

Robust numerical method for singularly perturbed differential equations having both large and small delay

Robust
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Abstract

Purpose – The purpose of this study is to develop stable, convergent and accurate numerical method for solving singularly perturbed differential equations having both small and large delay.

Design/methodology/approach – This study introduces a fitted nonpolynomial spline method for singularly perturbed differential equations having both small and large delay. The numerical scheme is developed on uniform mesh using fitted operator in the given differential equation.

Findings – The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points.

Originality/value – In this paper, the authors consider a new governing problem having both small delay on convection term and large delay. As far as the researchers' knowledge is considered numerical solution of singularly perturbed boundary value problem containing both small delay and large delay is first being considered.

Keywords Singularly perturbed, Small delay, Large delay, Fitted spline, ϵ -uniformly convergent

Paper type Research paper

1. Introduction

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments, and also, the highest derivative term is multiplied by a small parameter. Such type of delay, differential equations play a very important role in the mathematical models of science and engineering, such as the human pupil light reflex with mixed delay type [1], variational problems in control theory with small state problem [2], models of HIV infection [3] and signal transition [4].

Any system involving a feedback control almost involves time delay. The delay occurs because a finite time is required to sense the information and then react to it. Finding the solution of singularly perturbed delay differential equations, whose application mentioned above, is a challenging problem. In response to these, in recent years, there has been a growing interest in numerical methods on singularly perturbed delay differential equations. The

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authors of [5–7] have developed various numerical schemes on uniform meshes for singularly perturbed second-order differential equations having small delay on convection term. The authors of [8–12] have presented second-order differential equations having large delay.

In this paper, we consider a new governing problem having both small delay on convection term and large delay. Additionally, in recent years the correlative physical phenomena in-depth of the problem under consideration have been done by the authors [13–17]. As far as the researchers’ knowledge is considered numerical solution of singularly perturbed boundary value problem containing both small delay and large delay is first being considered. Thus, the purpose of this study is to develop stable, convergent and accurate numerical method for solving singularly perturbed differential-difference equations having both small and large delay.

Throughout our analysis C is generic positive constant that are independent of the parameter ε and number of mesh points $2N$. We assume that $\bar{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1)$, $\Omega_2 = (1, 2)$ and $\Omega^* = \Omega_1 \cup \Omega_2$. L_1 and L_2 are the linear operator associated to the domain Ω_1 and Ω_2 , respectively.

2. Statement of the problem

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) + d(x)y'(x-\delta) = f(x), \quad (1)$$

$$y(x) = \phi(x), \quad x \in [-1, 0], \quad y(2) = l, \quad (2)$$

where δ is small, that is $\delta = O(\varepsilon)$, $0 < \varepsilon \ll 1$, $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$ it is assumed that the sufficient smooth functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ satisfy at $a(x) \geq a_1 > a > 0$, $b(x) > b \geq 0$, $c(x) \leq \gamma < 0$, $d(x) \geq \zeta \geq 0$, and $2(a + \zeta) + 5b + 5\gamma > \eta > 0$, $a(a_1 - a) + 2\gamma > 0$. The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$.

The boundary value problem 1–2 exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$. Expand $y'(x - \delta)$ about x using the Taylor’s expansion and discard higher order terms. Then, the above problem can be approximated by

$$c_{\varepsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad (3)$$

where $c_{\varepsilon,\delta}(x) = -\varepsilon - \delta d(x)$ and $p(x) = a(x) + d(x)$,

$$y(x) = \phi(x), \quad x \in [-1, 0], \quad y(2) = l. \quad (4)$$

As we observed from Eqns (3) and (4), the values of $y(x - 1)$ is known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it is impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately. Eqns (3)–(4) are equivalent to

$$Ly(x) = R(x), \quad (5)$$

where

$$Ly(x) = \begin{cases} L_1 y(x) = c_{\varepsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x), & x \in \Omega_1, \\ L_2 y(x) = c_{\varepsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1), & x \in \Omega_2. \end{cases} \quad (6)$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2. \end{cases} \quad (7)$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \quad (8)$$

