

On copula moment: empirical likelihood based estimation method

Empirical
likelihood based
estimation
method

191

Jihane Abdelli and Brahim Brahimi
*Laboratory of Applied Mathematics, Mohamed Khider University,
Biskra, Algeria*

Received 23 January 2021
Revised 22 May 2021
Accepted 22 May 2021

Abstract

Purpose – In this paper, the authors applied the empirical likelihood method, which was originally proposed by Owen, to the copula moment based estimation methods to take advantage of its properties, effectiveness, flexibility and reliability of the nonparametric methods, which have limiting chi-square distributions and may be used to obtain tests or confidence intervals. The authors derive an asymptotically normal estimator of the empirical likelihood based on copula moment estimation methods (ELCM). Finally numerical performance with a simulation experiment of ELCM estimator is studied and compared to the CM estimator, with a good result.

Design/methodology/approach – In this paper we applied the empirical likelihood method which originally proposed by Owen, to the copula moment based estimation methods.

Findings – We derive an asymptotically normal estimator of the empirical likelihood based on copula moment estimation methods (ELCM). Finally numerical performance with a simulation experiment of ELCM estimator is studied and compared to the CM estimator, with a good result.

Originality/value – In this paper we applied the empirical likelihood method which originally proposed by Owen 1988, to the copula moment based estimation methods given by Brahimi and Necir 2012. We derive a new estimator of copula parameters and the asymptotic normality of the empirical likelihood based on copula moment estimation methods.

Keywords Archimedean copulas, Asymptotic distribution, Copula models, Method of moments, Semi-parametric models, Z-estimator

Paper type Research paper

1. Introduction

One of the main topics in multivariate statistical analysis is the statistical inference on the dependence parameter θ . Many researchers investigated the copula parameter estimation, namely the methods of concordance [1, 2] fully and the pseudo maximum likelihood [3], inference function of margins [4, 5], minimum distance [6] and recently the copula moment and L -moment based estimation methods given in [7, 8].

In this paper we applied the empirical likelihood method to the copula moment based estimation methods which originally proposed by [9–11]. Several authors investigated the empirical likelihood see for instance [12–16].

The advantage of this method is that the empirical likelihood has both effectiveness and flexibility of the likelihood method, and reliability of the non-parametric methods, and it helps

JEL Classification — Primary 62G05, Secondary 62G20

© Jihane Abdelli and Brahim Brahimi. Published in *Arab Journal of Mathematical Sciences*. Published by Emerald Publishing Limited. This article is published under the Creative Commons Attribution (CC BY 4.0) licence. Anyone may reproduce, distribute, translate and create derivative works of this article (for both commercial and non-commercial purposes), subject to full attribution to the original publication and authors. The full terms of this licence may be seen at <http://creativecommons.org/licenses/by/4.0/legalcode>

The authors are indebted to an anonymous referee for valuable remarks and suggestions.



Arab Journal of Mathematical
Sciences
Vol. 28 No. 2, 2022
pp. 191-202
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-01-2021-0025

us to construct confidence intervals without estimating the asymptotic variance, so the complexity of the asymptotic variance for some estimator especially the CM based estimators and the construction of non-parametric confidence intervals via estimating the asymptotic variance is usually inaccurate.

2. Empirical likelihood for CM based estimation method

We consider the Archimedean copula family defined by $C(\mathbf{u}) = \varphi^{-1}\left(\sum_{j=1}^d \varphi(u_j)\right)$, where $\varphi : [0, 1] \rightarrow \mathbb{R}$ is a twice differentiable function called the generator, satisfying: $\varphi(1) = 0$, $\varphi'(x) < 0$, $\varphi''(x) \geq 0$ for any $x \in (0, 1)$ and $\mathbf{u} = (u_1, \dots, u_d)$. The notation φ^{-1} stands for the inverse function of φ . Archimedean copulas are easy to construct and have nice properties. A variety of known copula families belong to this class, including the models of Gumbel, Clayton, Frank, . . .(see, Table 4.1 in [17], p. 116).

