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Periodic solutions for a class of fifth-order differential equations

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Received 18 July 2020
Revised 19 November 2020
30 November 2020
8 December 2020
Accepted 12 December 2020

Abstract

Purpose – This study aims to provide sufficient conditions for the existence of periodic solutions of the fifth-order differential equation.

Design/methodology/approach – The authors shall use the averaging theory, more precisely Theorem \$6\$.

Findings – The main results on the periodic solutions of the fifth-order differential equation (equation (1)) are given in the statement of Theorem 1 and 2.

Originality/value – In this article, the authors provide sufficient conditions for the existence of periodic solutions of the fifth-order differential equation.

Keywords Periodic orbit, Fifth-order differential equation, Averaging theory

Paper type Research paper

1. Introduction and statement of the main results

One of the main problems in the theory of differential equations is the study of their periodic orbits, their existence, their number and their stability. The goal of this paper is to study the periodic solutions of the fifth-order non-autonomous differential equation:

$$x^{(5)} - \lambda \ddot{\ddot{x}} + (p^2+1)\dddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = \varepsilon F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddot{\ddot{x}}\right), \quad (1)$$

where λ and ε are real parameters; p is a rational number different from $-1, 0, 1$, ε is sufficiently small; and F is a nonlinear non-autonomous periodic function.

There are many papers studying the periodic orbits of fifth-order differential equations, see for instance in Refs. [1–6]. But, our main tool for studying the periodic orbits of equation (1) is completely different from the tools mentioned papers, and consequently, the results obtained seem distinct and new. We shall use the averaging theory, more precisely Theorem 5. Many of the quoted papers dealing with the periodic orbits of fifth-order differential equations use Schauder's or Leray-Schauder's fixed point theorem, the non-local reduction method or variational methods. In Refs. [7–9], the authors studied the limit cycles of the fourth-, sixth- and eighth-order non-autonomous differential equations.

In general, to obtain analytically periodic solutions of a differential system is a very difficult task, usually impossible. Here, with the averaging theory, this difficult problem for the differential equation (1) is reduced to find the zeros of a nonlinear function. We must say that the averaging theory for finding periodic solutions in general does not provide all the periodic solutions of the system. For more information about the averaging theory, see Section 2 and the references quoted there.

JEL Classification — 37G15, 37C80, 37C30.

Arab Journal of Mathematical Sciences
Vol. 29 No. 1, 2023
pp. 2-13
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-07-2020-0024

Our main results on the periodic solutions of the fifth-order differential equation (1) are the following.

Theorem 1. *Assume that $p = m/n$ is a rational different from $-1, 0, 1$, $\lambda \neq 0$ in differential equation* (1). *Let*

$$\begin{aligned}
\mathcal{F}_1(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi n}\int_0^{2\pi n} \cos(t)F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_2(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi n}\int_0^{2\pi n} \sin(t)F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_3(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi n}\int_0^{2\pi n} \cos(pt)F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_4(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi n}\int_0^{2\pi n} \sin(pt)F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,
\end{aligned} \tag{2}$$

