

On classification of $(n + 6)$ -dimensional nilpotent n -Lie algebras of class 2 with $n \geq 4$

Classification
of nilpotent
 n -Lie algebras

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Abstract

Purpose – The purpose of this paper is to determine the structure of nilpotent $(n + 6)$ -dimensional n -Lie algebras of class 2 when $n \geq 4$.

Design/methodology/approach – By dividing a nilpotent $(n + 6)$ -dimensional n -Lie algebra of class 2 by a central element, the authors arrive to a nilpotent $(n + 5)$ dimensional n -Lie algebra of class 2. Given that the authors have the structure of nilpotent $(n + 5)$ -dimensional n -Lie algebras of class 2, the authors have access to the structure of the desired algebras.

Findings – In this paper, for each $n \geq 4$, the authors have found 24 nilpotent $(n + 6)$ dimensional n -Lie algebras of class 2. Of these, 15 are non-split algebras and the nine remaining algebras are written as direct additions of n -Lie algebras of low-dimension and abelian n -Lie algebras.

Originality/value – This classification of n -Lie algebras provides a complete understanding of these algebras that are used in algebraic studies.

Keywords Nilpotent n -lie algebra, Classification, Nilpotent n -Lie algebra of class 2

Paper type Research paper

1. Introduction

In 1985, Filippov [1] introduced the concept of n -Lie (Filippov) algebras, as an n -ary multilinear and skew-symmetric operation $[x_1, \dots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$. Beside presenting many examples of n -Lie algebras, he also extended the notions of simplicity and nilpotency and determined all $(n + 1)$ -dimensional n -Lie algebras over an algebraically closed field of characteristic zero.

The study of n -Lie algebras is important, since it is related to geometry and physics. Among other results, n -Lie algebras are classified in some cases. For example, Bai *et al.* [2] classified all n -Lie algebras of dimension $n + 1$ over a field of characteristic 2. Also, they showed that there is no simple n -Lie algebra of dimension $n + 2$. Then, Bai *et al.* [3] classified

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n -Lie algebras of dimension $n + 2$ on the algebraically closed fields with characteristic zero. (see [4–7] for more information on the Filippov algebras).

In 1986, Kasymov [8] studied some properties of nilpotent and solvable n -Lie algebras. An n -Lie algebra A is *nilpotent* if $A^s = 0$ for some nonnegative integer s , where A^i is defined inductively by $A^1 = A$ and $A^{i+1} = [A^i, A, \dots, A]$. The n -Lie algebra A is nilpotent of class c , if $A^{c+1} = 0$ and $A^i \neq 0$ for each $i \leq c$. The ideal $A^2 = [A, \dots, A]$ is called the *derived subalgebra* of A . The *center* of A is defined by

$$Z(A) = \{x \in A : [x, A, \dots, A] = 0\}.$$

Let $Z_0(A) = \langle 0 \rangle$. Then the i th center of A is defined inductively by

$$\frac{Z_i(A)}{Z_{i-1}(A)} = Z\left(\frac{A}{Z_{i-1}(A)}\right)$$

for all $i \geq 1$. Clearly, $Z_1(A) = Z(A)$.

The nilpotent theories of many algebras attract more and more attention. For example, in [9,10], and [11], the authors studied nilpotent Leibniz n -algebras, nilpotent Lie and Leibniz algebras, and nilpotent n -Lie algebras, respectively.

The $(n + 3)$ -dimensional nilpotent n -Lie algebras and $(n + 4)$ -dimensional nilpotent n -Lie algebras of class 2 were classified in [12]. Hoseini *et al.* [13] classified $(n + 5)$ -dimensional nilpotent n -Lie algebras of class 2.

In this paper, we have interest for algebras of class 2 (the minimal class for nonabelian case). The concept of filiform n -Lie algebras (maximal class) has been studied in some papers. For example, see [14].

The rest of our paper is organized as follows: Section 2 includes the results that are used frequently in the last section. In Section 3, we classify $(n + 6)$ -dimensional n -Lie algebras of class 2 when $n \geq 4$. For the case $n = 2$, this problem is dealt with by Yan *et al.* [15]. Also, the case $n = 3$ stated in [16].

2. Preliminaries

In this section, we introduce some known and necessary results. We denote d -dimensional abelian n -Lie algebra by $F(d)$. An important category of n -Lie algebras of class 2, which plays an essential role in classification of nilpotent n -Lie algebras, are algebras whose derived and center are equal. We call an n -Lie algebra A , a generalized Heisenberg of rank k , if $A^2 = Z(A)$ and $\dim A^2 = k$. The particular case $k = 1$, is called the special Heisenberg n -Lie algebras. The structure of this algebras defined as follows.