3. Properties of continuous solution

Lemma 3.1. (Maximum principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0$, $\psi(2) \geq 0$, $L_1\psi(x) \geq 0$, $\forall x \in \Omega_1$, $L_2\psi(x) \geq 0$, $\forall x \in \Omega_2$ and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$

Proof. For the proof refer [8] ■

Lemma 3.2. (Stability result) The solution $y(x)$ of problem (3)–(4), satisfies the bound

$$|y(x)| \leq C \max\{|y(0)|, |y(2)|, \sup_{x \in \bar{\Omega}} |Ly(x)|\}, x \in \bar{\Omega}.$$

Proof. For the proof refer [8] ■

Lemma 3.3. Let $y(x)$ be the solution of (3)–(4). Then we have the following bounds

$$\|y^{(k)}(x)\|_{\Omega^*} \leq C\epsilon^{-k}, \quad k = 1, 2, 3.$$

Proof. For the proof refer [8] ■

4. Numerical scheme formulation

We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h . Let $0 = x_0, x_1, \dots, x_N = 1, x_{N+1}, x_{N+2}, \dots, x_{2N} = 2$ be the mesh points. Then we have $x_i = ih, i = 0, 1, 2, \dots, 2N$.

Consider a uniform mesh with interval $[0, 1]$ in which $0 = x_0 < x_1 < \dots < x_N = 1$ where $h = \frac{1}{N}$ and $x_i = ih, i = 0, 1, 2, \dots, N$.

We can rewrite (3) as

$$c_{\epsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) = Q(x), x \in \Omega_1, \quad (9)$$

where $Q(x) = f(x) - c(x)y(x-1)$.

For each segment $[x_i, x_{i+1}], i = 1, 2, \dots, N-1$ the non-polynomial cubic spline $S_\Delta(x)$ has the following form

$$S_\Delta(x) = a_i + b_i(x - x_i) + c_i(e^{w(x-x_i)} + e^{-w(x-x_i)}) + d_i(e^{w(x-x_i)} - e^{-w(x-x_i)}), \quad (10)$$

where a_i, b_i, c_i and d_i are unknown coefficients, and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method.

To determine the unknown coefficients in (10) in terms of y_i, y_{i+1}, M_i and M_{i+1} first we define

$$\begin{cases} S_\Delta(x_i) = y_i, & S_\Delta(x_{i+1}) = y_{i+1}, \\ S''_\Delta(x_i) = M_i, & S''_\Delta(x_{i+1}) = M_{i+1}. \end{cases} \quad (11)$$

The coefficients in (10) are determined as

$$\begin{cases} a_i = y_i - \frac{M_i}{w^2}, \\ b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta}, \\ c_i = \frac{M_{i+1}}{w^2(e^\theta - e^{-\theta})} - \frac{M_i(e^\theta + e^{-\theta})}{2w^2(e^\theta - e^{-\theta})}, \\ d_i = \frac{M_i}{2w^2}, \end{cases} \quad (12)$$

where $\theta = wh$.

Using the continuity condition of the first derivative at x_i , $S'_{\Delta-1}(x_i) = S'_\Delta(x_i)$, we have

$$b_{i-1} + wc_{i-1}(e^\theta + e^{-\theta}) + wd_{i-1}(e^\theta - e^{-\theta}) = b_i + 2wc_i. \quad (13)$$