where m, n are positive integers, and

$$\begin{aligned}
\mathcal{A} &= -\frac{(X_0+\lambda Y_0)\cos t + (\lambda X_0 - Y_0)\sin t}{(p^2-1)(\lambda^2+1)} + \frac{(pZ_0+\lambda U_0)\cos(pt) - (pU_0 - \lambda Z_0)\sin(pt)}{p(p^2-1)(\lambda^2+p^2)}\\
\mathcal{B} &= \frac{-(\lambda X_0 - Y_0)\cos t + (X_0+\lambda Y_0)\sin t}{(p^2-1)(\lambda^2+1)} - \frac{(pU_0 - \lambda Z_0)\cos(pt) + (\lambda U_0 + pZ_0)\sin(pt)}{(p^2-1)(\lambda^2+p^2)}\\
\mathcal{C} &= \frac{(X_0+\lambda Y_0)\cos t + (\lambda X_0 - Y_0)\sin t}{(p^2-1)(\lambda^2-1)} + \frac{-(pZ_0+\lambda U_0)p\cos(pt) + (pU_0 - \lambda Z_0)p\sin(pt)}{(p^2-1)(\lambda^2+p^2)}\\
\mathcal{D} &= \frac{(\lambda X_0 - Y_0)\cos t - (X_0+\lambda Y_0)\sin t}{(p^2-1)(\lambda^2+1)} + \frac{(pU_0 - \lambda Z_0)p^2\cos(pt) + (\lambda U_0 + pZ_0)p^2\sin(pt)}{(p^2-1)(\lambda^2+p^2)}\\
\mathcal{J} &= -\frac{(X_0+\lambda Y_0)\cos t + (\lambda X_0 - Y_0)\sin t}{(p^2-1)(\lambda^2-1)} + \frac{(pZ_0+\lambda U_0)p^3\cos(pt) - (pU_0 - \lambda Z_0)p^3\sin(pt)}{(p^2-1)(\lambda^2+p^2)},
\end{aligned} \tag{3}$$

If the function F is $2\pi n-$periodic with respect to the variable t, then for every $\left(X_0^*, Y_0^*, Z_0^*, U_0^*\right)$ solution of the system:

$$\mathcal{F}_k(X_0, Y_0, Z_0, U_0) = 0,\ k = 1, ..., 4, \tag{4}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(X_0, Y_0, Z_0, U_0)}\right)\Big|_{(X_0,Y_0,Z_0,U_0)=\left(X_0^*,Y_0^*,Z_0^*,U_0^*\right)} \neq 0, \tag{5}$$

the differential equation (1) *has a periodic solution $x(t, \varepsilon)$ tending to the solution $x_0(t)$ given by:*

$$-\frac{\left(X_0^*+\lambda Y_0^*\right)\cos(t) + \left(\lambda X_0^* - Y_0^*\right)\sin(t)}{(p^2-1)(\lambda^2+1)} + \frac{\left(pZ_0^*+\lambda U_0^*\right)\cos(pt) - \left(pU_0^* - \lambda Z_0^*\right)\sin(pt)}{p(p^2-1)(\lambda^2+p^2)}$$

of $x^{(5)} - \lambda \dddot{x} + (p^2+1)\ddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$ when $\varepsilon \to 0$. *Note that this solution is periodic of period* $2\pi n$.

Theorem 1 is proved in Section 3. Its proof is based on the averaging theory for computing periodic orbits, see Section 2.

An application of Theorem 1 is the following.

Corollary 2. *If* $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = (1+\cos t)(ax^2 + b\dot{x})$ *with* $a \cdot b \neq 0$, *then the differential equation (1) with* $p = \frac{1}{2}$, $\lambda = 2$ *has one periodic solution* $x_2(t, \varepsilon)$ *tending to the periodic solution* $x_2(t)$ *given by:*

$$x_2(t) = -\frac{2b}{a}\sin(t),$$

of $x^{(5)} - 2\dddot{x} + \frac{5}{4}\ddot{x} - \frac{5}{2}\ddot{x} + \frac{1}{4}\dot{x} - \frac{1}{2}x = 0$ when $\varepsilon \to 0$.

Corollary 2 is proved in Section 5.

Theorem 3. *Assume that* $p = m/n$ *is a rational different from* $-1, 0, 1$, $\lambda = 0$ *in differential equation (1). Let*