Theorem 2.1. [17] Every special Heisenberg n -Lie algebra has dimension $mn + 1$ for some natural number m , and it is isomorphic to

$$H(n, m) = \langle x, x_1, \dots, x_{nm} : [x_{n(i-1)+1}, x_{n(i-1)+2}, \dots, x_{ni}] = x, i = 1, \dots, m \rangle.$$

Theorem 2.2. [18] Let A be a d -dimensional nilpotent n -Lie algebra, and let $\dim A^2 = 1$. Then, for some $m \geq 1$, it follows that

$$A \cong H(n, m) \oplus F(d - mn - 1).$$

Theorem 2.3. [18] Let A be a nonabelian nilpotent n -Lie algebra of dimension $d \leq n + 2$. Then A is isomorphic to $H(n, 1)$, $H(n, 1) \oplus F(1)$ or $A_{n,n+2,1}$, where $A_{n,n+2,1} = \langle e_1, \dots, e_{n+2} : [e_1, \dots, e_n] = e_{n+1}, [e_2, \dots, e_{n+1}] = e_{n+2} \rangle$.

For unification of notation in what follows, the t th d -dimensional n -Lie algebra is denoted by $A_{n,d,t}$.

Theorem 2.4. [12] The $(n+3)$ -dimensional nonabelian nilpotent n -Lie algebras for $n > 2$ over an arbitrary field are $A_{n,n+3,i}$ ($2 \leq i \leq 8$). Moreover nilpotent classes of $A_{n,n+3,2}$ and $A_{n,n+3,5}$ is two, nilpotent classes of $A_{n,n+3,3}$, $A_{n,n+3,4}$ and $A_{n,n+3,8}$ is three and finally, nilpotent classes of $A_{n,n+3,6}$ and $A_{n,n+3,7}$ is four (maximal class).

Theorem 2.5. [12] The only $(n+4)$ -dimensional nilpotent n -Lie algebras of class 2 are $H(n, 1) \oplus F(3)$, $A_{n,n+4,1}$, $A_{n,n+4,2}$, $A_{n,n+4,3}$, $H(2, 2) \oplus F(1)$, $H(3, 2)$, $L_{6,22}(\varepsilon)$, and $L_{6,7}^2(\eta)$.

Theorem 2.6. [13] The $(n+5)$ -dimensional nilpotent n -Lie algebras of class 2 for $n > 2$ over an arbitrary field are $H(n, 1) \oplus F(4)$, $A_{n,n+5,i}$ ($1 \leq i \leq 7$), $H(3, 2) \oplus F(1)$ and $H(4, 2)$.

Theorem 2.7. [19] Let A be a nilpotent n -Lie algebra of class 2. Then, there exist a generalized Heisenberg n -Lie algebra H and an abelian n -Lie algebra F such that $A = H \oplus F$.

3. Main results

In this section, we classify $(n+6)$ -dimensional nilpotent n -Lie algebras of class 2. If n -Lie algebra A is nilpotent of class 2, then A is nonabelian and $A^2 \subseteq Z(A)$. The nilpotent n -Lie algebra of class 2 plays an essential role in some geometry problems such as the commutative Riemannian manifold. Additionally, the classification of nilpotent Lie algebras of class 2 is one of the most important issues in Lie algebras.

The following theorems define the structure of generalized Heisenberg n -Lie algebras of rank 2 with dimension at most $2n+3$.

Theorem 3.1. [18] Let A be a nilpotent n -Lie algebra of dimension $d = n+k$ for $3 \leq k \leq n+1$ such that $A^2 = Z(A)$ and $\dim A^2 = 2$. Then

$$A \cong \langle e_1, \dots, e_{n+k} : [e_{k-1}, \dots, e_{n+k-2}] = e_{n+k}, [e_1, \dots, e_n] = e_{n+k-1} \rangle.$$

Remark. In the above theorem for $n=2$ and $k=3$, we obtain

$$A \cong \langle e_1, e_2, e_3, e_4, e_5 : [e_1, e_2] = e_4, [e_2, e_3] = e_5 \rangle.$$

This algebra appears many times in differential geometry in the study of Pfaffian systems. It was developed by P. Libermann and introduced in [20].