Reducing indices of Eqn (12) by one and substituting into Eqn (13), we obtain

$$\begin{aligned} \frac{y_i - y_{i-1}}{h} + \frac{M_i - M_{i+1}}{w\theta} + w \left(\frac{2M_i - (e^\theta + e^{-\theta})M_{i-1}}{2w^2(e^\theta - e^{-\theta})} \right) &= \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta} + 2w \left(\frac{M_{i+1}}{w^2(e^\theta - e^{-\theta})} - \frac{M_i(e^\theta + e^{-\theta})}{2w^2(e^\theta - e^{-\theta})} \right), \\ \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} &= \alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}, \end{aligned} \quad (14)$$

where

$$\alpha = \frac{1}{\theta^2} \left(1 - \frac{2\theta}{(e^\theta - e^{-\theta})} \right), \quad \beta = \frac{1}{\theta^2} \left(\frac{\theta(e^\theta + e^{-\theta})}{(e^\theta - e^{-\theta})} - 1 \right).$$

If $h \rightarrow 0$, then $\theta = hk \rightarrow 0$. Thus, using L'Hopitals rule we have $\lim_{h \rightarrow 0} \alpha = \frac{1}{6}$ and $\lim_{h \rightarrow 0} \beta = \frac{1}{3}$

Using $S''_\Delta(x_i) = y''_i = M_i$ in to (9), we get

$$\begin{cases} c_{e,\delta}(x)M_i = Q_i - p_i y'_i - b_i y_i, \\ c_{e,\delta}(x)M_{i-1} = Q_{i-1} - p_{i-1} y'_{i-1} - b_{i-1} y_{i-1}, \\ c_{e,\delta}(x)M_{i+1} = Q_{i+1} - p_{i+1} y'_{i+1} - b_{i+1} y_{i+1}. \end{cases} \quad (15)$$

Using Taylor's series expansions of y_{i-1} , y_{i+1} , y'_{i-1} and y'_{i+1} simplifying, we have

$$\begin{cases} y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1, \\ y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2, \\ y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2, \end{cases} \quad (16)$$

where

$$T_1 = -\frac{h^2}{6} y'''(\xi) \text{ and } T_2 = \frac{h^2}{12} y'''(\xi), \text{ for } \xi \in (x_{i-1}, x_i).$$

Substituting Eqn (16) in to Eqn (15), we obtain

$$\begin{cases} M_i = \frac{1}{c_{\epsilon,\delta}(x)} \left\{ Q_i - p_i \left(\frac{y_{i+1} - y_{i-1}}{2h} + T_1 \right) - b_i y_i \right\}, \\ M_{i-1} = \frac{1}{c_{\epsilon,\delta}(x)} \left\{ Q_{i-1} - p_{i-1} \left(\frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2 \right) - b_{i-1} y_{i-1} \right\}, \\ M_{i+1} = \frac{1}{c_{\epsilon,\delta}(x)} \left\{ Q_{i+1} - p_{i+1} \left(\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2 \right) - b_{i+1} y_{i+1} \right\}. \end{cases} \quad (17)$$

Substituting Eqn (17) into Eqn (14) and rearranging, we get

$$\begin{aligned} \frac{c_{\epsilon,\delta}(x)}{h^2} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h} (-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h} (y_{i+1} - y_{i-1}) \\ + \frac{\alpha p_{i+1}}{2h} (3y_{i+1} - 3y_i + y_{i-1}) = \alpha(Q_{i-1} - b_{i-1}y_{i-1} + Q_{i+1} - b_{i+1}y_{i+1}) \\ + 2\beta(Q_i - b_i y_i) + T, \end{aligned} \quad (18)$$

where, $T = (4\beta p_i - \alpha p_{i-1} - \alpha p_{i+1}) \frac{h^2}{12} y'''(\xi)$ is the local truncation error.

From the theory of singular perturbations described in [18] and the Taylor series expansion of $y(x)$ about the point '0' in the asymptotic solution of the problem in Eq. (9), we have

$$y(x_i) \approx y_0(x_i) + (\phi_0 - y_0(0))e^{-\rho(0)\frac{ih}{c_{\epsilon,\delta}(x)}},$$

and letting $\rho = \frac{h}{c_{\epsilon,\delta}(x)}$ we get

$$\lim_{h \rightarrow 0} y(ih) \approx y_0(ih) + (\phi_0 - y_0(0))e^{-\rho(0)ih},$$

since $x_i = x_0 + ih$.