Let $\mathbf{K}_C(s) := P(C(\mathbf{U}) \leq s)$, $s \in [0, 1]$, be the df of rv $C(\mathbf{U})$, where $\mathbf{U} = (U_1, \dots, U_d)$, then the k th-moment $M_k(C)$, called *copula moment*, of rv $C(\mathbf{U})$ given in [7] as the expectation of $(C(\mathbf{U}))^k$, that is

$$M_k(C) := \mathbb{E}\left[(C(\mathbf{U}))^k\right] = \int_{[0,1]^d} (C(\mathbf{u}))^k dC(\mathbf{u}), \quad k = 1, 2, \dots \tag{2.1}$$

Equation (2.1) may be rewritten into:

$$M_k(C) = \int_0^1 s^k d\mathbf{K}_C(s), \quad k = 1, 2, \dots$$

Suppose now, for unknown $\theta \in \mathcal{O}$, that $\varphi = \varphi_\theta$, it follows that $C = C_\theta$, $\mathbf{K}_C = \mathbf{K}_\theta$ and $M_k(C) = M_k(\theta)$, that is

$$M_k(\theta) = \int_0^1 s^k d\mathbf{K}_\theta(s), \quad k = 1, 2, \dots,$$

from Theorem 4.3.4 in [17] we have for any $s \in [0, 1]$, $\mathbf{K}_\theta(s) = s - \varphi_\theta(s)/\varphi'_\theta(s)$, it follows that the corresponding density is $\mathbf{K}'_\theta(s) = \varphi''_\theta(s)\varphi_\theta(s)/(\varphi'_\theta(s))^2$. Therefore (2.1), may be rewritten into

$$M_k(\theta) = \int_0^1 s^k \frac{\varphi''_\theta(s)\varphi_\theta(s)}{(\varphi'_\theta(s))^2} ds, \quad k = 1, 2, \dots \tag{2.2}$$

In terms of φ_θ .

The non-parametric likelihood of distribution function \mathbf{K}_C of rv $C(\mathbf{U})$ is defined by

$$\mathcal{L}(\mathbf{K}_C) = \prod_{i=1}^n \mathbf{K}'_C(C(\mathbf{U}_i)), \tag{2.3}$$

Where $\mathbf{U}_i := (U_{i1}, \dots, U_{id})$. We restrict \mathbf{K}_C to the one having the probability

$$p_i = \mathbf{K}'_C(C(\mathbf{U}_i)) > 0$$

on each observation \mathbf{U}_i . By a simple calculation, we find the maximizer of the non-parametric likelihood (2.3) turns to be the empirical distribution function \mathbf{K}_{C_n} , placing probability $1/n$ on each observation. Therefore, similar to the parametric case, non-parametric likelihood ratio of \mathbf{K}_C to the maximizer \mathbf{K}_{C_n} is defined by:

$$\mathcal{R}(\mathbf{K}_C) = \frac{\mathcal{L}(\mathbf{K}_C)}{\mathcal{L}(\mathbf{K}_{C_n})} = \frac{\prod_{i=1}^n p_i}{\prod_{i=1}^n 1/n} = \prod_{i=1}^n (np_i).$$

Suppose now that we are interested in a parameter $\theta \in \mathcal{O} \subset \mathbb{R}^r$, $C = C_\theta$ and $\mathbf{K}_C := \mathbf{K}_\theta$ that the parameter θ satisfies the following equations

$$\mathbb{E}[L_k(\mathbf{u}; \theta_0)] = 0, k = 1, 2, \dots, r \tag{2.4}$$

where

$$[L_k(\mathbf{u}; \theta)]_{k=1,2,\dots,r} := \left[(C_\theta(\mathbf{u}))^k - M_k(\theta) \right]_{k=1,2,\dots,r} \in \mathbb{R}^r \tag{2.5}$$

is a vector-valued function, called estimating function. Let the sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ from random vector $\mathbf{X} = (X_1, \dots, X_d)$, we define the corresponding joint empirical df by

$$F_n(\mathbf{x}) = n^{-1} \sum_{i=1}^n \mathbf{1}\{X_{1i} \leq x_1, \dots, X_{di} \leq x_d\},$$

with $\mathbf{x} := (x_1, \dots, x_d)$, and the marginal empirical df's pertaining to the sample (X_{j1}, \dots, X_{jn}) , from rv X_j , by

$$F_{jn}(x_j) = n^{-1} \sum_{i=1}^n \mathbf{1}\{X_{ji} \leq x_j\}, j = 1, \dots, d. \tag{2.6}$$