Theorem 3.2. [19] Let A be a generalized Heisenberg n -Lie algebra of rank 2 with dimension $2n+2$. Then

$$A \cong A_{n,2n+2,1} = \langle e_1, \dots, e_{2n+2} : [e_1, \dots, e_n] = e_{2n+1}, [e_{n+1}, \dots, e_{2n}] = e_{2n+2} \rangle.$$

Theorem 3.3. [19] Let A be a generalized Heisenberg n -Lie algebras of rank 2 with dimension $2n+3$. Then, A is isomorphic to one of the following n -Lie algebras:

$$A_{n,2n+3,1} = \langle e_1, \dots, e_{2n+3} : [e_1, \dots, e_n] = e_{2n+3}, [e_2, \dots, e_{n+1}] = [e_{n+2}, \dots, e_{2n+1}] = e_{2n+2} \rangle.$$

$$\begin{aligned} A_{n,2n+3,2} &= \langle e_1, \dots, e_{2n+3} : [e_1, \dots, e_n] = [e_{n+1}, \dots, e_{2n}] = e_{2n+3}, [e_2, \dots, e_{n+1}] \\ &= [e_{n+2}, \dots, e_{2n+1}] = e_{2n+2} \rangle. \end{aligned}$$

For $n=2$, we obtain also a Lie algebra of the previous type.

Now we are going to classify $(n + 6)$ -dimensional nilpotent n -Lie algebras of class 2.

According to Theorem 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n -Lie algebra of rank 2 and F is abelian. Therefore, first we classify the generalized Heisenberg n -Lie algebra of rank 2.

By the classification of nilpotent n -Lie algebras of class 2, we have the following theorem. All the algebras defined in theorem 3.4 and follow are in Table 1 at the end of the paper.

Theorem 3.4.

- (1) The only $(n + 4)$ -dimensional generalized Heisenberg n -Lie algebra of rank 3 is $A_{n,n+4,3}$.
- (2) The only $(n + 5)$ -dimensional generalized Heisenberg n -Lie algebras of rank 3 are $A_{n,n+5,4}$ and $A_{n,n+5,5}$.
- (3) The only $(n + 5)$ -dimensional generalized Heisenberg n -Lie algebra of rank 4 is $A_{n,n+5,6}$.

The following lemma defines the structure of $(n + 6)$ -dimensional generalized Heisenberg n -Lie algebras of rank 2.

Theorem 3.5. Let A be a generalized Heisenberg n -Lie algebra of rank 2 with dimension $n + 6$. Then

$$A \cong A_{n,n+6,1} = \langle e_1, \dots, e_{n+6} : [e_1, \dots, e_n] = e_{n+5}, [e_5, \dots, e_{n+4}] = e_{n+6} \rangle.$$

Proof. For $n = 4$, we have $n + 6 = 2n + 2$. Thus by Theorem 3.2, if $n \geq 5$, then $n + 3 < n + 6 \leq 2n + 1$. Applying Theorem 3.1 completes the proof. ■

Theorem 3.6. The only $(n + 6)$ -dimensional generalized Heisenberg n -Lie algebras of rank 3 are

$$A_{n,n+6,2}, A_{n,n+6,3}, A_{n,n+6,4}, A_{n,n+6,5}, \text{ and } A_{n,n+6,6}(\varepsilon).$$

Nilpotent n -Lie algebras of class 2	Nonzero multiplications
$A_{n,n+4,1}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4}$
$A_{n,n+4,2}$	$[e_1, \dots, e_n] = e_{n+3}, [e_3, \dots, e_{n+2}] = e_{n+4} \quad (n \geq 3)$
$A_{n,n+4,3}$	$[e_1, \dots, e_n] = e_{n+1}, [e_2, \dots, e_n, e_{n+2}] = e_{n+3},$ $[e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+4}$
$A_{n,n+5,1}$	$[e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}$
$A_{n,n+5,2}$	$[e_1, \dots, e_n] = e_{n+4}, [e_3, \dots, e_{n+2}] = e_{n+5} \quad (n \geq 3)$
$A_{n,n+5,3}$	$[e_1, \dots, e_n] = e_{n+5}, [e_4, \dots, e_{n+3}] = e_{n+4} \quad (n \geq 3)$
$A_{n,n+5,4}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_1, e_3, \dots, e_{n+1}] = e_{n+5}$
$A_{n,n+5,5}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_2, \dots, e_n, e_{n+2}] = e_{n+5}$
$A_{n,n+5,6}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_3, \dots, e_{n+2}] = e_{n+5}$
$A_{n,n+5,7}$	$[e_1, \dots, e_n] = e_{n+1}, [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+5},$ $[e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+4}, [e_2, \dots, e_n, e_{n+2}] = e_{n+3}$

Table 1.

