $A_\varphi = W M_\varphi = 0$. Let $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. Then by Lemma 5.11, we have $\varphi(z) = \sum_{\xi \in S} z^\xi \varphi_\xi(z^2)$ where $\varphi_\xi(z) = \sum_{k \in \mathbb{Z}^n} a_{2k+\xi} z^k$. So for $t \in S$ we have $\varphi(z) z^t = \sum_{\xi \in S} z^{\xi+t} \varphi_\xi(z^2)$, and since by Lemma 5.4, $\xi + t$ is even only for $\xi = t$, so $W(z^{\xi+t} \varphi_\xi(z^2)) = 0$ if $\xi \neq t$. Hence we have

$$\begin{aligned} W M_\varphi e_t(z) &= W(\varphi(z) z^t) = W(z^{2t} \varphi_t(z^2)), \\ &= W\left(\sum_{k \in \mathbb{Z}^n} a_{2k+t} z^{2(k+t)}\right) = z^t \sum_{k \in \mathbb{Z}^n} a_{2k+t} z^k = z^t \varphi_t(z) \end{aligned}$$

Therefore, $W M_\varphi = 0$ implies that $\varphi_t(z) = 0 \forall t \in S$, from which we can conclude that $\varphi(z) = 0$. $\square$

Theorem 6.7. $M_\varphi A_\psi$ is a slant Toeplitz operator and $M_\varphi A_\psi = A_{\varphi(z^2) \cdot \psi(z)}$.

Proof. For each $k \in \mathbb{Z}^n$ we have $M_{z^k}(M_\varphi A_\psi) = M_\varphi M_{z^k} A_\psi = (M_\varphi A_\psi) M_{z^{2k}}$. So, by Theorem 6.2, $M_\varphi A_\psi$ is a slant Toeplitz operator. Again, as A_ψ is slant Toeplitz, so $M_{z^k} A_\psi = A_\psi M_{z^{2k}} \forall k \in \mathbb{Z}^n$ which implies that $M_{\varphi(z)} A_\psi = A_\psi M_{\varphi(z^2)} \forall \varphi \in L^\infty(\mathbb{T}^n)$.

Therefore, $M_\varphi A_\psi = A_\psi M_{\varphi(z^2)} = W M_\psi M_{\varphi(z^2)} = W M_{\psi(z) \varphi(z^2)} = A_{\psi(z) \varphi(z^2)}$. $\square$

Theorem 6.8. $A_\psi M_\varphi = M_\varphi A_\psi$ if and only if $\psi(z) \varphi(z) = \psi(z) \varphi(z^2)$ for any $z \in \mathbb{T}^n$. In particular, if ψ is invertible, then $A_\psi M_\varphi = M_\varphi A_\psi$ if and only if $\varphi(z) = \text{constant}$.

Proof. $A_\psi M_\varphi = W M_\psi M_\varphi = W M_\psi \varphi = A_{\psi \varphi}$, and by Theorem 6.7, $M_\varphi A_\psi = A_{\varphi(z^2) \cdot \psi(z)}$. Thus $A_\psi M_\varphi = M_\varphi A_\psi$ if and only if $A_{\psi(z) \varphi(z)} = A_{\varphi(z^2) \cdot \psi(z)}$. In other words, $A_\psi M_\varphi = M_\varphi A_\psi$ if and only if $\psi(z) \varphi(z) = \psi(z) \varphi(z^2)$ for any $z \in \mathbb{T}^n$.

If ψ is invertible, then there exists $\psi^{-1} \in L^\infty(\mathbb{T}^n)$ such that $\psi \psi^{-1} = I = \psi^{-1} \psi$, so that $\varphi(z) \psi(z) = \varphi(z^2) \psi(z)$ if and only if $\varphi(z) = \varphi(z^2)$.

Now if $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$, then $\varphi(z^2) = \sum_{k \in \mathbb{Z}^n} a_k z^{2k}$, and $\text{so } \varphi(z) = \varphi(z^2)$ implies $\sum_{k \in \mathbb{Z}^n, k \text{ odd}} a_k z^k + \sum_{0 \neq k \in \mathbb{Z}^n} (a_{2k} - a_k) z^{2k} = 0$

This gives $a_k = 0 \forall k \text{ odd}$ and $a_{2k} = a_k \forall 0 \neq k \in \mathbb{Z}^n$

Therefore $a_k = 0 \forall 0 \neq k \in \mathbb{Z}^n$, which yields $\varphi(z) = a_0$, a constant.