According to [18] the empirical copula function is defined by

$$C_n(\mathbf{u}) := F_n(F_{1n}^{-1}(u_1), \dots, F_{dn}^{-1}(u_d)), \text{ for } \mathbf{u} \in [0, 1]^d, \tag{2.7}$$

where $F_{jn}^{-1}(s) := \inf\{x : F_{jn}(x) \geq s\}$ denotes the empirical quantile function pertaining to F_{jn} . For each $j = 1, \dots, d$, we compute $\hat{U}_{ji} := F_{jn}(X_{ji})$, then set

$$\hat{\mathbf{U}}_i := \left(\hat{U}_{1i}, \dots, \hat{U}_{di} \right), i = 1, \dots, n. \tag{2.8}$$

and for each $k = 1, \dots, r$, we compute

$$\hat{M}_k := n^{-1} \sum_{i=1}^n \left(C_n(\hat{\mathbf{U}}_i) \right)^k \tag{2.9}$$

By substitution of M_k by \hat{M}_k and solving system (2.4) in θ we obtain the solution $\hat{\theta}^{CM} := (\hat{\theta}_1, \dots, \hat{\theta}_r)$, called the CM estimator for θ .

Assume that the following assumptions [H.1] – [H.3] hold.

- (1) [H.1] $\theta_0 \in \mathcal{O} \subset \mathbb{R}^r$ is the unique zero of the mapping $\theta \rightarrow \int_{[0,1]^d} \mathbf{L}(\mathbf{u}; \theta) dC_{\theta_0}(\mathbf{u})$ which is defined from \mathcal{O} to \mathbb{R}^r , where $\mathbf{L}(\mathbf{u}; \theta) = (L_1(\mathbf{u}; \theta), \dots, L_r(\mathbf{u}; \theta))$.
- (2) [H.2] $\mathbf{L}(\cdot; \theta)$ is differentiable with respect to θ with the Jacobean matrix denoted by

$$\dot{\mathbf{L}}(\mathbf{u}; \theta) := \left[\frac{\partial L_k(\mathbf{u}; \theta)}{\partial \theta_\ell} \right]_{r \times r},$$

$\dot{\mathbf{L}}(\mathbf{u}; \theta)$ is continuous both in \mathbf{u} and θ , and the Euclidean norm $\left| \dot{\mathbf{L}}(\mathbf{u}; \theta) \right|$ is dominated by a dC_θ -integrable function $h(\mathbf{u})$.

(3) [H.3] The $r \times r$ matrix $A_0 := \int_{[0,1]^d} \dot{\mathbf{L}}(\mathbf{u}; \boldsymbol{\theta}_0) d\mathbf{C}_{\boldsymbol{\theta}_0}(\mathbf{u})$ is non-singular.

Theorem 2.1. Assume that assumptions [H.1] – [H.3] hold. Then with probability tending to one as $n \rightarrow \infty$, the solution $\widehat{\boldsymbol{\theta}}^{CM}$ converges to $\boldsymbol{\theta}_0$. Moreover

$$\sqrt{n} \left(\widehat{\boldsymbol{\theta}}^{CM} - \boldsymbol{\theta}_0 \right) \xrightarrow{D} \mathcal{N} \left(\mathbf{0}, A_0^{-1} D_0 \left(A_0^{-1} \right)^T \right), \text{ as } n \rightarrow \infty,$$

where $D_0 := \text{var}\{\mathbf{L}(\boldsymbol{\xi}; \boldsymbol{\theta}_0) + \mathbf{V}(\boldsymbol{\xi}; \boldsymbol{\theta}_0)\}$ and $\mathbf{V}(\boldsymbol{\xi}; \boldsymbol{\theta}_0) = (V_1(\boldsymbol{\xi}; \boldsymbol{\theta}_0), \dots, V_r(\boldsymbol{\xi}; \boldsymbol{\theta}_0))$ with

$$V_k(\boldsymbol{\xi}; \boldsymbol{\theta}_0) := \sum_{j=1}^d \int_{[0,1]^d} \frac{\partial (\mathbf{C}_{\boldsymbol{\theta}_0}(\mathbf{u}))^k}{\partial u_j} (\mathbf{1}\{\xi_j \leq u_j\} - u_j) d\mathbf{C}_{\boldsymbol{\theta}_0}(\mathbf{u}), \quad k = 1, \dots, r,$$

where $\boldsymbol{\xi} := (\xi_1, \dots, \xi_d)$ is a $(0, 1)^d$ -uniform random vector with joint df $\mathbf{C}_{\boldsymbol{\theta}_0}$.