$$\begin{aligned}
\mathcal{F}_1(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi n}\int_0^{2\pi n} \cos(t) F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_2(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi n}\int_0^{2\pi n} \sin(t) F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_3(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi n}\int_0^{2\pi n} \cos(pt) F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_4(X_0, Y_0, Z_0, U_0) &= -\frac{1}{2\pi n}\int_0^{2\pi n} \sin(pt) F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,\\
\mathcal{F}_5(X_0, Y_0, Z_0, U_0) &= \frac{1}{2\pi n}\int_0^{2\pi n} F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J})dt,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\mathcal{A} &= \frac{-(X_0\cos t - Y_0 \sin t)p^2 + Z_0\cos(pt) - U_0\sin(pt) + \left(p^2-1\right)V_0}{p^2\left(p^2-1\right)},\\
\mathcal{B} &= \frac{(Y_0\cos t + X_0\sin t)p - U_0\cos(pt) - Z_0\sin(pt)}{p(p^2-1)},\\
\mathcal{C} &= \frac{X_0\cos t - Y_0\sin t - Z_0\cos(pt) + U_0\sin(pt)}{p^2-1},\\
\mathcal{D} &= \frac{-Y_0\cos t - X_0\sin t + p(U_0\cos(pt) + Z_0\sin(pt))}{p^2-1},\\
\mathcal{J} &= \frac{-X_0\cos t + Y_0\sin t + p^2(Z_0\cos(pt) - U_0\sin(pt))}{(p^2-1)},
\end{aligned} \tag{7}$$

If the function F *is* $2\pi n-$*periodic with respect to the variable* t, *then for every* $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*\right)$ *solution of the system:*

$$\mathcal{F}_k(X_0, Y_0, Z_0, U_0, V_0) = 0,\ k = 1, \ldots, 5, \tag{8}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(X_0, Y_0, Z_0, U_0, V_0)}\right)\Big|_{(X_0,Y_0,Z_0,U_0,V_0)=\left(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*\right)} \neq 0, \tag{9}$$

the differential equation (1) *has a periodic solution* $x(t, \varepsilon)$ *tending to the solution* $x_0(t)$ *given by:*

$$\frac{-\left(X_0^* \cos t - Y_0^* \sin t\right)p^2 + Z_0^* \cos(pt) - U_0^* \sin(pt) + (p^2 - 1) V_0^*}{p^2(p^2 - 1)} \tag{10}$$

of $x^{(5)} - \lambda \ddddot{x} + (p^2 + 1)\dddot{x} - \lambda(p^2 + 1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$ *when* $\varepsilon \to 0$. *Note that this solution is periodic of period* $2\pi n$.

Theorem 5 is proved in Section 4. Its proof is based on the averaging theory for computing periodic orbits, see Section 2. An application of Theorem 3 is given in the following corollary:

Corollary 4. *If* $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = (2x^2 - \dot{x}2 + x - 2\dot{x})\sin t$ *then the differential Eqn (1) with* $p = \frac{1}{2}, \lambda = 0$ *has six periodic solutions* $x_k(t, \varepsilon)$ *for* $k = 1, \ldots, 6$ *tending to the periodic solutions:*

$$x_1(t) = -\frac{1}{4}\sin t - \frac{1}{28}\sqrt{42}\cos\left(\frac{1}{2}t\right) - \frac{1}{4},\ x_2(t) = -\frac{1}{4}\sin t + \frac{1}{28}\sqrt{42}\cos\left(\frac{1}{2}t\right) - \frac{1}{4},$$

$$x_3(t) = \frac{1}{4}\sin t + \frac{1}{28}\sqrt{42}\sin\left(\frac{1}{2}t\right) - \frac{1}{4},\ x_4(t) = \frac{1}{4}\sin t - \frac{1}{28}\sqrt{42}\sin\left(\frac{1}{2}t\right) - \frac{1}{4},$$

$$x_5(t) = -\frac{1}{10}\sqrt{10}\sin t - \frac{1}{4},\ x_6(t) = \frac{1}{10}\sqrt{10}\sin t - \frac{1}{4},$$

of $x^{(5)} + \frac{5}{4}\dddot{x} + \frac{1}{4}\dot{x} = 0$ *when* $\varepsilon \to 0$.

Corollary 4 is proved in Section 5.

