

# Nonessential sum graph of an Artinian ring

Graph of an Artinian ring

Bikash Barman and Kukil Kalpa Rajkhowa  
*Cotton University, Guwahati, India*

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## Abstract

**Purpose** – The authors study the interdisciplinary relation between graph and algebraic structure ring defining a new graph, namely “non-essential sum graph”. The nonessential sum graph, denoted by  $NES(R)$ , of a commutative ring  $R$  with unity is an undirected graph whose vertex set is the collection of all nonessential ideals of  $R$  and any two vertices are adjacent if and only if their sum is also a nonessential ideal of  $R$ .

**Design/methodology/approach** – The method is theoretical.

**Findings** – The authors obtain some properties of  $NES(R)$  related with connectedness, diameter, girth, completeness, cut vertex,  $r$ -partition and regular character. The clique number, independence number and domination number of  $NES(R)$  are also found.

**Originality/value** – The paper is original.

**Keywords** Nonessential ideal, Nonessential sum graph, Minimal ideal

**Paper type** Research paper

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## 1. Introduction

The growth of interdisciplinary study of graph and algebra took place after the introduction of zero-divisor graph by Istvan Back [1]. Some of the interesting graphs are comaximal graph of commutative ring [2], intersection graph of ideals of rings [3], total graph of commutative ring [4], etc. In [5], Atani *et al.* introduced a graph associated to proper nonsmall ideals of a commutative ring, namely, small intersection graph. The small intersection graph of a ring  $R$ , denoted by  $G(R)$ , is an undirected graph with vertex set is the collection of all nonsmall proper ideals of  $R$  and any two distinct vertices are adjacent if and only if their intersection is not small in  $R$ . Taking this insight of small intersection graph of a ring, we, in this paper, define nonessential sum graph of an Artinian ring.

To continue this sequel, we are going to remember some definitions and notations from ring and graph. Let  $R$  be a commutative ring with unity. An ideal  $I$  of  $R$  is said to be essential in  $R$  if  $I \cap J \neq 0$ , whenever  $J$  is a nonzero ideal of  $R$ . The sum of all minimal ideals of  $R$  is known as socle of  $R$ , denoted by  $\text{soc}(R)$ . We use  $\min(R)$  to denote the collection of all minimal ideals of  $R$ . The ring  $R$  is said to be an Artinian ring if every descending chain of  $R$  terminates. In an Artinian ring, every ideal contains a minimal ideal.

Let  $G$  be an undirected simple graph with vertex set  $V(G)$  and edge set  $E(G)$ .  $G$  is said to be a null graph if  $V(G) = \emptyset$  and that  $G$  is said to be empty if  $E(G) = \emptyset$ . We denote degree of  $v \in V(G)$  by  $\deg(v)$ . If  $\deg(v) = 1$ , then  $v$  is called an end vertex.  $G$  is complete if any two vertices are adjacent.  $G$  is said to be  $r$ -regular if degree of each vertex of  $G$  is  $r$ . A walk in  $G$  is an alternating sequence of vertices and edges,  $v_0x_1v_1 \dots x_nv_n$  in which each edge  $x_i$  is  $v_{i-1}v_i$ . A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a circuit is a closed walk which all its vertices are distinct (except the first and last).