Proof. See [7]. □

Now we define the empirical likelihood ratio function for $\boldsymbol{\theta}$ by

$$\mathcal{L}(\boldsymbol{\theta}) := \sup_{\mathbf{p}} \left\{ \prod_{i=1}^n (np_i) : \sum_{i=1}^n p_i L_k(\mathbf{U}_i, \boldsymbol{\theta}) = \mathbf{0}, p_i > 0, \sum_{i=1}^n p_i = 1 \right\}$$

where $\mathbf{p} = (p_1, \dots, p_n)$. This is the maximum of the non-parametric likelihood ratio with the restriction that the mean of the estimating function is zero under the distribution $\mathbf{K}_{\boldsymbol{\theta}}$.

Let, for $k = 1, \dots, r$

$$L_i^{(k)} = (\mathbf{C}_{\boldsymbol{\theta}}(\mathbf{U}_i))^k - M_k(\boldsymbol{\theta})$$

and

$$\widehat{L}_{i,n}^{(k)} = \left(\mathbf{C}_n(\widehat{\mathbf{U}}_i) \right)^k - M_k(\boldsymbol{\theta})$$

where \widehat{M}_k is defined in (2.9). Then, the empirical likelihood evaluated at $\boldsymbol{\theta}$ is defined as

$$\widetilde{\mathcal{L}}(\boldsymbol{\theta}) = \sup_{\mathbf{p}} \left\{ \prod_{i=1}^n (np_i) : \sum_{i=1}^n p_i L_i^{(k)} = \mathbf{0}, \sum_{i=1}^r p_i = 1 \right\}$$

Since the $L_i^{(k)}$'s depend on $\mathbf{C}_{\boldsymbol{\theta}}$, for an unknown $\boldsymbol{\theta}$, we replace them by the $\widehat{L}_{i,n}^{(k)}$'s. Therefore, an estimated empirical likelihood evaluated at $\boldsymbol{\theta}$ is defined by

$$\mathcal{L}(\boldsymbol{\theta}) = \sup_{\mathbf{p}} \left\{ \prod_{i=1}^n (np_i) : \sum_{i=1}^n p_i \widehat{L}_{i,n}^{(k)} = \mathbf{0}, \sum_{i=1}^r p_i = 1 \right\}$$

Now, by introducing a vector of Lagrange multipliers $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r$, to find the optimal p_i 's i.e. maximizing

$$G = \sum_{i=1}^n \log(np_i) - n \sum_{k=1}^r \lambda_k \sum_{i=1}^n p_i \widehat{L}_{i,n}^{(k)} + \gamma \left(\sum_{i=1}^r p_i - 1 \right).$$

So, setting $\frac{\partial G}{\partial p_i} = 0$ gives

$$\frac{\partial G}{\partial p_i} = \frac{1}{p_i} - n\lambda_k \widehat{L}_{i,n}^{(k)} + \gamma.$$

Therefore, the equation $\sum_{i=1}^n p_i \frac{\partial G}{\partial p_i} = 0$ gives $\gamma = -n$. Then, p_i is given by

$$p_i = \frac{1}{n} \left(1 + \lambda_k \widehat{L}_{i,n}^{(k)} \right)^{-1} =: p_{i,k}.$$

We have the problem that all the solutions $p_{1,k}, p_{2,k}, \dots, p_{n,k}, \lambda_k$ and γ are not obtained in a closed form. Note that for $k = 1, 2, \dots, r$: $\prod_{i=1}^n p_{i,k}$, subject to $\sum_{i=1}^n p_{i,k} = 1$, attains its maximum n^{-n} at $p_{i,k} = 1/n$. So we define the empirical likelihood ratio for θ as

$$\mathcal{R}(\theta) = \prod_{i=1}^n (np_{i,k}) = \prod_{i=1}^n \left(1 + \lambda_k \widehat{L}_{i,n}^{(k)} \right)^{-1},$$

and the corresponding empirical log-likelihood ratio is defined as

$$\mathcal{L}(\theta) = -2 \log \mathcal{R}(\theta) = 2 \sum_{i=1}^n \log \left(1 + \lambda_k \widehat{L}_{i,n}^{(k)} \right), k = 1, 2, \dots, r, \quad (2.10)$$

where the vector λ is the solution of the system of r equations given by

$$\frac{1}{n} \sum_{i=1}^n \frac{\widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = 0, k = 1, 2, \dots, r. \quad (2.11)$$

Since (2.11) is an implicit function of λ , we may solve (2.11) with respect to λ by the iterative procedure such as the Newton-Raphson optimization method or a simple grid search.