2. Basic results on the averaging theory

In this section, we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form:

$$\dot{\mathrm{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{11}$$

with $\varepsilon > 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \mathbf{\Omega} \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \mathbf{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the variable t, and $\mathbf{\Omega}$ is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system:

$$\dot{\mathrm{x}} = F_0(t, \mathbf{x}), \tag{12}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (12) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as:

$$\dot{\mathrm{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{13}$$

In what follows, we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (13), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates, i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We assume that there exists a k-dimensional submanifold $\mathcal{Z}$ of Ω filled with T-periodic solutions of (12). Then, an answer to the problem of bifurcation of T-periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (11) is given in the following result.

Theorem 5. *Let W be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{CL}(W) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that:*

(1) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \quad \alpha \in \mathrm{CL}(W)\} \subset \Omega$, and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$, the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (12) is T-periodic;*

(2) *For each $\mathbf{z}_\alpha \in \mathcal{Z}$, there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (13) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the upper-right corner the $k \times (n-k)$ zero matrix, and in the lower-right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{CL}(W) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right). \tag{14}$$

If there exists $a \in W$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\alpha)(a)) \neq 0$, then there is a T-periodic solution $\varphi(t, \varepsilon)$ of system (11) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 5 goes back to Malkin [10] and Roseau [11]; for a shorter proof, see Ref. [12].

We assume that there exists an open set V with $CL(V) \subset \Omega$ such that for each $\mathbf{z} \in CL(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T-periodic, where $\mathbf{x}(t, \mathbf{z}, 0)$ denotes the solution of the unperturbed system (12) with $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$. The set $CL(V)$ is *isochronous* for the system (11), i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $CL(V)$ is given in the following result.

Theorem 6. [Perturbations of an isochronous set] *We assume that there exists an open and bounded set V with $CL(V) \subset \Omega$ such that for each $\mathbf{z} \in CL(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T-periodic, then we consider the function $\mathcal{F} : \mathrm{CL}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M^{-1}_{\mathbf{z}}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}))dt. \tag{15}$$

If there exists $a \in V$ with $\det((d\mathcal{F}/d\alpha)(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t, \varepsilon)$ of system (11) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

For a shorter proof of Theorem 6, see Corollary 1 of [12]. In fact, this result goes back to Malkin [10] and Roseau [11].

3. Proof of Theorem 1

If $y = \dot{x}, z = \ddot{x}, u = \dddot{x}, v = \ddddot{x}$, then system (1) can be written as:

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= u,\\
\dot{u} &= v,\\
\dot{v} &= \lambda p^2 x - p^2 y + \lambda(p^2+1)z - (p^2+1)u + \lambda v + \varepsilon F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}\right),
\end{aligned} \tag{16}$$

The unperturbed system has a unique singular point, the origin. The eigenvalues of the linearized system at this singular point are $\pm i$, $\pm pi$ and λ. By the linear invertible transformation:

$$(X, Y, Z, U, V)^T = B(x, y, z, u, v)^T, \tag{17}$$

where

$$B = \begin{pmatrix} 0 & -\lambda p^2 & p^2 & -\lambda & 1 \\ -\lambda p^2 & p^2 & -\lambda & 1 & 0 \\ 0 & -\lambda & 1 & -\lambda & 1 \\ -\lambda p & p & -\lambda p & p & 0 \\ p^2 & 0 & p^2+1 & 0 & 1 \end{pmatrix},$$

we transform the system (16) such that its linear part is real Jordan normal form of the linear part of system (16) with $\varepsilon = 0$, i.e.:

$$\begin{cases} \dot{X} = -Y + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{Y} = X, \\ \dot{Z} = -pU + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{U} = pZ, \\ \dot{V} = \lambda V + \varepsilon G(t, X, Y, Z, U, V), \end{cases} \tag{18}$$

where

$$G = F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J}) = G(t, X, Y, Z, U, V),$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$,and $\mathcal{J}$ as in the statement of Theorem 1.