Introducing a fitting factor $\sigma(\rho)$ in to Eq. (18), we get

$$\begin{aligned} \frac{\sigma(\rho)c_{\epsilon,\delta}(x)}{h^2} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h} (-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h} (y_{i+1} - y_{i-1}) \\ + \frac{\alpha p_{i+1}}{2h} (3y_{i+1} - 3y_i + y_{i-1}) = \alpha(Q_{i-1} - b_{i-1}y_{i-1} + Q_{i+1} - b_{i+1}y_{i+1}) \\ + 2\beta(Q_i - b_i y_i) + T. \end{aligned} \quad (19)$$

Multiplying Eqn (19) by h and taking a limit as $h \rightarrow 0$ we get

$$\begin{aligned} \frac{\sigma}{\rho} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p(0)}{h} \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) \\ + \beta p(0) \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) + \frac{\alpha p(0)}{2} \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = 0. \end{aligned} \quad (20)$$

Thus, we consider two cases of the boundary layers.

Case 1: For $p(x) > 0$ (Left-end boundary layer), we have

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) = (\phi_0 - y_0(0))e^{-p(0)ih} (e^{p(0)\rho} + e^{-p(0)\rho} - 2), \\ \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)ih} (-3e^{p(0)\rho} - e^{-p(0)\rho} + 4), \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)ih} (e^{p(0)\rho} + 3e^{-p(0)\rho} - 4), \\ \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)ih} (e^{-p(0)\rho} - e^{p(0)\rho}). \end{cases} \quad (21)$$

Substituting Eqn (21) into Eqn (20) and simplifying, we get

$$\sigma_0 = \rho p(0)(\alpha + \beta) \coth\left(\frac{p(0)\rho}{2}\right).$$

Case 2: For $p(x) < 0$ (Right-end boundary layer), we have

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) = (\varphi - y_0(1))e^{-p(1)ih} (e^{p(1)\rho} + e^{-p(1)\rho} - 2), \\ \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) = (\varphi - y_0(1))e^{-p(1)ih} (-3e^{p(1)\rho} - e^{-p(1)\rho} + 4), \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (\varphi - y_0(1))e^{-p(1)ih} (e^{p(1)\rho} + 3e^{-p(1)\rho} - 4), \\ \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = (\varphi - y_0(1))e^{-p(1)ih} (e^{-p(1)\rho} - e^{p(1)\rho}). \end{cases} \quad (22)$$

Substituting Eq. (22) into Eq. (20) and simplifying, we obtain

$$\sigma_1 = \rho p(1)(\alpha + \beta) \coth\left(\frac{p(1)\rho}{2}\right). \quad (23)$$

In general, we take a variable fitting parameter as

$$\sigma_i = \rho_i p(x_i)(\alpha + \beta) \coth\left(\frac{p(x_i)\rho_i}{2}\right), \quad (24)$$

where, $\rho_i = \frac{h}{c_{e,\delta}(x)}$

Thus, Eqn (19) can be written as

$$\begin{aligned} & \left\{ \frac{c_{e,\delta}(x)\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h} \right\} y_{i-1} - \left\{ \frac{2c_{e,\delta}(x)\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta b_i + \frac{2\alpha p_{i+1}}{h} \right\} y_i \\ & + \left\{ \frac{c_{e,\delta}(x)\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h} \right\} y_{i+1} \\ & = \alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i. \end{aligned} \quad (25)$$

Simplifying Eqn (25), for the domain $\Omega_1 = (0, 1)$, we get the tri-diagonal system of the equation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \dots, N - 1, \quad (26)$$

where

$$\begin{cases} E_i = \frac{c_{\epsilon,\delta}(x)\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}, \\ F_i = \frac{2c_{\epsilon,\delta}(x)\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta b_i + \frac{2\alpha p_{i+1}}{h}, \\ G_i = \frac{c_{\epsilon,\delta}(x)\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}, \\ H_i = \alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i. \end{cases}$$