Theorem 2.2. *Assume conditions [H.1]–[H.3] hold. Then the limiting distribution of $\mathcal{L}(\theta)$ is a scaled chi-square distribution with r degrees of freedom, that is,*

$$\mathcal{L}(\theta) \rightarrow \chi_{(r)}^2$$

3. Illustrative example and simulation study

We consider the transformed Gumbel copula given by

$$C_{\alpha,\beta}(u_1, u_2) := \left(\left((u_1^{-\alpha} - 1)^\beta + (u_2^{-\alpha} - 1)^\beta \right)^{1/\beta} + 1 \right)^{-1/\alpha}, \quad (3.12)$$

which is also a two-parameter Archimedean copula with generator $\varphi_{\alpha,\beta}(t) := (t^{-\alpha} - 1)^\beta$. Here $\theta = (\alpha, \beta)$ then $r = 2$, and $\mathbf{U} = (U_1, U_2)$. By an elementary calculation we get the k th CM:

$$M_k(\alpha, \beta) = \frac{(k+1)\beta + \alpha\beta - k}{(k+1)^2\beta + (k+1)\alpha\beta}.$$

In particular the first two CM's are

$$M_1(\alpha, \beta) := \frac{2\beta + \alpha\beta - 1}{4\beta + 2\alpha\beta} \text{ and } M_2(\alpha, \beta) := \frac{3\beta + \alpha\beta - 2}{9\beta + 3\alpha\beta}.$$

Then

$$L_{in}^{(1)}(\mathbf{U}_i; \alpha, \beta) := (C_n(U_{1i}, U_{2i})) - M_1(\alpha, \beta)$$

and

$$L_{in}^{(2)}(\mathbf{U}_i; \alpha, \beta) := (C_n(U_{1i}, U_{2i}))^2 - M_2(\alpha, \beta)$$

Then

$$p_{i,1} = \frac{1}{n} (1 + \lambda_1(C_n(U_{1i}, U_{2i}) - M_1(\alpha, \beta)))^{-1}$$

and

$$p_{i,2} = \frac{1}{n} \left(1 + \lambda_2 \left((C_n(U_{1i}, U_{2i}))^2 - M_2(\alpha, \beta) \right)\right)^{-1}$$

where the vector $\lambda = (\lambda_1, \lambda_2)$ satisfies two equations given by

$$\begin{aligned} \sum_{i=1}^n (1 + \lambda_1(C_n(U_{1i}, U_{2i}) - M_1(\alpha, \beta)))^{-1} &= 0, \\ \sum_{i=1}^n \left(1 + \lambda_2 \left((C_n(U_{1i}, U_{2i}))^2 - M_2(\alpha, \beta) \right)\right)^{-1} &= 0 \end{aligned}$$

Finally, we get

$$\begin{aligned} &\mathcal{R}(\alpha, \beta) \\ &= \left(\sum_{i=1}^n (1 + \lambda_1(C_n(U_{1i}, U_{2i}) - M_1(\alpha, \beta)))^{-1}, \sum_{i=1}^n \left(1 + \lambda_2 \left((C_n(U_{1i}, U_{2i}))^2 - M_2(\alpha, \beta) \right)\right)^{-1} \right) \end{aligned}$$

To evaluate and compare the performance of empirical likelihood for CM's estimator is called the empirical likelihood copula moment (ELCM) estimator with the CM's and PML's estimator, a simulation study is carried out by considering the above example of bivariate Gumbel copula family $C_{\alpha, \beta}$. The evaluation of the performance is based on the bias and the RMSE defined as follows:

$$\text{Bias} = \frac{1}{R} \sum_{i=1}^R \hat{\theta}_i - \theta, \quad \text{RMSE} = \left(\frac{1}{R} \sum_{i=1}^R (\hat{\theta}_i - \theta)^2 \right)^{1/2}, \quad (3.13)$$

where $\hat{\theta}_i$ is an estimator (from the considered method) of θ from the i th samples for R generated samples from the underlying copula. In both parts, we selected $R = 1000$. To assess the improvement in the bias and RMSE of the estimators we repeat the following steps:

Step 1: For a given sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ from random vector $\mathbf{X} = (X_1, \dots, X_d)$, we define the corresponding joint empirical df by

$$C_n(\mathbf{x}) = n^{-1} \sum_{i=1}^n \mathbf{1}\{X_{1i} \leq x_1, \dots, X_{di} \leq x_d\}.$$

with $\mathbf{x} := (x_1, \dots, x_d)$. For each $j = 1, \dots, d$, compute (2.8).