Note that the linear part of the differential system (18) at the origin is in its real Jordan normal form, and that the change of variables (17) is defined when p is a rational different from -1, $0, 1$, because the determinant of the matrix of the change is $p(p^2-1)^2(\lambda^2+1)(\lambda^2+p^2)$.

We shall apply Theorem 5 to the differential system (18). We note that system (18) can be written as system (11) taking

$$x = \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix}, F_0(x,t) = \begin{pmatrix} -Y \\ X \\ -pU \\ pZ \\ \lambda V \end{pmatrix}, F_1(x,t) = \begin{pmatrix} G \\ 0 \\ G \\ 0 \\ G \end{pmatrix}.$$

We shall study the periodic solutions of system (18) in our case, i.e. the periodic solutions of system (18) with $\varepsilon = 0$. These periodic solutions are:

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \\ U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} X_0 \cos(t) - Y_0 \sin(t) \\ Y_0 \cos(t) + X_0 \sin(t) \\ Z_0 \cos(pt) - U_0 \sin(pt) \\ U_0 \cos(pt) + Z_0 \sin(pt) \\ 0 \end{pmatrix}.$$

This set of periodic orbits has dimension four, all having the same period $2\pi n$, where n is defined in the statement of Theorem 1. To look for the periodic solutions of our equation (1) we must calculate the zeros $z = (X_0, Y_0, Z_0, U_0, V_0)$ of the system $\mathcal{F}(z) = 0$, where $\mathcal{F}(z)$ is given by (14). The fundamental matrix $M(t)$ of the differential system (18) with $\varepsilon = 0$, along any periodic solution is:

$$M(t) = M_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) & 0 \\ 0 & 0 & \sin(pt) & \cos(pt) & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

The inverse matrix of $M(t)$ is:

$$MI(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & \sin(pt) & 0 \\ 0 & 0 & -\sin(pt) & \cos(pt) & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda t} \end{pmatrix}.$$

Moreover, an easy computation shows that:

$$MI(0) - MI(2\pi n) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{-2\pi n\lambda} \end{pmatrix}.$$

We obtain $(1 - \exp(-2\pi n\lambda)) \neq 0$, because $\lambda \neq 0$. Consequently, all the assumptions of Theorem 5 are satisfied. Therefore, we must study the zeros in W of the system $\mathcal{F}(\mathbf{z}) = 0$ of four equations with four unknowns, where W and $\mathcal{F}$ are given in the statement of Theorem 5. More precisely, we have $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(z), \mathcal{F}_2(z), \mathcal{F}_3(z), \mathcal{F}_4(z))$, such that $z = (X_0, Y_0, Z_0, U_0)$, where the functions $\mathcal{F}_1$, $\mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ are the ones given in (2). The zeros $\left(X_0^*, Y_0^*, Z_0^*, U_0^*\right)$ of system (4) with respect to the variables X_0, Y_0, Z_0 and U_0 provide periodic orbits of system (18) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. if the condition (5) is satisfied. Going back through the change of variables, for every simple zero $\left(X_0^*, Y_0^*, Z_0^*, U_0^*\right) \in \mathbb{R}^4 - \{(0,0,0,0)\}$ of system (4), we obtain a $2\pi n$ periodic solution $x(t)$ of the differential equation (1) for $\varepsilon \neq 0$ sufficiently small such that $x(t)$ tends to the periodic solution, where $x(t)$ is defined in the statement of Theorem 1, of $x^{(5)} - \lambda \ddddot{x} + (p^2+1)\dddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$ when $\varepsilon \to 0$. Note that this solution is periodic of period $2\pi n$. This completes the proof of Theorem 1.

4. Proof of Theorem 3

We want to study the periodic orbits of the class of fifth-order differential equation:

$$x^{(5)} + (p^2+1)\dddot{x} + p^2\dot{x} = \varepsilon F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}\right). \tag{19}$$

This is the case of equation (1) when $\lambda = 0$, and p is a rational number different from $-1, 0, 1$.