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The length of a circuit is the number of edges in the circuit. The length of the smallest circuit of  $G$  is called the girth of  $G$ , denoted by  $\text{girth}(G)$ .  $G$  is connected if there is a path between every two distinct vertices.  $G$  is disconnected if it is not connected. A vertex of the connected graph  $G$  is said to be a cut vertex if removal of it makes  $G$  disconnected. If  $x$  and  $y$  are two distinct vertices of  $G$ , then  $d(x, y)$  is the length of the shortest path from  $x$  to  $y$  and if there is no such path then  $d(x, y) = \infty$ . The diameter of  $G$  is the maximum distance among distances between all pair of vertices of  $G$ , denoted by  $\text{diam}(G)$ .  $G$  is said to be a bipartite graph if the vertex set of  $G$  can be partitioned into two disjoint subsets  $V_1$  and  $V_2$  such that every edge of  $G$  joins  $V_1$  and  $V_2$ . If  $|V_1| = m$ ,  $|V_2| = n$  and if every vertex of  $V_1$  (or  $V_2$ ) is adjacent to all vertices of  $V_2$ , then the bipartite graph is said to be complete and is denoted by  $K_{m,n}$ . If either  $m$  or  $n$  is equal to 1, then  $K_{m,n}$  is said to be a star. An  $r$ -partite graph is a graph whose vertex set is partitioned into  $r$  subsets with no edge has both ends in any one subset. If each vertex of a partite subset is joined to every vertex that is not in that partite subset, then the  $r$ -partite graph is said to be complete. A complete subgraph of  $G$  is called a clique. The number of vertices in the largest clique of  $G$  is called the clique number of  $G$ , denoted by  $\omega(G)$ . The neighborhood  $N(v)$  of a vertex  $v$  in  $G$  is the set of vertices which are adjacent to  $v$ . For each  $S \subseteq V(G)$ ,  $N(S) = \cup_{v \in S} N(v)$  and  $N[S] = N(S) \cup S$ . A set of vertices  $S$  in  $G$  is a dominating set, if  $N[S] = V$ . The domination number,  $\gamma(G)$ , of  $G$  is the minimum cardinality of a dominating set of  $G$ . An independent set of  $G$  is a set of vertices of  $G$  such that no two vertices are adjacent in that vertex set. The independence number of  $G$  is the number of vertices in the largest independence set in  $G$ , denoted by  $\alpha(G)$ .

In this paper, we introduce nonessential sum graph of commutative ring with unity. Let  $R$  be a commutative ring with unity. The nonessential sum graph of  $R$ , denoted by  $\text{NES}(R)$ , is an undirected graph with vertex set as the collection of all nonessential ideals of  $R$  and any two vertices  $A$  and  $B$  are adjacent if and only if  $A \cap B$  is also a nonessential ideal of  $R$ . In this article, we are mainly interested in nonessential sum graph of Artinian ring.

Any undefined terminology can be obtained in [7–8, 15–20].

## 2. Connectedness of nonessential sum graph

In this section, we obtain some results related to connectedness, diameter, girth, completeness, cut vertex, partiteness and regular character. We start with a remark.

**Remark 2.1.** An ideal  $A$  is nonessential ideal in  $R$  if and only if  $A \subseteq \text{soc}(R)$ . If  $B$  is a nonessential ideal of  $R$  then every ideal which is contained in  $B$  is also a nonessential ideal of  $R$ . If  $m$  is a minimal ideal of  $R$  and if  $A$  and  $B$  are two ideals such that  $m \subseteq A + B$ , then  $m \subseteq A$  or  $m \subseteq B$ .

**Lemma 2.2.** If  $\text{min}(R) = \{m_i\}_{i \in \lambda}$  where  $\lambda$  is an index set and  $\mu$  is a finite subset of  $\lambda$ , then  $\sum_{\mu} m_i$  is a nonessential ideal of  $R$ .

*Proof.* If possible suppose  $K = \sum_{\mu} m_i$  is an essential ideal of  $R$ . Since each  $m_j \neq (0)$ , so  $K \cap m_j \neq (0)$  for  $j \notin \mu$ , which implies that  $m_j \subseteq K$ . But it is a contradiction by Remark 2.1. Hence the lemma.  $\square$

From this onwards,  $R$  is an Artinian ring.

**Theorem 2.3.**  $\text{NES}(R)$  is a null graph if and only if  $R$  contains exactly one minimal ideal.

*Proof.* First consider that  $\text{NES}(R)$  is a null graph. On the contrary, assume that  $m_1$  and  $m_2$  are two distinct minimal ideals of  $R$ . So  $m_1 \cap m_2 = 0$  and this provides that both  $m_1$  and  $m_2$  are nonessential ideals of  $R$ , a contradiction. Conversely, suppose that  $R$  has exactly one minimal ideal  $m$ , say. If  $m$  is the only nontrivial proper ideal of  $R$ , then obviously  $\text{NES}(R)$  is a null graph. If  $A$  is a nontrivial proper ideal of  $R$  with  $A \neq m$ , then it is easy to observe that  $A$  is essential in  $R$ . The proof is complete.  $\square$

**Theorem 2.4.**  $\text{NES}(R)$  is an empty graph if and only if  $R$  has exactly two minimal ideals, which are the only nonessential ideals of  $R$ .