Step 2: Solve the following system for $k = 1, \dots, r$,

$$\sum_{i=1}^n \left(1 + \lambda_k \left(\left(C_n(\hat{\mathbf{U}}_i) \right)^k - M_k(\boldsymbol{\theta}) \right) \right)^{-1} = 0,$$

The obtained solution $\hat{\boldsymbol{\theta}}^{ELCM} := (\hat{\theta}_1, \dots, \hat{\theta}_r)$.

For different sample sizes n with $n = 50, 100, 200, 500$ with increasing sample size and a large set of parameters of the true copula $C_{\alpha,\beta}$. The choice of the true values of the parameter (α, β) has to be meaningful, in the sense that each couple of parameters assigns a value of one of the dependence measure, that is weak, moderate and strong dependence. The selected values of the true parameters are summarized in Table 1, the results are summarized in Table 2.

4. Comments and conclusions

From Table 2, by considering three dependence cases: weak ($\tau = 0.01$), moderate ($\tau = 0.5$) and strong ($\tau = 0.8$), the performance of the ELCM estimator remains quite good in small sample size. We show that the ELCM estimator is performs better than the CM based estimator in large one. Moreover, in time-consuming point of view, we observe that for a sample size $n = 30$ with $N = 1000$ replications, the central processing unit (CPU) time to apply ELCM method took 1.442 hours, which takes approximately the same time with the PLM method and is relatively big to the CM method, which is measured in seconds 22.013. For only one replication, the CPU times (in seconds), for different sample sizes, are summarized as follows: $(n, CPU) = (30, 5.2613), (100, 10.891), (200, 16.965), (500, 25.995)$, (see Table 3). Which opens the door to new applications in copulas estimation framework.

5. Proofs

5.1 Proof of Theorem 2.2

For the proof we need the following Lemmas

Lemma 5.1. Under the same conditions as in Theorem 2.1

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{L}_{i,n}^{(k)} \rightarrow \mathcal{N}(0, \sigma_1^2(\boldsymbol{\theta})).$$

Proof. Follows straightaway from Theorem 2.1, see [7].

Lemma 5.2. Under the same conditions as in Theorem 2.1, for $k = 1, 2, \dots, r$ we have

$$(1) \frac{1}{n} \sum_{i=1}^n \left(\hat{L}_{i,n}^{(k)} \right)^2 = O_p(1), \quad (2) \hat{\sigma}_k^2(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^n \left(\hat{L}_{i,n}^{(k)} \right)^2 + o_p(1).$$

	τ	α	β	
Weak	0.01	0.1	1.059	The true parameters of transformed gumbel copula used for the simulation study
Moderate	0.5	0.5	1.600	
Strong	0.8	0.9	3.450	

Table 1.

The true parameters of transformed gumbel copula used for the simulation study

Table 2.
Bias and RMSE of
ELCM estimator of
two-parameter
transformed gumbel
copula

n	$\tau = 0.01$		$\alpha = 0.5$		$\tau = 0.5$		$\beta = 1.6$		$\alpha = 0.9$		$\tau = 0.8$		$\beta = 3.45$		CPU
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	
50	0.047	0.103	0.024	0.138	0.033	0.118	0.012	0.213	0.036	0.113	0.073	0.349	0.073	0.349	1.462 hours
100	0.023	0.088	0.022	0.100	0.019	0.057	0.009	0.126	0.021	0.071	0.014	0.294	0.014	0.294	3.025 hours
200	0.009	0.079	0.011	0.057	0.007	0.046	0.008	0.097	0.004	0.064	0.026	0.105	0.026	0.105	4.710 hours
500	0.003	0.035	0.009	0.040	0.001	0.029	0.003	0.092	0.002	0.018	0.007	0.112	0.007	0.112	7.221 hours

Proof. (1) From the law of large number, it follows that

$$\frac{1}{n} \sum_{i=1}^n \left(L_i^{(k)} \right)^2 = \mathbb{E} \left[\left(L_1^{(k)} \right)^2 \right] + o_p(1) = O_p(1)$$

(2) Let

$$\mathbb{T} = \left| \frac{1}{n} \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} \right)^2 - \frac{1}{n} \sum_{i=1}^n \left(L_i^{(k)} \right)^2 \right|.$$