Similarly, if we consider the domain $\Omega_2 = (1, 2)$, from Eqs. (3), we have

$$\begin{cases} c_{\epsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), & x \in \Omega_2 \\ y(1) = \theta, & y(2) = l. \end{cases} \quad (27)$$

Using $S''_{\Delta}(x_i) = y''_i = M_i$ for Eqn (27), we get

$$\begin{cases} c_{\epsilon,\delta}(x_i)M_i = f_i - p_i y'_i - b_i y_i - c_i y(x_i - 1), \\ c_{\epsilon,\delta}(x_i)M_{i-1} = f_{i-1} - p_{i-1} y'_{i-1} - b_{i-1} y_{i-1} - c_{i-1} y(x_{i-1} - 1), \\ c_{\epsilon,\delta}(x_i)M_{i+1} = f_{i+1} - p_{i+1} y'_{i+1} - b_{i+1} y_{i+1} - c_{i+1} y(x_{i+1} - 1). \end{cases} \quad (28)$$

Substituting Eq. (16) in to Eq. (28), we obtain

$$\begin{cases} M_i = \frac{1}{c_{\epsilon,\delta}(x_i)} \left\{ f_i - p_i \left(\frac{y_{i+1} - y_{i-1}}{2h} + T_1 \right) - b_i y_i - c_i y(x_i - 1) \right\}, \\ M_{i-1} = \frac{1}{c_{\epsilon,\delta}(x_i)} \left\{ f_{i-1} - p_{i-1} \left(\frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2 \right) - b_{i-1} y_{i-1} - c_{i-1} y(x_{i-1} - 1) \right\}, \\ M_{i+1} = \frac{1}{c_{\epsilon,\delta}(x_i)} \left\{ f_{i+1} - p_{i+1} \left(\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2 \right) - b_{i+1} y_{i+1} - c_{i+1} y(x_{i+1} - 1) \right\}. \end{cases} \quad (29)$$

Substituting Eqn (29) in to Eqn (14), introducing fitting factor and rearranging, we get

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} + T_i = H_i, \quad i = N + 1, N + 2, \dots, 2N - 1, \quad (30)$$

where

$$\begin{cases} E_i = \frac{c_{\epsilon,\delta}(x_i)\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}, \\ F_i = \frac{2c_{\epsilon,\delta}(x_i)\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta b_i + \frac{2\alpha p_{i+1}}{h}, \\ G_i = \frac{c_{\epsilon,\delta}(x_i)\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}, \\ H_i = \alpha(f_{i-1} + f_{i+1}) + 2\beta f_i, \\ T_i = \alpha\{c_{i-1}y(x_{i-1} - 1) + c_{i+1}y(x_{i+1} - 1) + 2\beta c_i y(x_i - 1)\}. \end{cases}$$

Therefore, on the whole domain $\bar{\Omega} = [0, 2]$, the basic schemes to solve Eqs (1)–(2) are the schemes given in Eqn (26) and Eqn (30).

5. Stability and convergence analysis

5.1 Truncation error

Let expand the terms $y_{i\pm 1}$ and $M_{i\pm 1}$ from Eqn (14), using Taylor’s series as

$$\begin{cases} y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y^{(4)}_i + \frac{h^5}{5!}y^{(5)}_i + \frac{h^6}{6!}y^{(6)}_i + O(h^7), \\ y_{i-1} = y_i - hy'_i + \frac{h^2}{2!}y''_i - \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y^{(4)}_i - \frac{h^5}{5!}y^{(5)}_i + \frac{h^6}{6!}y^{(6)}_i + O(h^7), \\ M_{i+1} = y''_{i+1} = y''_i + hy'''_i + \frac{h^2}{2!}y^{(4)}_i + \frac{h^3}{3!}y^{(5)}_i + \frac{h^4}{4!}y^{(6)}_i + O(h^7), \\ M_{i-1} = y''_{i-1} = y''_i - hy'''_i + \frac{h^2}{2!}y^{(4)}_i - \frac{h^3}{3!}y^{(5)}_i + \frac{h^4}{4!}y^{(6)}_i + O(h^7). \end{cases} \tag{31}$$