*Proof.* Let  $\text{NES}(R)$  be an empty graph. Then by [Theorem 2.3](#)  $|\min(R)| \neq 1$ . If  $|\min(R)| \geq 3$  and  $m_1, m_2, m_3 \in \min(R)$ , then  $m_1$  and  $m_2$  are adjacent by [Lemma 2.2](#). Therefore,  $|\min(R)| = 2$  and so we take  $|\min(R)| = \{m_1, m_2\}$  with  $m_1 \neq m_2$ . Clearly  $m_1$  and  $m_2$  are nonessential. If  $I$  is any other nonessential ideal which is different from  $m_1$  and  $m_2$ , then  $m_i \subset I$  for  $i = 1, 2$ . This gives that  $I$  and  $m_i$  are adjacent, a contradiction. Thus  $m_1$  and  $m_2$  are the only nonessential ideals of  $R$ . For the other direction, we consider  $R$  has exactly two minimal ideals, which are the only nonessential ideals of  $R$ . Then  $m_1 + m_2 = \text{soc}(R)$  is essential. So,  $\text{NES}(R)$  is an empty graph. This completes the proof.  $\square$

**Theorem 2.5.** The following statements are equivalent:

- (1)  $\text{NES}(R)$  is disconnected.
- (2)  $|\min(R)| = 2$ .
- (3)  $\text{NES}(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $\text{NES}(R)$ .

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $\text{NES}(R)$  is disconnected. We consider  $G_1$  and  $G_2$  are two components of  $\text{NES}(R)$  and  $I, J$  be two ideals such that  $I \in V(G_1)$  and  $J \in V(G_2)$ . Take the minimal ideals  $m_1$  and  $m_2$  with  $m_1 \subseteq I$  and  $m_2 \subseteq J$ . If  $m_1 = m_2$ , then  $I - m_1 - J$  is a path, a contradiction. This asserts that  $m_1 \neq m_2$ . Again, if  $|\min(R)| \geq 3$ , then  $m_1 + m_2$  is nonessential in  $R$ . From this we get  $I - m_1 - m_2 - J$  is a path, a contradiction. Therefore  $|\min(R)| = 2$ .

(ii)  $\Rightarrow$  (iii) Assume that  $|\min(R)| = 2$ . Then we obtain  $\text{soc}(R) = m_1 + m_2$ , where  $m_1$  and  $m_2$  are the minimal ideals of  $R$ . Let  $G_i = \{I \subseteq R : m_i \subseteq I \text{ and } I \text{ is nonessential in } R\}$ . Let  $I$  and  $J$  be two nonadjacent vertices in  $G_1$ , then  $I + J$  is essential in  $R$ , which implies  $\text{soc}(R) \subseteq I + J$ . Hence  $m_2 \subseteq I$  or  $m_2 \subseteq J$ , a contradiction because in that case either  $I$  is essential or  $J$  is essential. So,  $G_1$  is complete subgraph of  $\text{NES}(R)$ . In the same way,  $G_2$  is also a complete subgraph of  $\text{NES}(R)$ . Suppose  $K$  and  $L$  are two adjacent vertices where  $K \in V(G_1)$  and  $L \in V(G_2)$ . Since  $\text{soc}(R) = m_1 + m_2 \subseteq K + L$ , so  $K + L$  is essential, a contradiction. Thus  $\text{NES}(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $\text{NES}(R)$ .

(iii)  $\Rightarrow$  (i) The proof is obvious.