Proof. Suppose that A is an $(n+6)$ -dimensional generalized Heisenberg n -Lie algebra of rank 3 with basis $\{e_1, \dots, e_{n+6}\}$, which $n \geq 4$. Also, suppose that $A^2 = \langle e_{n+4}, e_{n+5}, e_{n+6} \rangle$. In this case, $A/\langle e_{n+6} \rangle$ is an $(n+5)$ -dimensional nilpotent n -Lie algebra of class 2 with derived algebra of dimension 2. By Theorem 2.6, we have three possibilities for $A/\langle e_{n+6} \rangle$: Case 1: Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,1}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \dots, e_n] = e_{n+4} + \alpha e_{n+6}, \\ [e_2, \dots, e_{n+1}] = e_{n+5} + \beta e_{n+6}, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+3}] = \gamma_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+2}, e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, \widehat{e}_k, \dots, e_n, e_{n+1}, e_{n+2}, e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \leq i < j < k \leq n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n -Lie algebras, we conclude that one of the coefficients

$$\lambda_{ij} (1 \leq i < j \leq n), \quad \phi_{ijk} (1 \leq i < j < k \leq n)$$

is equal to one, and the others are zero. We have four possibilities:

- (1) $\lambda_{12} = 1$, $\lambda_{ij} = 0 (1 \leq i < j \leq n, (i, j) \neq (1, 2))$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n)$. In this case, the brackets in A can be written as

$$[e_1, \dots, e_n] = e_{n+4}, \quad [e_2, \dots, e_{n+1}] = e_{n+5}, \quad [e_3, \dots, e_n, e_{n+2}, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,2}$.

- (2) $\lambda_{23} = 1$, $\lambda_{ij} = 0 (1 \leq i < j \leq n, (i, j) \neq (2, 3))$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n)$. In this case, the brackets in A can be written as

$$[e_1, \dots, e_n] = e_{n+4}, \quad [e_2, \dots, e_{n+1}] = e_{n+5}, \quad [e_1, e_4, \dots, e_n, e_{n+2}, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,3}$.

- (3) $\lambda_{ij} = 0 (1 \leq i < j \leq n)$, $\phi_{123} = 1$, and $\phi_{ijk} = 0 (1 \leq i < j < k \leq n, (i, j, k) \neq (1, 2, 3))$. In this case, the brackets in A can be written as

$$[e_1, \dots, e_n] = e_{n+4}, \quad [e_2, \dots, e_{n+1}] = e_{n+5}, \quad [e_4, \dots, e_{n+3}] = e_{n+6}.$$

One can easily see that this algebra is isomorphic to $A_{n+6,3}$.

- (4) $\lambda_{ij} = 0 (1 \leq i < j \leq n)$, $\phi_{234} = 1$, $\phi_{ijk} = 0 (1 \leq i < j < k \leq n, (i, j, k) \neq (2, 3, 4))$. In this case, the brackets in A can be written as

$$[e_1, \dots, e_n] = e_{n+4}, \quad [e_2, \dots, e_{n+1}] = e_{n+5}, \quad [e_1, e_5, \dots, e_{n+3}] = e_{n+6},$$

which we denote it by $A_{n,n+6,4}$.

Case 2. Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,2}$. Then the brackets in A can be written as

$$\left\{ \begin{array}{ll} [e_1, \dots, e_n] = e_{n+4} + \alpha e_{n+6}, & \\ [e_2, \dots, e_{n+1}] = e_{n+5} + \beta e_{n+6}, & \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+3}] = \gamma_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ & (i, j) \neq (1, 2), \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+2}, e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, \widehat{e}_k, \dots, e_n, e_{n+1}, e_{n+2}, e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \leq i < j < k \leq n. \end{array} \right.$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n -Lie algebras and $Z(A) = \langle e_{n+4}, e_{n+5}, e_{n+6} \rangle$, we conclude that one of the coefficients

$$\begin{array}{ll} \gamma_i (1 \leq i \leq n), & \delta_{ij} (1 \leq i < j \leq n), \\ \lambda_{ij} (1 \leq i < j \leq n), & \phi_{ijk} (1 \leq i < j < k \leq n) \end{array}$$

is equal to one, and the others are zero. Similar to case 1, up to isomorphism, we have the following algebras:

$$\begin{array}{l} [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_2, \dots, e_n, e_{n+3}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_1, e_2, e_4, \dots, e_n, e_{n+3}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_2, e_4, \dots, e_{n+1}, e_{n+3}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_1, e_2, e_5, \dots, e_{n+1}, e_{n+3}] = e_{n+6}. \end{array}$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,2}$ and $A_{n,n+6,3}$, respectively. The third and fourth algebras are denoted by $A_{n,n+6,5}$ and $A_{n,n+6,6}$, respectively, that is,

$$\begin{array}{l} A_{n,n+6,5} = \langle e_1, \dots, e_{n+6} : [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_2, e_4, \dots, e_{n+1}, e_{n+3}] = e_{n+6} \rangle, \\ A_{n,n+6,6} = \langle e_1, \dots, e_{n+6} : [e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5}, [e_1, e_2, e_5, \dots, e_{n+1}, e_{n+3}] = e_{n+6} \rangle, \end{array}$$