The local truncation error $T_i(h)$ obtained from Eqn (14) as

$$T_i(h) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \alpha(M_{i-1} + M_{i+1}) - 2\beta M_i. \tag{32}$$

Substituting the series of $y_{i\pm 1}$ and $M_{i\pm 1}$ from Eqn (31)–(32) and collecting like terms gives

$$T_i(h) = (1 - 2(\alpha + \beta))y''_i + h^2\left(\frac{1}{12} - \alpha\right)y^{(4)}_i + O(h^4). \tag{33}$$

But from the values of $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$ Eqn (33) becomes

$$T_i(h) = h^2\left(-\frac{1}{12}\right)y^{(4)}_i + O(h^4),$$

which implies

$$\|T_i(h)\| \leq Ch^2, \tag{34}$$

where $C = \frac{1}{12}|y^{(4)}_i|$.

This establishes that the developed method is second order accurate or its order of convergence is $O(h^2)$.

5.2 Convergence analysis

Local truncation errors refer to the differences between the original differential equation and its finite difference approximation at a mesh points. Finite difference scheme is called consistent if the limit of truncation error ($T_i(h)$) is equal to zero as the mesh size h goes to zero. Hence, the proposed method in Eqn (26) with local truncation error in Eqn (34) satisfies the definition of consistency as

$$\lim_{h \rightarrow 0} T_i(h) = \lim_{h \rightarrow 0} Ch^2 = 0. \tag{35}$$

Thus, the proposed scheme is consistent.

5.3 Stability analysis

Consider the developed scheme in Eqn (26),

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{36}$$

where the coefficients E_i, F_i and G_i are as in Eqn (26). If we multiply both sides of Eqn (26) by h^2 and consider the values of E_i, F_i and G_i for sufficiently small h , we get

$$E_i = G_i = \epsilon\sigma, F_i = 2\epsilon\sigma, \tag{37}$$

Considering Eqn (37) into Eqn (26) the one which is multiplied by h^2 the developed scheme can be written in a matrix form

$$AY = B \tag{38}$$

where the matrices $A = \begin{pmatrix} -2\epsilon\sigma & \epsilon\sigma & 0 & \dots & 0 \\ \epsilon\sigma & -2\epsilon\sigma & \epsilon\sigma & \dots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & \epsilon\sigma \\ 0 & - & - & \epsilon\sigma & -2\epsilon\sigma \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix}$ and

$$B = \begin{pmatrix} h^2 H_1 - E_1 y_0 \\ h^2 H_2 \\ \vdots \\ h^2 H_{N-2} \\ h^2 H_{N-1} - G_{N-1} y_N \end{pmatrix}.$$

Here, the coefficient matrix A is a tri-diagonal matrix with size $(N - 1) \times (N - 1)$. Matrix A is irreducible if its codiagonals contain nonzero elements only. The codiagonals contain E_i and G_i . It is clearly seen that, for sufficiently small h both $E_i \neq 0$ and $G_i \neq 0$ for $i = 1, 2, \dots, N - 1$. Hence, A is irreducible.

Again we can see that all $|E_i|, |F_i|, |G_i| > 0$ for $i = 1, 2, \dots, N - 1$ and in each row of A , the modulus of diagonal element is greater than or equal to the sum of the modulus of the two codiagonal elements (i.e. $|F_i| \geq |E_i| + |G_i|$). This implies that A is diagonally dominant. Under this condition, the Thomas algorithm is stable for sufficiently small h .