On the other hand, if $\varphi(z)$ is constant then $\varphi(z) = \varphi(z^2)$, and so we have $A_\psi M_\varphi = M_\varphi A_\psi$. $\square$

Theorem 6.9. $W A_\varphi$ is a slant Toeplitz operator if and only if $\varphi = 0$.

Proof. The result is obvious if $\varphi = 0$. Conversely suppose $W A_\varphi$ is a slant Toeplitz operator with $\varphi(z) = \sum_{\tau \in \mathbb{Z}^n} \delta_\tau z^\tau$. Then by Theorem 4.7, we get $\langle W A_\varphi e_{k+2\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle W A_\varphi e_k, e_t \rangle \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$.

Now $\langle W A_\varphi e_{k+2\epsilon_j}, e_{t+\epsilon_j} \rangle = \langle A_\varphi e_k, e_{2t+\epsilon_j} \rangle = \langle W M_\varphi e_k, e_{2t+\epsilon_j} \rangle = \langle M_\varphi e_k, e_{4t+2\epsilon_j} \rangle$, and similarly, $\langle W A_\varphi e_k, e_t \rangle = \langle M_\varphi e_k, e_{4t} \rangle$.

So, $W A_\varphi$ is slant Toeplitz implies that $\langle M_\varphi e_k, e_{4t+2\epsilon_j} \rangle = \langle M_\varphi e_k, e_{4t} \rangle$, which yields $\langle \varphi(z) z^k, z^{4t+2\epsilon_j} \rangle = \langle \varphi(z) z^k, z^{4t} \rangle$. Consequently $\langle \sum_\tau \delta_\tau z^{\tau+k}, z^{4t+2\epsilon_j} \rangle = \langle \sum_\tau \delta_\tau z^{\tau+k}, z^{4t} \rangle$ showing that $\delta_{4t-k+2\epsilon_j} = \delta_{4t-k} \forall k, t \in \mathbb{Z}^n, 1 \leq j \leq n$.

Thus $\delta_{k+2\epsilon_j} = \delta_k \forall k \in \mathbb{Z}^n, 1 \leq j \leq n$.

So, for each $k \in \mathbb{Z}^n$, and $j = 1, 2, \dots, n$, we have $\delta_k = \delta_{k+2\epsilon_j} = \delta_{k+4\epsilon_j} = \delta_{k+6\epsilon_j} = \dots$. But $|k + 2\lambda\epsilon_j| \rightarrow \infty$ as $\lambda \rightarrow \infty$, and as $\varphi \in L^\infty(\mathbb{T}^n)$ so $\delta_{k+2\lambda\epsilon_j} \rightarrow 0$ as $\lambda \rightarrow \infty$.

Hence $\delta_k = 0$. As this is true for all $k \in \mathbb{Z}^n$, so we must have $\varphi = 0$. $\square$

Theorem 6.10. For $\varphi, \psi \in L^\infty(\mathbb{T}^n)$ the following are equivalent:

- (i) $A_\varphi A_\psi$ is a slant Toeplitz operator.

$$(ii) \varphi(z^2) \cdot \psi(z) = 0.$$

$$(iii) A_\varphi A_\psi = 0.$$

Proof. (i) $\Leftrightarrow$ (ii) Using Theorem 6.7 we get $A_\varphi A_\psi = W M_\varphi A_\psi = W A_{\varphi(z^2) \cdot \psi(z)}$. Therefore by Theorem 6.9, $A_\varphi A_\psi$ is a slant Toeplitz operator if and only if $\varphi(z^2) \cdot \psi(z) = 0$.