Case 3. Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,3}$. Then the brackets in A can be written as

$$\left\{ \begin{array}{ll} [e_1, \dots, e_n] = e_{n+4} + \alpha e_{n+6}, & \\ [e_2, \dots, e_{n+1}] = e_{n+5} + \beta e_{n+6}, & \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+3}] = \gamma_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+3}] = \delta_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+2}, e_{n+3}] = \lambda_{ij} e_{n+6}, & 1 \leq i < j \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, \widehat{e}_k, \dots, e_n, e_{n+1}, e_{n+2}, e_{n+3}] = \phi_{ijk} e_{n+6}, & 1 \leq i < j < k \leq n, \\ & (i, j, k) \neq (1, 2, 3). \end{array} \right.$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$.

Since $\dim(A/\langle e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(3)$. According to the structure of n -Lie algebra, we conclude that one of the coefficients

$$\alpha_i, \beta_i, \gamma_i (1 \leq i \leq n), \quad \chi_{ij}, \delta_{ij}, \lambda_{ij} (1 \leq i < j \leq n), \\ \phi_{ijk} (1 \leq i < j < k \leq n, (i, j, k) \neq (1, 2, 3))$$

is equal to one, and the others are zero. Similar to case 1, up to isomorphism, we have the following algebras:

$$\begin{aligned} [e_1, \dots, e_n] &= e_{n+5}, [e_4, \dots, e_{n+3}] = e_{n+4}, [e_2, e_3, \dots, e_{n+1}] = e_{n+6}, \\ [e_1, \dots, e_n] &= e_{n+5}, [e_4, \dots, e_{n+3}] = e_{n+4}, [e_1, e_2, e_3, e_5, \dots, e_{n+1}] = e_{n+6}, \\ [e_1, \dots, e_n] &= e_{n+5}, [e_4, \dots, e_{n+3}] = e_{n+4}, [e_2, e_3, e_5, \dots, e_{n+2}] = e_{n+6}. \end{aligned}$$

One can easily see that these algebras are isomorphic to $A_{n,n+6,3}$, $A_{n,n+6,4}$ and $A_{n,n+6,6}$, respectively. Therefore, there is no new algebra in this case. ■

Theorem 3.7. The only $(n+6)$ -dimensional generalized Heisenberg n -Lie algebras of rank 4 are

$$A_{n,n+6,7}, A_{n,n+6,8}, A_{n,n+6,9}, A_{n,n+6,10}, A_{n,n+6,11}, A_{n,n+6,12} \text{ and } A_{n,n+6,13}.$$

Proof. Suppose that A is an $(n+6)$ -dimensional generalized Heisenberg n -Lie algebra of rank 4 with basis $\{e_1, \dots, e_{n+6}\}$, which $n \geq 4$. Also, suppose that $A^2 = \langle e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6} \rangle$. In this case, $A/\langle e_{n+6} \rangle$ is an $(n+5)$ -dimensional nilpotent n -Lie algebra of class 2 with derived algebra of dimension 3. By Theorem LABEL:?, we have three possibilities for $A/\langle e_{n+6} \rangle$:

Case 4. Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,4}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3} + \alpha e_{n+6}, \\ [e_2, \dots, e_{n+1}] = e_{n+4} + \beta e_{n+6}, \\ [e_1, e_3, \dots, e_{n+1}] = e_{n+5} + \gamma e_{n+6}, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 3 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n -Lie algebras, we conclude that one of the coefficients

$$\alpha_i (3 \leq i \leq n), \quad \beta_i (1 \leq i \leq n), \quad \chi_{ij} (1 \leq i < j \leq n)$$

is equal to one, and the others are zero. According to $Z(A) = \{e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6}\}$, the coefficients $\alpha_i (3 \leq i \leq n)$ cannot be equal to one. We have three possibilities:

$$(1) \quad \beta_1 = 1, \beta_i = 0 (2 \leq i \leq n), \alpha_i = 0 (3 \leq i \leq n), \chi_{ij} = 0 (1 \leq i < j \leq n).$$

In this case, the brackets in A can be written as

$$\begin{aligned} [e_1, \dots, e_n] &= e_{n+3}, \quad [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \dots, e_{n+1}] &= e_{n+5}, \quad [e_2, \dots, e_n, e_{n+2}] = e_{n+6}, \end{aligned}$$

which we denote it by $A_{n,n+6,7}$.

$$(2) \quad \beta_3 = 1, \beta_i = 0 (1 \leq i \leq n, n \neq 3), \alpha_i = 0 (3 \leq i \leq n), \chi_{ij} = 0 (1 \leq i < j \leq n).$$

In this case, the brackets in A can be written as

$$\begin{aligned} [e_1, \dots, e_n] &= e_{n+3}, & [e_2, \dots, e_{n+1}] &= e_{n+4}, \\ [e_1, e_3, \dots, e_{n+1}] &= e_{n+5}, & [e_1, e_2, e_4, \dots, e_n, e_{n+2}] &= e_{n+6}, \end{aligned}$$

which we denote it by $A_{n,n+6,8}$.