**Theorem 2.6.** The diameter of  $\text{NES}(R)$  is 1, 2 or  $\infty$ .

*Proof.* If  $\text{NES}(R)$  is disconnected then  $\text{diam}(\text{NES}(R)) = \infty$ . Suppose that  $\text{NES}(R)$  is connected. If  $I$  and  $J$  are two nonadjacent vertices of  $\text{NES}(R)$  then  $I + J$  is essential in  $R$ . Consider the minimal ideals  $m_1$  and  $m_2$  with  $m_1 \subseteq I$  and  $m_2 \subseteq J$ . If  $m_1 + J$  is nonessential, then  $I - m_1 - J$  is a path, which gives  $d(I, J) = 2$ . Similarly, if  $m_2 + I$  is nonessential in  $R$ , then  $d(I, J) = 2$ . Suppose that  $m_1 + J$  and  $m_2 + I$  are both essential in  $R$ . Since  $\text{NES}(R)$  is connected, so  $|\min(R)| \geq 3$ . Let  $m_3 \in \min(R)$ . Since  $I + J$  is essential in  $R$ , therefore  $m_3 \subseteq I + J$ . This implies  $m_3 \subseteq I$  or  $m_3 \subseteq J$ . If we take  $m_3 \subseteq I$  then obviously  $m_3 + I$  is nonessential in  $R$ . We assert that  $m_3 + J$  is nonessential. If possible,  $m_3 + J$  is essential in  $R$ , then  $m_1 \subseteq \text{soc}(R) \subseteq m_3 + J$ , which gives  $m_1 \subseteq J$ . Hence  $m_1 + J = J$  is nonessential, a contradiction. Therefore  $I - m_3 - J$  is a path. Thus  $\text{diam}(I, J) = 2$ .  $\square$

**Theorem 2.7.** If  $\text{NES}(R)$  contains a cycle, then  $\text{girth}(\text{NES}(R)) = 3$

*Proof.* First if we consider  $|\min(R)| = 2$ , then by [Theorem 2.5](#)  $\text{NES}(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $\text{NES}(R)$ . Therefore in this case,  $\text{girth}(\text{NES}(R)) = 3$ , whenever  $\text{NES}(R)$  contains a cycle. Next, when  $|\min(R)| \geq 3$ ,  $m_1 + m_2, m_2 + m_3, m_3 + m_1$  are nonessential in  $R$  where  $m_i \in \min(R)$ ,  $i = 1, 2, 3$ . Thus  $m_1 - m_2 - m_3 - m_1$  is a cycle. Hence  $\text{girth}(\text{NES}(R)) = 3$ .  $\square$

**Theorem 2.8.** Let  $R$  contain finitely many minimal ideals, then the following holds:

- (1) There exists no vertex in  $NES(R)$  which is adjacent to every other vertex.
- (2)  $NES(R)$  is not a complete graph.

*Proof.* To prove (i), let  $\min(R) = \{m_1, m_2, \dots, m_t\}$ . Assume that there exists a vertex  $I$  in  $NES(R)$  such that  $I$  is adjacent to every other vertex. Let  $m_i \subseteq I$  for some  $i$ . Let  $K = \sum_{j \neq i} m_j$ , which is nonessential in  $R$ . Thus  $K$  is a vertex in  $NES(R)$ . Now,  $K + I \supseteq \sum_{j \neq i} m_j + m_i = \text{soc}(R)$ . Hence  $K + I$  is essential, a contradiction to the fact that  $I$  is adjacent to every other vertex. Hence the result.

- (ii) Clearly  $NES(R)$  is not complete by (i).  $\square$

**Theorem 2.9.** If  $NES(R)$  is connected, then  $NES(R)$  has no cut vertex.

*Proof.* On the contrary assume that  $I$  is a cut vertex of  $NES(R)$ . Then  $NES(R) \setminus \{I\}$  is disconnected. Thus, there are vertices  $J$  and  $K$  with  $I$  lies in every path joining  $K$  to  $J$ . By [Theorem 2.6](#),  $d(K, J) = 2$  and therefore  $J - I - K$  is a path. We claim that  $I$  is a minimal ideal of  $R$ . If not, there exists an ideal  $L$  of  $R$  such that  $L \subseteq I$ . As  $I$  is nonessential in  $R$ , therefore  $L$  is also nonessential in  $R$ . Since  $J + L \subseteq J + I$  and  $J + I$  is nonessential in  $R$ , so  $J + L$  is nonessential in  $R$ . In the same direction,  $K + L$  is also nonessential in  $R$ . So,  $J - L - K$  is a path in  $NES(R) \setminus \{I\}$ , which is a contradiction. Thus,  $I$  is a minimal ideal of  $R$ . Now, we assert that there exist a minimal ideal  $m_i \neq I$  of  $R$  such that  $m_i \not\subseteq J$ . If not then  $m_i \subseteq J$  for each  $I(\neq m_i) \in \min(R)$  and so  $\sum_{m_i \neq I} m_i \subseteq J$ . This gives that  $\text{soc}(R) = I + \sum_{m_i \neq I} m_i \subseteq I + J$ , a contradiction to the fact that  $I + J$  is nonessential. Similarly, there exists  $m_j (\neq I)$  such that  $m_j \not\subseteq K$ . Now we see that for each  $m_t \in \min(R)$  either  $m_t \subseteq J$  or  $m_t \subseteq K$ . Since  $J + K$  is essential,  $m_t \subseteq \text{soc}(R) \subseteq J + K$ , which implies  $m_t \subseteq J$  or  $m_t \subseteq K$ . Let  $I \neq m_i, m_j \in \min(R)$  such that  $m_i \not\subseteq J$  and  $m_j \not\subseteq K$ . Therefore,  $m_i \subseteq K$  and  $m_j \subseteq J$ . So,  $K - m_i - m_j - J$  is a path in  $NES(R) \setminus \{I\}$ , a contradiction. Therefore,  $NES(R)$  has no cut vertex.  $\square$