If $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$, $v = \ddddot{x}$, we write the fifth-order differential equation (19) as the following first-order differential system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= u, \\ \dot{u} &= v, \\ \dot{v} &= -p^2 y - \left(p^2 + 1\right)u + \varepsilon F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}\right), \end{aligned} \tag{20}$$

The unperturbed system has a unique singular point, the origin. The eigenvalues of the linearized system at this singular point are $\pm i$, $\pm pi$ and 0. By the linear invertible transformation:

$$(X, Y, Z, U, V)^T = B(x, y, z, u, v)^T,$$

where

$$B = \begin{pmatrix} 0 & 0 & p^2 & 0 & 1 \\ 0 & p^2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & p & 0 & p & 0 \\ p^2 & 0 & p^2 + 1 & 0 & 1 \end{pmatrix},$$

we transform the system (20) such that its linear part is real Jordan normal form of the linear part of system (20) with $\varepsilon = 0$, i.e.:

$$\begin{cases} \dot{X} = -Y + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{Y} = X, \\ \dot{Z} = -pU + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{U} = pZ, \\ \dot{V} = \varepsilon G(t, X, Y, Z, U, V), \end{cases} \tag{21}$$

where

$$G = F(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{J}) = G(t, X, Y, Z, U, V),$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$,and $\mathcal{J}$ as in the statement of Theorem 3.

Note that the linear part of the differential system (21) at the origin is in its real Jordan normal form. We shall apply Theorem 6 to the differential system (21). We note that system (21) can be written as system (11) taking

$$x = \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix}, F_0(x, t) = \begin{pmatrix} -Y \\ X \\ -pU \\ pZ \\ 0 \end{pmatrix}, F_1(x, t) = \begin{pmatrix} G \\ 0 \\ G \\ 0 \\ G \end{pmatrix}.$$

We shall study the periodic solutions of system (21) in our case, i.e. the periodic solutions of system (21) with $\varepsilon = 0$. These periodic solutions are:

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \\ U(t) \\ V(t) \end{pmatrix} = \begin{pmatrix} X_0 \cos t - Y_0 \sin t \\ Y_0 \cos t + X_0 \sin t \\ Z_0 \cos(pt) - U_0 \sin(pt) \\ U_0 \cos(pt) + Z_0 \sin(pt) \\ V_0 \end{pmatrix}.$$

This set of periodic orbits has dimension five, all having the same period $2\pi n$, where n is defined in the statement of Theorem 3. To look for the periodic solutions of our equation (19), we must calculate the zeros $z = (X_0, Y_0, Z_0, U_0, V_0)$ of the system $\mathcal{F}(z) = 0$, where $\mathcal{F}(z)$ is given by (15). The fundamental matrix $M(t)$ of the differential system (21) with $\varepsilon = 0$, along any periodic solution is

$$M(t) = M_z(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 & 0 \\ \sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) & 0 \\ 0 & 0 & \sin(pt) & \cos(pt) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse matrix of $M(t)$ is:

$$MI(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & \sin(pt) & 0 \\ 0 & 0 & -\sin(pt) & \cos(pt) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now computing the function $\mathcal{F}(z)$ given in (15), we got that the system $\mathcal{F}(z) = 0$, can be written as system (8) with the function $\mathcal{F}_k(X_0, Y_0, Z_0, U_0, V_0)$ given in (6). The zeros $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*\right)$ of system (8) with respect to the variables X_0, Y_0, Z_0, U_0, and V_0, provide periodic orbits of system (21) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. if (9) holds. Going back through the change of variables, for every simple zero $X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*$ of system (8), we obtain a $2\pi n$ periodic solution $x(t)$ of the differential equation (1) for $\varepsilon \neq 0$ sufficiently small such that $x(t)$ tends to the periodic solution (10) of $x^{(5)} + (p^2+1)\dddot{x} + p^2\dot{x} = 0$ when $\varepsilon \to 0$. Note that this solution is periodic of period $2\pi n$. This completes the proof of Theorem 3.