(ii) $\Leftrightarrow$ (iii)

As $A_\varphi A_\psi = W A_{\varphi(z^2) \cdot \psi(z)}$, so $A_\varphi A_\psi = 0$ implies that $W A_{\varphi(z^2) \cdot \psi(z)}$ is a slant Toeplitz operator. Therefore, by Theorem 6.9, $\varphi(z^2) \cdot \psi(z) = 0$. Conversely, $\varphi(z^2) \cdot \psi(z) = 0$ implies $M_{\varphi(z^2) \cdot \psi(z)} = 0$, and so $A_\varphi A_\psi = W A_{\varphi(z^2) \cdot \psi(z)} = W^2 M_{\varphi(z^2) \cdot \psi(z)} = 0$. $\square$

The following theorem gives a necessary and sufficient condition for two slant Toeplitz operators to commute.

Theorem 6.11. For $\varphi, \psi \in L^\infty(\mathbb{T}^n)$, $A_\varphi A_\psi = A_\psi A_\varphi$ if and only if $\varphi(z^2)\psi(z) = \psi(z^2)\varphi(z)$ for any $z \in \mathbb{Z}^n$.

Proof. Suppose $A_\varphi A_\psi = A_\psi A_\varphi$. Now $A_\varphi A_\psi = W M_\varphi A_\psi$ and by Theorem 6.7, $M_\varphi A_\psi = A_{\varphi(z^2)\psi(z)}$. Thus, $A_\varphi A_\psi = A_\psi A_\varphi$ implies $W A_{\varphi(z^2)\psi(z)} = W A_{\psi(z^2)\varphi(z)}$. In view of Theorem 6.6 we can infer that $W A_{\varphi(z^2)\psi(z) - \psi(z^2)\varphi(z)} = 0$, which in turn implies that $W A_{\varphi(z^2)\psi(z) - \psi(z^2)\varphi(z)}$ is a slant Toeplitz operator. Therefore by Theorem 6.9 we get $\varphi(z^2)\psi(z) - \psi(z^2)\varphi(z) = 0$ which implies $\varphi(z^2)\psi(z) = \psi(z^2)\varphi(z)$.

Conversely, suppose $\varphi(z^2)\psi(z) = \psi(z^2)\varphi(z)$. Then by Theorem 6.9, we have $W A_{\varphi(z^2)\psi(z) - \psi(z^2)\varphi(z)}$ is slant Toeplitz operator and equals zero. This together with Theorems 6.6 and 6.7 implies that $A_\varphi A_\psi = A_\psi A_\varphi$. $\square$

Theorem 6.12. $A_\varphi^2 = A_\varphi$ if and only if $\varphi = 0$. In other words, there are no non-zero idempotents among slant Toeplitz operators.

Proof. Suppose $A_\varphi^2 = A_\varphi$. Then $A_\varphi^2 = A_\varphi A_\varphi$ is slant Toeplitz, so that by Theorem 6.10 we have $A_\varphi^2 = 0$. Thus $A_\varphi = 0 = W A_\varphi$. This in turn implies that $W A_\varphi$ is slant Toeplitz, and by Theorem 6.9 we get $\varphi = 0$.

Conversely, $\varphi = 0 \Rightarrow M_\varphi = 0 \Rightarrow A_\varphi = W M_\varphi = 0 \Rightarrow A_\varphi^2 = A_\varphi$. $\square$

Theorem 6.13. A_φ^* is a slant Toeplitz operator if and only if $\varphi = 0$.

Proof. If $\varphi = 0$ then $A_\varphi = 0 = A_\varphi^*$ and so A_φ^* is slant Toeplitz.

Conversely, suppose A_φ^* is slant Toeplitz. Then, by Theorem 4.7,

$$\langle A_\varphi^* e_{k+2\epsilon_j}, e_{l+\epsilon_j} \rangle = \langle A_\varphi^* e_k, e_l \rangle \quad \forall k, t \in \mathbb{Z}^n, \quad 1 \leq j \leq n. \text{ This implies that}$$

$$\langle A_\varphi e_{l+\epsilon_j}, e_{k+2\epsilon_j} \rangle = \langle A_\varphi e_l, e_k \rangle \quad \forall k, t \in \mathbb{Z}^n, \quad 1 \leq j \leq n, \text{ which in turn implies } \langle W M_\varphi e_{l+\epsilon_j}, e_{k+2\epsilon_j} \rangle = \langle W M_\varphi e_l, e_k \rangle \quad \forall k, t \in \mathbb{Z}^n, \quad 1 \leq j \leq n.$$