**Theorem 2.10.**  $NES(R)$  is not a complete  $r$ -partite graph.

*Proof.* If possible assume that  $NES(R)$  is a complete  $r$ -partite graph with  $r$  parts  $V_1, V_2, \dots, V_r$ . Since two minimal ideals are always adjacent, by [Remark 2.1](#), so each  $V_i$  contains at most one minimal ideal. Thus we get  $|\min(R)| \leq r$ . Our claim is  $|\min(R)| = r$ . Suppose  $\min(R) = \{m_1, m_2, \dots, m_t\}$  and  $t < r$ . Without loss of generality we can take  $m_i \in V_i$  for  $1 \leq i \leq t$ . So,  $V_{t+1}$  contains no minimal ideal. Since  $\min(R)$  is finite, so  $\sum_{j \neq i} m_j$  is nonessential in  $R$ . Now,  $\sum_{j \neq i} m_j + m_i = \text{soc}(R)$ , so  $\sum_{j \neq i} m_j$  and  $m_i$  are not adjacent. Thus  $\sum_{j \neq i} m_j \in V_i$  as  $m_i \in V_i$ . Let  $I \in V_{t+1}$  and  $m_k \subseteq I$  for some  $m_k \in \min(R)$ . So,  $I$  is adjacent to  $m_k$ . Since  $NES(R)$  is assumed to be complete  $r$ -partite and  $m_k \in V_k$ , so  $I$  is adjacent to every element of  $V_k$ , which implies  $I$  is adjacent to  $\sum_{i \neq k} m_i$ , a contradiction. Therefore,  $|\min(R)| = r$ . Now, consider  $J = \sum_{i=3}^r m_i$ . Clearly  $J$  is nonessential in  $R$  by [Remark 2.1](#). As  $J$  is adjacent to  $m_1$  and  $m_2$ , so  $J \notin V_1, V_2$ . Moreover,  $J + m_i = J$  for  $3 \leq i \leq r$ . So,  $J$  is adjacent to all minimal ideals of  $R$ . We get that  $J \notin V_i$  for each  $i$ , a contradiction. Hence the theorem.  $\square$

**Theorem 2.11.** The following statements holds:

- (1)  $NES(R)$  contains an end vertex if and only if  $NES(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $NES(R)$  and  $|V(G_i)| = 2$  for some  $i = 1, 2$ .
- (2)  $NES(R)$  is not a star graph.

*Proof.* (i) Let  $I$  be an end vertex of  $NES(R)$ . So,  $\deg(I) = 1$ . Suppose  $|\min(R)| \geq 3$ . For each  $m_i \in \min(R)$ ,  $m_i$  is adjacent to every other minimal ideal of  $R$ , so  $\deg(m_i) \geq 2$ . Hence  $I$  is not a

minimal ideal. We can assume  $m_1 \subseteq I$ . Hence  $I$  and  $m_1$  are adjacent. Since  $\deg(I) = 1$ , so the only vertex adjacent to  $I$  is  $m_1$  and  $m_j \notin I, j \neq 1$ . Again  $I$  and  $m_2$  are not adjacent, so  $I + m_2$  is essential. So we get,  $m_j \subseteq \text{soc}(R) \subseteq I + m_2$  for  $j \neq 1, 2$ , which implies  $m_j \subseteq I$  for  $j \neq 1$ , a contradiction. So,  $|\min(R)| = 2$ . By [Theorem 2.5](#),  $\text{NES}(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $\text{NES}(R)$ . Let  $I \in V(G_i)$ . Since  $G_i$  is a complete subgraph and  $\deg(I) = 1$ , so  $|V(G_i)| = 2$ . The converse part is clear.