5. Proof of Corollaries 2 and 4

Proof of Corollary 2. Consider the function

$$F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}\right) = \left(ax^2 + b\dot{x}\right)(1 + \cos t),$$

which corresponds to the case $p = \frac{1}{2}$, $\lambda = 2$. The functions $\mathcal{F}_i = \mathcal{F}_i(X_0, Y_0, Z_0, U_0)$ for $i = 1, \ldots, 4$ of Theorem 1 are:

$$\begin{aligned}\mathcal{F}_1 &= \frac{2048}{2601}aU_0^2 - \frac{1024}{2601}aU_0Z_0 + \frac{14}{225}aX_0^2 + \frac{16}{225}aX_0Y_0 + \frac{26}{225}aY_0^2 \\ &+ \frac{128}{2601}aZ_0^2 + \frac{4}{15}bX_0 - \frac{2}{15}bY_0, \\ \mathcal{F}_2 &= \frac{2}{15}bX_0 + \frac{4}{15}bY_0 + \frac{512}{2601}aU_0^2 - \frac{8}{225}aX_0^2 + \frac{8}{225}aY_0^2 - \frac{512}{2601}aZ_0^2 \\ &- \frac{640}{867}aU_0Z_0 - \frac{4}{75}aX_0Y_0, \\ \mathcal{F}_3 &= \frac{2}{17}bU_0 - \frac{8}{17}bZ_0 - \frac{64}{255}aU_0X_0 - \frac{32}{45}aU_0Y_0 - \frac{64}{153}aX_0Z_0, \\ \mathcal{F}_4 &= -\frac{32}{765}aY_0Z_0 - \frac{8}{51}bU_0 - \frac{2}{51}bZ_0 + \frac{256}{765}aU_0X_0 - \frac{128}{765}aU_0Y_0 + \frac{64}{765}aX_0Z_0,\end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$ has only real solution:

$$\left(X_0^*, Y_0^*, Z_0^*, U_0^*\right) = \left(-\frac{6b}{a}, \frac{3b}{a}, 0, 0\right).$$

Since the Jacobian

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(X_0, Y_0, Z_0, U_0)}\right)\Big|_{(X_0,Y_0,Z_0,U_0)=\left(X_0^*,Y_0^*,Z_0^*,U_0^*\right)} = \frac{208b^4}{135} \neq 0, b \neq 0$$

by Theorem 1 equation (1) has the periodic solution of the statement of the corollary. □

Proof of Corollary 4. Consider the function:

$$F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}\right) = \left(2x^2 - \dot{x}2 + x - 2\dot{x}\right)\sin t,$$

which corresponds to the case $p = \frac{1}{2}$, $\lambda = 0$. The functions $\mathcal{F}_i = \mathcal{F}_i(X_0, Y_0, Z_0, U_0, V_0)$ for $i = 1, \ldots, 5$ of Theorem 2 are:

$$\begin{aligned}\mathcal{F}_1 &= -\frac{4}{3}X_0Y_0, \\ \mathcal{F}_2 &= -\frac{112}{9}U_0^2 - 16V_0^2 + \frac{2}{9}X_0^2 - \frac{10}{9}Y_0^2 - \frac{112}{9}Z_0^2 - 2V_0, \\ \mathcal{F}_3 &= \frac{4}{3}U_0 - \frac{4}{3}Z_0 + \frac{64}{3}U_0V_0 + \frac{16}{9}X_0U_0 + \frac{64}{9}Y_0Z_0, \\ \mathcal{F}_4 &= \frac{4}{3}U_0 + \frac{4}{3}Z_0 + \frac{64}{9}Y_0U_0 + \frac{64}{3}V_0Z_0 - \frac{16}{9}X_0Z_0, \\ \mathcal{F}_5 &= \frac{4}{3}X_0 - \frac{2}{3}Y_0 - 32\,U_0Z_0 - \frac{32}{3}V_0Y_0.\end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = 0$ has the six solutions $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*\right)$ given by:

$$\left(0, \frac{3}{16}, \frac{3}{448}\sqrt{42}, 0, -\frac{1}{16}\right), \left(0, \frac{3}{16}, -\frac{3}{448}\sqrt{42}, 0, -\frac{1}{16}\right), \left(0, -\frac{3}{16}, 0, \frac{3}{448}\sqrt{42}, -\frac{1}{16}\right)$$
$$\left(0, -\frac{3}{16}, 0, -\frac{3}{448}\sqrt{42}, -\frac{1}{16}\right), \left(0, \frac{3}{40}\sqrt{10}, 0, 0, -\frac{1}{16}\right), \left(0, -\frac{3}{40}\sqrt{10}, 0, 0, -\frac{1}{16}\right).$$

Since the Jacobian:

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5)}{\partial(X_0, Y_0, Z_0, U_0, V_0)}\right)\Big|_{(X_0,Y_0,Z_0,U_0,V_0)=\left(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*\right)},$$

for six solutions $(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*)$ is:

$$\frac{85}{252}, \frac{85}{252}, -\frac{85}{252}, -\frac{85}{252}, -\frac{32}{225}\sqrt{10}, -\frac{32}{225}\sqrt{10},$$

Respectively, we obtain using Theorem 3 the ten solutions given in statement of the corollary. □

References

1. El-Gamel M. Sinc and the numerical solution of fifth-order boundary value problems. Appl Math Comput. 2007; 187(2): 1417-1433.
2. Ezeilo JOC. A further instability theorem for a certain fifth-order differential equation. Math Proc Camb Phil Soc. 1979; 86: 491-493.
3. Odda SN. Existence solution for 5th order differential equations under some conditions. Appl Math. 2010; 1(4): 279-282.
4. Sellami N, Makhlouf A. Limit cycles for a class of fifth-order differential equations. Ann of Diff Eqs. 2012; 28(2): 202-219.
5. Tiryaki A. On the periodic solutions of certain fourth and fifth order differential equations. Pure Appl Math Sci. 1990; 32(1–2): 11-14.
6. Tunç C. On the periodic solutions of certain fourth and fifth order vector differential equations. Math Commun. 2005; 10(2): 135-141.

Further reading

7. Berhail CE, Bouslah Z, Makhlouf A. On the limit cycles for a class of eighth-order differential equations. Moroccan J Pure Appl Anal. 2020; 6(1): 53-61.
8. Llibre J, Makhlouf A. Periodic orbits of the fourth-order non-autonomous differential equation $\ddddot{u} + q\ddot{u} + pu = \epsilon F(t, u, \dot{u}, \ddot{u}, \dddot{u})$. Appl Math Comput. 2012; 219: 827-836.
9. Makhlouf A, Berhail CE. Limit cycles of the sixth-order non-autonomous differential equation. Arab J Math Sci. 2012; 18: 177-187. doi: 10.1016/j.ajmsc.2012.03.003.
10. Malkin IG. Some Problems of the theory of nonlinear oscillations. Moscow: Gosudarstv. Izdat. Tehn-Teor. Lit.; 1956. (in Russian).
11. Roseau M. Vibrations non linéaires et théorie de la stabilité, Springer Tracts in Natural Philosophy. New York: Springer; 1985; 8.
12. Buica A, Françoise JP, Llibre J. Periodic solutions of nonlinear periodic differential systems with a small parameter. Commun Pure Appl Anal. 2006; 6: 103-111.

13. Sanders JA, Verhulst F. Averaging methods in nonlinear dynamical systems, applied mathematical sciences. New York: Springer; 1985; 59.
14. Verhulst F. Nonlinear differential equations and dynamical systems, universitext. New York: Springer; 1996.

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