- (3) Only one of $\chi_{ij}(1 \leq i < j \leq n)$ is equal to one and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \dots, e_{n+1}] = e_{n+5}, & [e_3, \dots, e_{n+2}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \dots, e_{n+1}] = e_{n+5}, & [e_2, e_4, \dots, e_{n+2}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_1, e_3, \dots, e_{n+1}] = e_{n+5}, & [e_1, e_2, e_5, \dots, e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,7}$ and $A_{n,n+6,8}$, respectively. The third algebra is denoted by $A_{n,n+6,9}$.

Case 5. Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,5}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3} + \alpha e_{n+6}, \\ [e_2, \dots, e_{n+1}] = e_{n+4} + \beta e_{n+6}, \\ [e_2, \dots, e_n, e_{n+2}] = e_{n+5} + \gamma e_{n+6}, \\ [e_1, \dots, \widehat{e_i}, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \dots, \widehat{e_i}, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \dots, \widehat{e_i}, \dots, \widehat{e_j}, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n. \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n -Lie algebras, we conclude that one of the coefficients

$$\alpha_i (3 \leq i \leq n), \quad \beta_i (1 \leq i \leq n), \quad \chi_{ij} (1 \leq i < j \leq n)$$

is equal to one, and the others are zero. We have two possibilities:

- (1) Only one of $\alpha_i (3 \leq i \leq n)$ and $\beta_i (1 \leq i \leq n)$ is equal to one and the others are zero. Without loss of generality, we assume $\alpha_2 = 1$. Thus, the brackets in A can be written as

$$\begin{aligned} [e_1, \dots, e_n] &= e_{n+3}, & [e_2, \dots, e_{n+1}] &= e_{n+4}, \\ [e_2, \dots, e_n, e_{n+2}] &= e_{n+5}, & [e_1, e_3, \dots, e_{n+1}] &= e_{n+6}. \end{aligned}$$

One can easily see that this algebra is isomorphic to $A_{n,n+6,7}$.

- (2) Only one of $\chi_{ij}(1 \leq i < j \leq n)$ is equal to one and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_2, \dots, e_n, e_{n+2}] = e_{n+5}, & [e_3, \dots, e_{n+2}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_2, \dots, e_n, e_{n+2}] = e_{n+5}, & [e_1, e_4, \dots, e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first algebra is isomorphic to $A_{n,n+6,7}$. The second algebra is denoted by $A_{n,n+6,10}$.

Case 6. Let $A/\langle e_{n+6} \rangle \cong A_{n,n+5,6}$. Then the brackets in A can be written as

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3} + \alpha e_{n+6}, \\ [e_2, \dots, e_{n+1}] = e_{n+4} + \beta e_{n+6}, \\ [e_3, \dots, e_{n+2}] = e_{n+5} + \gamma e_{n+6}, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+1}] = \alpha_i e_{n+6}, & 2 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, & 1 \leq i \leq n, \\ [e_1, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \chi_{ij} e_{n+6}, & 1 \leq i < j \leq n, (i, j) \neq (1, 2). \end{cases}$$

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$.

Since $\dim(A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle)^2 = 1$, we have $A/\langle e_{n+3}, e_{n+4}, e_{n+5} \rangle \cong H(n, 1) \oplus F(2)$. According to the structure of n -Lie algebras, we conclude that one of the coefficients

$$\alpha_i \ (2 \leq i \leq n), \quad \beta_i \ (1 \leq i \leq n), \quad \chi_{ij} \ (1 \leq i < j \leq n, (i, j) \neq (1, 2))$$

is equal to one, and the others are zero. Up to isomorphism, we have the following algebras:

$$\begin{cases} [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_3, \dots, e_{n+2}] = e_{n+5}, & [e_1, e_3, \dots, e_{n+1}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_3, \dots, e_{n+2}] = e_{n+5}, & [e_1, e_2, e_4, \dots, e_{n+1}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_3, \dots, e_{n+2}] = e_{n+5}, & [e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_3, \dots, e_{n+2}] = e_{n+5}, & [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+6}, \\ [e_1, \dots, e_n] = e_{n+3}, & [e_2, \dots, e_{n+1}] = e_{n+4}, \\ [e_3, \dots, e_{n+2}] = e_{n+5}, & [e_1, e_2, e_5, \dots, e_{n+2}] = e_{n+6}. \end{cases}$$