(ii) Suppose that  $\text{NES}(R)$  is a star graph. So,  $\text{NES}(R)$  contains an end vertex. By the previous part  $|\min(R)| = 2$  and then by [Theorem 2.5](#), the graph is disconnected. Hence,  $\text{NES}(R)$  is not a star graph.  $\square$

**Theorem 2.12.** The following statements holds:

- (1) If  $I$  and  $J$  are two vertices of  $\text{NES}(R)$  such that  $I \subseteq J$ , then  $\deg(I) \geq \deg(J)$ .
- (2) If  $\text{NES}(R)$  is an  $r$ -regular graph then  $|V(\text{NES}(R))| = 2r + 2$ .

*Proof.* (i) Suppose  $I$  and  $J$  are two vertices of  $\text{NES}(R)$  such that  $I \subseteq J$ . Let  $K$  be a vertex adjacent to  $J$ . So,  $J + K$  is nonessential in  $R$ . As  $I + K \subseteq J + K$ , so  $I + K$  is nonessential in  $R$ . Thus, each vertex adjacent to  $J$  is also adjacent to  $I$ . Hence  $\deg(I) \geq \deg(J)$ .

(ii) Let  $\text{NES}(R)$  be an  $r$ -regular graph. So, for each  $m_i \in \min(R)$ ,  $\deg(m_i) = r$ . Since  $m_i$  is adjacent to each minimal ideal, by [Remark 2.1](#), so  $\min(R)$  is finite. Suppose,  $|\min(R)| \geq 3$ , so  $\deg(m_1 + m_2) \leq \deg(m_1)$  by (i). Also,  $\deg(m_1 + m_2) \neq \deg(m_1)$ , since  $\sum_{j \neq 2} m_j$  is adjacent to  $m_1$  but not to  $m_1 + m_2$ . Thus,  $\deg(m_1 + m_2) < \deg(m_1)$ , a contradiction. So,  $|\min(R)| \leq 2$ . If  $|\min(R)| = 1$  then  $\text{NES}(R)$  is null. Therefore,  $|\min(R)| = 2$ . By [Theorem 2.5](#),  $\text{NES}(R) = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are two disjoint complete subgraphs of  $\text{NES}(R)$ . Let  $\min(R) = \{m_1, m_2\}$  and  $m_1 \in G_1$ . Since  $\deg(m_1) = r$ , so  $|G_1| = r + 1$ . In the same direction,  $|G_2| = r + 1$ . Hence,  $|V(\text{NES}(R))| = 2r + 2$ .  $\square$

### 3. Clique number, independence number, domination number of nonessential sum graph

In this section, we will find clique number, independence number, domination number of  $\text{NES}(R)$ .

**Theorem 3.1.** The following holds:

- (1)  $\omega(\text{NES}(R)) \geq |\min(R)|$ .
- (2) If  $\omega(\text{NES}(R)) < \infty$ , then number of minimal ideals of  $R$  is finite.
- (3)  $\omega(\text{NES}(R)) = 1$  if and only if  $\min(R) = \{m_1, m_2\}$  and these two are the only nonessential ideals in  $R$ .
- (4) If the number of minimal ideals of  $R$  is finite, then  $\omega(\text{NES}(R)) \geq 2^{|\min(R)|-1} - 1$ .

*Proof.* (i) Since any two minimal ideals of  $R$  are adjacent, by [Lemma 2.2](#), the subgraph with vertex set  $\{m_i\}_{m_i \in \min(R)}$  of  $\text{NES}(R)$  is complete. So,  $\omega(\text{NES}(R)) \geq |\min(R)|$ .

(ii) If  $\omega(\text{NES}(R)) < \infty$ , then by (i) the number of minimal ideals of  $R$  is finite.

(iii) It is clear from [Theorem 2.4](#).