So we can write

$$\begin{aligned} \mathbb{T} &= \left| \frac{1}{n} \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)} \right) \left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)} + 2L_i^{(k)} \right) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)} \right)^2 + 2 \left| \frac{1}{n} \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} - L_i^{(k)} \right) L_i^{(k)} \right| \equiv \mathbb{T}_1 + 2\mathbb{T}_2. \end{aligned}$$

We have

$$\mathbb{T}_1 = \frac{1}{n} \sum_{i=1}^n \left(\left(C_{\theta}(\widehat{\mathbf{U}}_i) \right)^k - \left(C_n(\widehat{\mathbf{U}}_i) \right)^k \right)^2 = O_p(1),$$

and

$$\begin{aligned} \mathbb{T}_2 &= \left| \frac{1}{n} \sum_{i=1}^n \left(\left(C_{\theta}(\widehat{\mathbf{U}}_i) \right)^k - \left(C_n(\widehat{\mathbf{U}}_i) \right)^k \right) L_i^{(k)} \right| \\ &\leq \sup_{\mathbf{t}} \left(\left(C_{\theta}(\mathbf{t}) \right)^k - \left(C_n(\mathbf{t}) \right)^k \right) \left| \frac{1}{n} \sum_{i=1}^n L_i^{(k)} \right| = O_p(1). \end{aligned}$$

Hence $\mathbb{T} = O_p(1)$. It follows

$$\begin{aligned} \left| \widehat{\sigma}_k^2(\boldsymbol{\theta}) - \frac{1}{n} \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} \right)^2 \right| &= \left| \frac{1}{n} \sum_{i=1}^n \left(\left(C_n(\widehat{\mathbf{U}}_i) \right)^k - \widehat{M}_k \right)^2 - \frac{1}{n} \sum_{i=1}^n \left(\left(C_n(\widehat{\mathbf{U}}_i) \right)^k - M_k(\boldsymbol{\theta}) \right)^2 \right| \\ &= \left| \frac{1}{n} \sum_{i=1}^n \left(\widehat{M}_k - M_k(\boldsymbol{\theta}) \right) \left(\widehat{M}_k - 2 \left(C_n(\widehat{\mathbf{U}}_i) \right)^k + M_k(\boldsymbol{\theta}) \right) \right| \\ &\leq \left| \widehat{M}_k - M_k(\boldsymbol{\theta}) \right| \left| \frac{1}{n} \sum_{i=1}^n \left(\widehat{M}_k - \left(C_n(\widehat{\mathbf{U}}_i) \right)^k \right) \right| \\ &\quad + \left| \frac{1}{n} \sum_{i=1}^n \left(M_k(\boldsymbol{\theta}) - \left(C_n(\widehat{\mathbf{U}}_i) \right)^k \right) \right| \\ &\leq \left(\widehat{M}_k - M_k(\boldsymbol{\theta}) \right)^2 = o_p(1). \end{aligned}$$

The proof of [Lemma 5.2](#) is completed. □

As showing in [19]; for $k = 1, 2, \dots, r$,

$$\lambda_k = O_p(n^{-1/2}).$$

Now applying Taylor's expansion to $\mathcal{L}(\boldsymbol{\theta})$, we have

$$\mathcal{L}(\boldsymbol{\theta}) \simeq 2 \left(\sum_{i=1}^n \left(\lambda_k \widehat{L}_{i,n}^{(k)} - \frac{1}{2} \left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2 \right) \right) \quad (5.14)$$

Note that from (2.11), for $k = 1, 2, \dots, r$

$$\begin{aligned} \sum_{i=1}^n \frac{\widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} &= 0 = \sum_{i=1}^n \widehat{L}_{i,n}^{(k)} \left(1 - \lambda_k \widehat{L}_{i,n}^{(k)} + \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} \right) \\ &= \sum_{i=1}^n \widehat{L}_{i,n}^{(k)} - \lambda_k \sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} \right)^2 + \sum_{i=1}^n \frac{\lambda_k^2 \left(\widehat{L}_{i,n}^{(k)} \right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} \end{aligned} \quad (5.15)$$

From, Lemma 5.2 it follows that for $k = 1, 2, \dots, r$

$$\lambda_k = \left(\sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} \right)^2 \right)^{-1} \sum_{i=1}^n \widehat{L}_{i,n}^{(k)} + o_p(n^{-1/2}). \quad (5.16)$$