As discussed in [19] the eigenvalues of a tri-diagonal matrix A are given by

$$\lambda_s = -2\epsilon\sigma + 2\left\{ \sqrt{(\epsilon\sigma)(\epsilon\sigma)} \right\} \cos \frac{s\pi}{N}, s = 1(1)N - 1.$$

Hence, the eigenvalues of matrix A in Eqn (38) are

$$\lambda_s = -2\epsilon\sigma + 2\left\{ \sqrt{(\epsilon\sigma)^2} \right\} \cos \frac{s\pi}{N} = -2\epsilon\sigma \left(1 - \cos \frac{s\pi}{N} \right), s = 1(1)N - 1.$$

But from trigonometric identity, we have $1 - \cos \frac{s\pi}{N} = 2\sin^2 \frac{s\pi}{2N}$. Thus, the eigenvalues of A

$$\lambda_s = -2\epsilon\sigma \left(2\sin^2 \frac{s\pi}{2N} \right) = -4\epsilon\sigma \sin^2 \frac{s\pi}{2N} \leq -4\epsilon\sigma. \tag{39}$$

A finite difference method for the boundary value problems is stable if A is nonsingular and $\|A^{-1}\| \leq C$, for $0 < h < h_0$. where, C and h_0 are two constants that are independent of h .

Since A is real and symmetric it follows that A^{-1} is also real and symmetric so that, its eigenvalues are real and given by $\frac{1}{\lambda_s}$. Hence, as [19] the stability condition of the method will be satisfied when; $\|A^{-1}\| = \left| \frac{1}{\lambda_s} \right| = \left| \frac{-1}{4\epsilon\sigma} \right| = \frac{1}{4\epsilon\sigma} \leq C$, where, C is independent of h . Thus the developed scheme in Eqn (26) is stable. A consistent and stable finite difference method is convergent by [20]. Hence as we have shown above, the proposed method is satisfying both the criteria of consistency and stability which are equivalent to convergence of the method.

6. Numerical examples and results

In this section, one example is given to illustrate the numerical method discussed above. The exact solutions of the test problem is not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_\epsilon^N = \max_{0 \leq i \leq 2N} |Y_i^N - Y_{2i}^{2N}|, \quad (40)$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} components of the numerical solutions on meshes of N and $2N$, respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_\epsilon E_\epsilon^N, \text{ and } R^N = \log_2 \left(\frac{E^N}{E^{2N}} \right). \quad (41)$$

The numerical results are presented for the values of the perturbation parameter $\epsilon \in \{10^{-4}, 10^{-8}, \dots, 10^{-20}\}$.

Example 6.1. Consider the model singularly perturbed boundary value problem:

$$-\epsilon y''(x) + 10y'(x) - y(x - 1) + y'(x - \epsilon) = x \quad x \in (0, 1) \cup (1, 2),$$

Subject to the boundary conditions

$$y(x) = 1, \quad x \in [-1, 0], \quad y(2) = 2.$$

7. Discussion and conclusion

This study introduces a fitted nonpolynomial spline method for singularly perturbed differential equations having both small and large delay. The numerical scheme is developed on uniform mesh using fitted operator in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Table 1). Further, behavior of the numerical solution (Figure 1), point-wise absolute error (Figure 2) and the ϵ -uniform convergence of the method is shown by the log-log plot (Figure 3). The method is shown to be ϵ -uniformly convergent

with order of convergence $O(h^2)$. The proposed method gives accurate, stable and ε -uniform numerical result.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
10^{-4}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-8}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-12}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-16}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-20}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
E^N	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
R^N	0.9849	0.9925	0.9962	0.9982	

Table 1. Maximum absolute errors and rate of convergence for [Example 6.1](#) at different perturbation parameter and number of mesh points