Taking $\varphi(z) = \sum_{\tau \in \mathbb{Z}^n} a_\tau z^\tau$ in the above expression, we get

$$\langle \sum_{\tau} a_\tau z^{\tau+l+\epsilon_j}, z^{2k+4\epsilon_j} \rangle = \langle \sum_{\tau} a_\tau z^{\tau+l}, z^{2k} \rangle \quad \forall k, t \in \mathbb{Z}^n, \quad 1 \leq j \leq n, \text{ which yields } a_{2k-t+3\epsilon_j} = a_{2k-t} \quad \forall k, t \in \mathbb{Z}^n, \quad 1 \leq j \leq n$$

Thus, $a_{k+3\epsilon_j} = a_k \quad \forall k \in \mathbb{Z}^n, 1 \leq j \leq n$, and as in Theorem 6.9, this implies that $\varphi = 0$. $\square$

Corollary 6.14. There is no non-zero self adjoint slant Toeplitz operator.

Theorem 6.15. A_φ is hyponormal if and only if $\varphi = 0$.

Proof. A_φ is hyponormal if and only if $\|A_\varphi f\|^2 \geq \|A_\varphi^* f\|^2 \quad \forall f \in L^2(\mathbb{T}^n)$. Equivalently, $\|W(\varphi(z)f(z))\|^2 \geq \|W^* f\|^2 \quad \forall f \in L^2(\mathbb{T}^n)$

Let $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. If $\varphi = 0$ then clearly $A_\varphi = 0$ is hyponormal. For the converse we consider the following two cases.

Case 1: Let $f(z) = 1 = e_0(z)$. Then by Remark 4.4(ii), we have $W^*f(z) = 1$

$$\begin{aligned} \text{So } A_\varphi \text{ is hyponormal} &\Rightarrow \left\| W \left(\sum_{k \in \mathbb{Z}^n} a_k z^k \right) \right\|^2 \geq \left\| \sum_{k \in \mathbb{Z}^n} \bar{a}_k z^{-k} \right\|^2 \\ &\Rightarrow \left\| \sum_{k \in \mathbb{Z}^n} a_{2k} z^k \right\|^2 \geq \sum_{k \in \mathbb{Z}^n} |\bar{a}_k|^2 \\ &\Rightarrow \sum_{k \in \mathbb{Z}^n} |a_{2k}|^2 \geq \sum_{k \in \mathbb{Z}^n} |a_k|^2 \\ &\Rightarrow \sum_{k \in \mathbb{Z}^n, k \text{ odd}} |a_k|^2 \leq 0 \\ &\Rightarrow a_k = 0 \quad \forall k \in \mathbb{Z}^n, k \text{ odd} \end{aligned}$$

Therefore, $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_{2k} z^{2k}$.

Case 2: Let $f(z) = z$ so that $W^*f(z) = z^2$, and

$$\begin{aligned} A_\varphi \text{ is hyponormal} &\Rightarrow \left\| W \left(\sum_{k \in \mathbb{Z}^n} a_{2k} z^{2k+1} \right) \right\|^2 \geq \left\| \sum_{k \in \mathbb{Z}^n} \bar{a}_{2k} z^{2-2k} \right\|^2 \\ &\Rightarrow 0 \geq \sum_{k \in \mathbb{Z}^n} |a_{2k}|^2, \text{ by Remark 4.4(i)} \\ &\Rightarrow a_{2k} = 0 \quad \forall k \in \mathbb{Z}^n. \end{aligned}$$

Thus, $\varphi = 0$. $\square$

Theorem 6.16. *A slant Toeplitz operator cannot be an isometry.*

Proof. Let, if possible A_φ be an isometry. Then $A_\varphi^* A_\varphi = I$. Or equivalently, $\|A_\varphi f\| = \|f\| \quad \forall f \in L^2(\mathbb{T}^n)$

By Theorem 6.3, $A_\varphi A_\varphi^* = M_\psi$ where $\psi = W|\varphi|^2$

Also by Corollary 6.4, $\|\psi\|_\infty = \|A_\varphi\|^2 = \|A_\varphi^* A_\varphi\| = 1$. Thus, $\|W|\varphi|^2\|_\infty = 1$