One can easily see that the first and second algebras are isomorphic to $A_{n,n+6,7}$ and $A_{n,n+6,8}$, respectively. The third, fourth and fifth algebras are denoted by $A_{n,n+6,11}$, $A_{n,n+6,12}$ and $A_{n,n+6,13}$, respectively. ■

Theorem 3.8. The only $(n+6)$ -dimensional nilpotent n -Lie algebras of class 2 where $n \geq 4$ are

$$\begin{cases} H(n, 1) \oplus F(5) \ (n \geq 4), \ H(4, 2) \oplus F(1), \ H(5, 2), \\ A_{n,n+5,1} \oplus F(1), \ A_{n,n+4,1} \oplus F(2), \ A_{n,n+3,1} \oplus F(3), \ A_{n,n+5,4} \oplus F(1) \\ A_{n,n+5,5} \oplus F(1), \ A_{n,n+4,3} \oplus F(2), \ A_{n,n+5,6} \oplus F(1), \text{ and } A_{n,n+6,i} \ (1 \leq i \leq 14). \end{cases}$$

Proof. Assume that A is an $(n+6)$ -dimensional nilpotent n -Lie algebra of class 2, where $n \geq 4$ and $A = \langle e_1, \dots, e_{n+6} \rangle$. If $\dim A^2 = 1$, then by Theorem 2.2, A is isomorphic to one of the following algebras:

$$H(n, 1) \oplus F(5) \ (n \geq 4), \quad H(4, 2) \oplus F(1), \quad H(5, 2).$$

Now, assume that $\dim A^2 \geq 2$ and that $\langle e_{n+5}, e_{n+6} \rangle \subset A^2$. Therefore, $A/\langle e_{n+6} \rangle$ is an $(n+5)$ -dimensional nilpotent n -Lie algebra of class 2. It follows from Theorem 2.5 that $A/\langle e_{n+6} \rangle$ is one of the following forms:

$$H(n, 1) \oplus F(4), \quad H(4, 2), \quad A_{n,n+5,i} \ (1 \leq i \leq 7).$$

If $A/\langle e_{n+6} \rangle$ is isomorphic to $H(n, 1) \oplus F(4)$ or $H(4, 2)$, then $\dim A^2 = 2$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n -Lie algebra of

rank 2 and F is abelian. The center of A has a dimension at most 5; thus the possible cases of A are H_0 , $H_1 \oplus F(1)$, $H_2 \oplus F(2)$, $H_3 \oplus F(3)$, where H_0 , H_1 , H_2 , H_3 are generalized Heisenberg n -Lie algebras of rank 2 with dimensions $n+6$, $n+5$, $n+4$, $n+3$, respectively. These algebras read as follows:

$$A_{n,n+6,1}, \quad A_{n,n+5,1} \oplus F(1), \quad A_{n,n+4,1} \oplus F(2), \quad A_{n,n+3,1} \oplus F(3).$$

If $A/\langle e_{n+6} \rangle$ is isomorphic to $A_{n,n+5,1}$, $A_{n,n+5,2}$ or $A_{n,n+5,3}$, then $\dim A^2 = 3$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n -Lie algebra of rank 3 and F is abelian. According to [?], these algebras read as follows:

$$A_{n,n+6,2}, \quad A_{n,n+6,3}, \quad A_{n,n+6,4}, \quad A_{n,n+6,5}, \quad A_{n,n+6,6}, \\ A_{n,n+5,4} \oplus F(1), \quad A_{n,n+5,5} \oplus F(1), \quad A_{n,n+4,3} \oplus F(2).$$

Also, If $A/\langle e_{n+6} \rangle$ is isomorphic to $A_{n,n+5,4}$, $A_{n,n+5,5}$ or $A_{n,n+5,6}$, then $\dim A^2 = 4$. According to Lemma 2.7, we can write $A = H \oplus F$, where H is a generalized Heisenberg n -Lie algebra of rank 4 and F is abelian. According to [?], these algebras read as follows:

$$A_{n,n+6,7}, \quad A_{n,n+6,8}, \quad A_{n,n+6,9}, \quad A_{n,n+6,10}, \\ A_{n,n+6,11}, \quad A_{n,n+6,12}, \quad A_{n,n+6,13}, \quad A_{n,n+5,6} \oplus F(1).$$