(iv) Let  $\min(R) = \{m_1, m_2, \dots, m_t\}$  and for each  $1 \leq i \leq t$ , take  $A_i = \{m_1, m_2, \dots, m_{i-1}, m_{i+1}, \dots, m_t\}$ . Let  $P(A_i)$  be the power set of  $A_i$ . For each  $X (\neq \{\}) \in P(A_i)$ , consider  $R_X = \sum_{T \in X} T$ . Clearly  $T_X$  is nonessential. Also, subgraph with vertex set  $\{R_X\}_{X \in P(A_i)}$  is a complete subgraph which is clear by [Lemma 2.2](#). Now,  $|P(A_i) \setminus \{\}| = 2^{|\min(R)|-1} - 1$ . Therefore  $|\{R_X\}_{X \in P(A_i)}| = 2^{|\min(R)|-1} - 1$ . Hence,  $\omega(\text{NES}(R)) \geq 2^{|\min(R)|-1} - 1$ .  $\square$

**Theorem 3.2.** The following holds:

- (1)  $\gamma(\text{NES}(R)) \leq 2$ .
- (2)  $\min(R)$  is finite if and only if  $\gamma(\text{NES}(R)) = 2$  and  $\min(R)$  is infinite if and only if  $\gamma(\text{NES}(R)) = 1$ .

*Proof.* (i) Since  $\text{NES}(R)$  is not a null graph,  $|\min(R)| \geq 2$ . Consider  $T = \{m_1, m_2\}$ , where  $m_1, m_2 \in \min(R)$ . Take a vertex  $I$  in  $\text{NES}(R)$ . If  $m_1 \subseteq I$  or  $m_2 \subseteq I$ , then  $m_1 + I$  or  $m_2 + I$  is non-essential in  $R$ . Then  $I$  is adjacent to  $m_1$  or  $m_2$ . Suppose that  $m \not\subseteq I$  and  $m \not\subseteq J$ . If  $I$  is not adjacent to  $m_1$ , then  $m_1 + I$  is essential in  $R$ . So,  $m_2 \subseteq \text{soc}(R) \subseteq m_1 + I$ , which implies  $m_2 \subseteq I$ , a contradiction. Therefore  $I$  is adjacent to  $m_1$ . In the same way,  $I$  is adjacent to  $m_2$ . Thus  $\gamma(\text{NES}(R)) \leq 2$ .

(ii) If  $\min(R)$  is finite, then by [Theorem 2.8](#), there exists no vertex which is adjacent to every other vertex. So,  $\gamma(\text{NES}(R)) \neq 1$ . Therefore,  $\gamma(\text{NES}(R)) = 2$  by part (i). In the opposite direction, let  $\gamma(\text{NES}(R)) = 1$ . So, the graph has a vertex which is adjacent to every other vertex. So the graph does not contain finite minimal ideals. Hence the result.  $\square$

**Theorem 3.3.** Let  $R$  contain finite number of minimal ideals. Then  $\alpha(\text{NES}(R)) = |\min(R)|$ .

*Proof.* Let  $\min(R)$  be finite and  $\min(R) = \{m_1, m_2, \dots, m_t\}$ . Since  $\{\sum_{j=1, j \neq i}^t m_j\}_{i=1}^t$  is an independent set in  $\text{NES}(R)$ , therefore  $t \leq \alpha(\text{NES}(R))$ . Assume that  $\alpha(\text{NES}(R))$  is equal to  $p$  and  $S = \{I_1, I_2, \dots, I_p\}$  is the maximal independent set. For each  $I \in S$ ,  $I$  is nonessential in  $R$ . So, there exists a minimal ideal  $m$  such that  $m \not\subseteq I$ . If  $p > t$ , then there exists  $1 \leq i, j \leq p$  and  $m \in \min(R)$  such that  $m \not\subseteq I_i$  and  $m \not\subseteq I_j$ . Thus  $m \not\subseteq I_i + I_j$ . Otherwise,  $m \subseteq I_i + I_j$  which leads to a contradiction. As  $S$  is independent, so  $I_i + I_j$  is essential, which implies  $m \subseteq \text{soc}(R) \subseteq I_i + I_j$ , a contradiction. Therefore,  $\alpha(\text{NES}(R)) = |\min(R)|$ . If we take  $\alpha(\text{NES}(R)) = \infty$ , then by similar argument we get a contradiction. Hence,  $\alpha(\text{NES}(R)) = |\min(R)|$ .  $\square$

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**Corresponding author**

Kukil Kalpa Rajkhowa can be contacted at: [kukilrajkhowa@yahoo.com](mailto:kukilrajkhowa@yahoo.com)

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