Let $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. Then $\|\varphi\| = (\sum_{k \in \mathbb{Z}^n} |a_k|^2)^{\frac{1}{2}} = [\sum_{t \in S} (\sum_{k \in \mathbb{Z}^n} |a_{2k+t}|^2)]^{\frac{1}{2}}$, by Lemma 5.2. Therefore,

$$\|W|\varphi|^2\|_\infty = \|W M_\varphi \bar{\varphi}\| = \|A_\varphi \bar{\varphi}\| = \|\bar{\varphi}\| = \|\varphi\| = \left[\sum_{t \in S} \left(\sum_{k \in \mathbb{Z}^n} |a_{2k+t}|^2 \right) \right]^{\frac{1}{2}} \quad (6.2)$$

For $t \in S$, $A_\varphi \bar{z}^t = W M_\varphi \bar{z}^t = W \sum_{k \in \mathbb{Z}^n} a_k z^{k-t} = W \sum_{\tau \in \mathbb{Z}^n} a_{\tau+t} z^\tau$. So by Remark 4.4(i), $A_\varphi \bar{z}^t = \sum_{\tau \in \mathbb{Z}^n} a_{2\tau+t} z^\tau$. Combining this along with the fact that $\|A_\varphi \bar{z}^t\|^2 = \|\bar{z}^t\|^2 = 1$, we get $\|\sum_{\tau \in \mathbb{Z}^n} a_{2\tau+t} z^\tau\|^2 = 1$, which implies that $\sum_{\tau \in \mathbb{Z}^n} |a_{2\tau+t}|^2 = 1$.

Using this in Eq. (6.2) we get $\|W|\varphi|^2\|_\infty = 2^{\frac{n}{2}}$, since $o(S) = 2^n$.

So, $\|W|\varphi|^2\|_\infty = 1$ means $2^{\frac{n}{2}} = 1$, which is only possible if $n = 0$. n being a positive integer this is not possible, and so we conclude that A_φ cannot be an isometry. $\square$

Theorem 6.17. *A_φ is compact if and only if $\varphi = 0$.*

Proof. We know that the set of compact operators on a Hilbert space H is an ideal of the space of bounded linear operators on H . Thus,

A_φ is compact $\Rightarrow A_\varphi M_{z^t}$ is compact $\Rightarrow (A_\varphi M_{z^t})^*$ is compact $\forall t \in S$.

For $t \in S$, $(A_\varphi M_{z^t})^* = (W M_\varphi M_{z^t})^* = (W M_{z^t \varphi})^* = A_{z^t \varphi}^*$.

So, $W(A_\varphi M_{z^t})^* = W A_{z^t \varphi}^* = M_\psi$, where $\psi = W(\overline{z^t \varphi})$ by Theorem 5.9

But $A_\varphi M_{z^t}$ is compact implies that $W(A_\varphi M_{z^t})^* = M_\psi$ is compact, and consequently $\psi = 0$.

Hence $\langle W(\overline{z^t \varphi}), z^m \rangle = \langle \psi, z^m \rangle = 0 \forall m \in \mathbb{Z}^n$, and so $\langle \overline{\varphi}, z^{2m+t} \rangle = 0 \forall m \in \mathbb{Z}^n, t \in S$. Thus, if $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$, then we must have $a_{2m+t} = 0 \forall m \in \mathbb{Z}^n, t \in S$, and now, by Lemma 5.2, we get $a_k = 0 \forall k \in \mathbb{Z}^n$, so that $\varphi = 0$. $\square$

7. Remarks

In Section 3 we have shown that an operator on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if it can be represented as a Toeplitz matrix of level n . Using this relation, we can define the operator B_φ on $H^2(\mathbb{D}^n)$ as $B_\varphi = W T_\varphi$, calling it the compression of the slant Toeplitz operator to the Hardy space of the polydisk. A study of the properties of B_φ vis-a-vis that of A_φ should yield interesting results. For $H^2(\mathbb{D})$, similar studies have been conducted in [2], Section 3 [18], and [22].

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