Finally, If $A/\langle e_{n+6} \rangle \cong A_{n,n+5,7}$, then $A^2 = Z(A) = \langle e_{n+1}, e_{n+3}, e_{n+4}, e_{n+5}, e_{n+6} \rangle$. The brackets in A can be written as

$$\begin{cases} [e_1, \dots, e_n] = e_{n+1} + \alpha e_{n+6}, \\ [e_2, \dots, e_n, e_{n+2}] = e_{n+3} + \beta e_{n+6}, \\ [e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+4} + \gamma e_{n+6}, \\ [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+5} + \phi e_{n+6}, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, \end{cases} \quad 4 \leq i \leq n.$$

With a suitable change of basis, one can assume that $\alpha = \beta = \gamma = \phi = 0$. Thus, the brackets in A are

$$\begin{cases} [e_1, \dots, e_n] = e_{n+1}, \\ [e_2, \dots, e_n, e_{n+2}] = e_{n+3}, \\ [e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+4}, \\ [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+5}, \\ [e_1, \dots, \widehat{e}_i, \dots, e_n, e_{n+2}] = \beta_i e_{n+6}, \end{cases} \quad 4 \leq i \leq n.$$

By $\dim Z(A)$, we must have $\beta_i \neq 0$ for some $4 \leq i \leq n$. Without loss of generality, assume that $\beta_4 \neq 0$. By applying the transformations

$$e'_4 = e_4 + \sum_{j=5}^n (-1)^j \frac{\beta_j}{\beta_4} e_j, \quad e'_i = e_i \quad (1 \leq i \leq n+5, i \neq 4), \quad e'_{n+6} = \beta_4 e_{n+6},$$

we conclude that

$$A = \langle e_1, \dots, e_{n+6} : [e_1, \dots, e_n] = e_{n+1}, [e_2, \dots, e_n, e_{n+2}] = e_{n+3}, \\ = e_{n+4}, [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+5}, e_{n+6} \rangle,$$

which we denote it by $A_{n,n+6,14}$. ■

In Table 1, we show all $(n+4)$ -dimensional and $(n+5)$ -dimensional nilpotent n -Lie algebras of class 2.

In Table 2, we show all n -Lie algebras obtained in this paper.

Table 2.

Nilpotent n -Lie algebras of class 2	Nonzero multiplications
$A_{n,n+6,1}$	$[e_1, \dots, e_n] = e_{n+5}, [e_5, \dots, e_{n+4}] = e_{n+6}$
$A_{n,n+6,2}$	$[e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5},$ $[e_3, \dots, e_n, e_{n+2}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,3}$	$[e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5},$ $[e_1, e_4, \dots, e_n, e_{n+2}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,4}$	$[e_1, \dots, e_n] = e_{n+4}, [e_2, \dots, e_{n+1}] = e_{n+5},$ $[e_1, e_5, \dots, e_{n+3}] = e_{n+6}$
$A_{n,n+6,5}$	$[e_1, \dots, e_n] = e_{n+4}, [e_3, \dots, e_{n+2}] = e_{n+5},$ $[e_2, e_4, \dots, e_{n+1}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,6}$	$[e_1, \dots, e_n] = e_{n+4}, [e_3, \dots, e_{n+2}] = e_{n+5},$ $[e_1, e_2, e_5, \dots, e_{n+1}, e_{n+3}] = e_{n+6}$
$A_{n,n+6,7}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_1, e_3, \dots, e_{n+1}] = e_{n+5}, [e_3, \dots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,8}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_1, e_3, \dots, e_{n+1}] = e_{n+5}, [e_2, e_4, \dots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,9}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_1, e_3, \dots, e_{n+1}] = e_{n+5}, [e_1, e_2, e_5, \dots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,10}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_2, \dots, e_n, e_{n+2}] = e_{n+5}, [e_1, e_4, \dots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,11}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_3, \dots, e_{n+2}] = e_{n+5}, [e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+6}$
$A_{n,n+6,12}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_3, \dots, e_{n+2}] = e_{n+5}, [e_1, e_2, e_4, \dots, e_n, e_{n+2}] = e_{n+6}$
$A_{n,n+6,13}$	$[e_1, \dots, e_n] = e_{n+3}, [e_2, \dots, e_{n+1}] = e_{n+4},$ $[e_3, \dots, e_{n+2}] = e_{n+5}, [e_1, e_2, e_5, \dots, e_{n+2}] = e_{n+6}$
$A_{n,n+6,14}$	$[e_1, \dots, e_n] = e_{n+1}, [e_2, \dots, e_n, e_{n+2}] = e_{n+3},$ $[e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+4}, [e_3, \dots, e_{n+2}] = e_{n+5},$ $[e_1, e_3, \dots, e_n, e_{n+2}] = e_{n+6}$

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