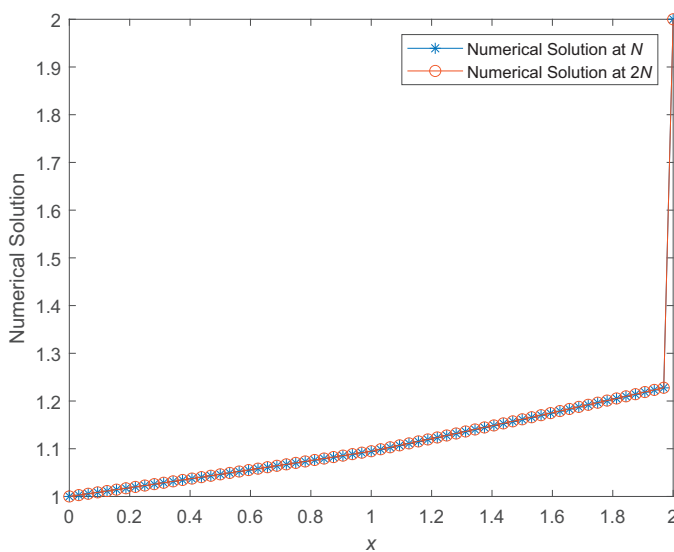


Figure 1. The behavior of the numerical solution for [Example 6.1](#) at $\varepsilon = 10^{-12}$ and $N = 32$

Figure 2.
Point wise absolute error of Example 6.1 at $\varepsilon = 10^{-12}$ with different mesh point N

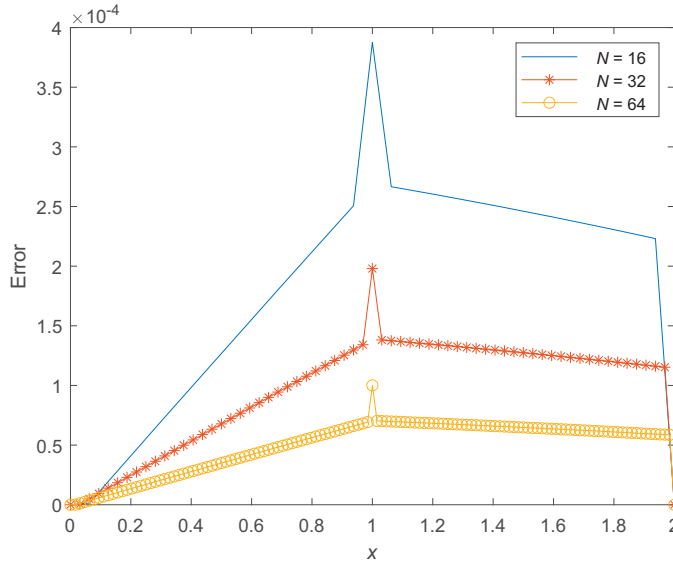
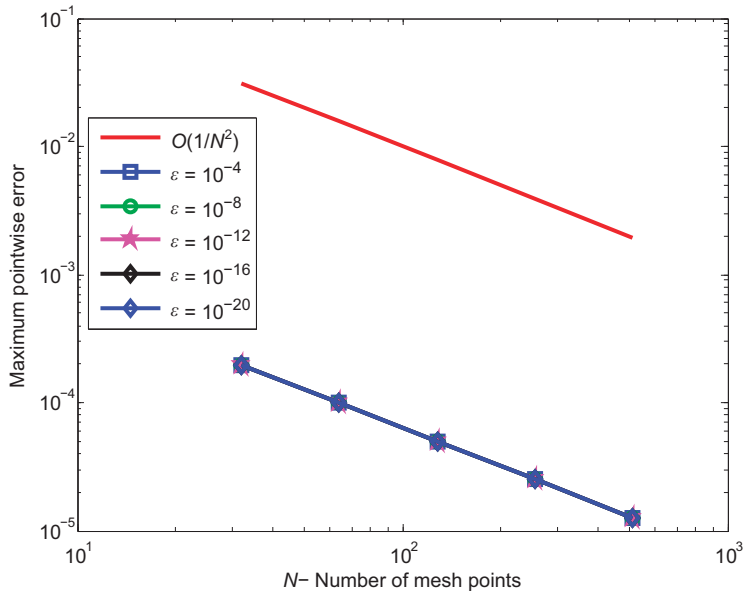


Figure 3.
 ε -uniform convergence with fitted operator in log-log scale for Example 6.1



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