By (5.15) we get

$$\sum_{i=1}^n \frac{\lambda_k \widehat{L}_{i,n}^{(k)}}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = \sum_{i=1}^n \lambda_k \widehat{L}_{i,n}^{(k)} - \sum_{i=1}^n \left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2 + \sum_{i=1}^n \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = 0$$

Note that

$$\sum_{i=1}^n \frac{\left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^3}{1 + \lambda_k \widehat{L}_{i,n}^{(k)}} = O_p(n^{-1/2})$$

then

$$\sum_{i=1}^n \lambda_k \widehat{L}_{i,n}^{(k)} = \sum_{i=1}^n \left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2 + o_p(1),$$

Therefore, it follows from (5.14) and Lemmas 5.1 and (5.16) that

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}) &= 2 \left(\sum_{i=1}^n \left(\lambda_k \widehat{L}_{i,n}^{(k)} - \frac{1}{2} \left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2 \right) \right) \\ &= \left(\sum_{i=1}^n \left(\lambda_k \widehat{L}_{i,n}^{(k)} \right)^2 \right) + o_p(1) \\ &= \left(\sum_{i=1}^n \left(\widehat{L}_{i,n}^{(k)} \right)^2 \right)^{-1} \left(\sum_{i=1}^n \widehat{L}_{i,n}^{(k)} \right)^2 + o_p(1) \\ &\rightarrow \chi_1^2. \end{aligned}$$

The proof of Theorem 2.2 is completed.

References

1. Genest C. Frank's family of bivariate distributions. *Biometrika*, 1987; 74: 549-55.
2. Oakes D. A model for association in bivariate survival data. *J Roy Statist Soc Ser B*, 1982; 44(3): 414-22.
3. Genest C, Ghoudi K, Rivest LP. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 1995; 82: 543-52.
4. Joe H. *Multivariate models and dependence concepts*, London: Chapman & Hall; 1997.
5. Joe H. Asymptotic efficiency of the two-stage estimation method for copula-based models. *J Multivariate Anal*, 2005; 94: 401-19.
6. Tsukahara H. Semiparametric estimation in copula models. *Canad J Statist*, 2005; 33: 357-75.
7. Brahimi B, Necir A. A semi-parametric estimation of copula models based on the method of moments. *Stat Method*, 2012; 9(4): 467-77.
8. Brahimi B, Chebana F, Necir A. Copula representation of bivariate L-moments: a new estimation method for multiparameter two-dimensional copula models. *Statistics*, 2015; 49(3): 497-521. doi: [10.1080/02331888.2014.932792](https://doi.org/10.1080/02331888.2014.932792).
9. Owen AB. Empirical likelihood ratio confidence intervals for a single functional. *Biometrika*, 1988; 75: 237-49.
10. Owen AB. Empirical likelihood confidence regions. *Ann Statist*, 1990; 18: 90-120.
11. Owen, A.B. *Empirical likelihood*, Boca Raton, London, New York, NY, Washington: Chapman and Hall/CRC, 2001.
12. Newey WK, Smith RJ. Higher order properties of GMM and generalized empirical likelihood estimators. *Econometrica*, 2004; 72: 219-55.
13. Qin J, Lawless, J. Empirical likelihood and general estimating equations. *Ann Statist* , 1994; 22: 300-25.
14. Kitamura Y. Empirical likelihood methods with weakly dependent processes. *Ann Statist*, 1997; 25: 2084-102.
15. Monti AC. Empirical likelihood confidence regions in time series models. *Biometrika*, 1997; 80: 329-38.
16. Ogata H, Taniguchi M. An empirical likelihood approach for non-Gaussian vector stationary processes and its application to minimum contrast estimation. *Aust N Z J Stat*, 2010; 52: 451-68.
17. Nelsen RB. *An introduction to copulas*, 2nd ed., New York, NY: Springer; 2006.
18. Deheuvels P. La fonction de dépendance empirique et ses propriétés. *Acad Roy Belg Bull Cl Sci*, 1979; 65: 274-92.
19. Owen, A. Empirical likelihood for linear models. *Ann Statist*, 1991; 19(4): 1725-47.

Corresponding author

Brahim Brahimi can be contacted at: brah.brahim@gmail.com

For instructions on how to order reprints of this article, please visit our website:

www.emeraldgroupublishing.com/licensing/reprints.htm

Or contact us for further details: permissions@emeraldinsight.com