

Screen Cauchy–Riemann (SCR)-lightlike submanifolds of metallic semi-Riemannian manifolds

Arab Journal of
Mathematical
Sciences

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Received 12 November 2021
Revised 2 February 2022
9 June 2022
21 May 2023
16 January 2024
27 March 2024
Accepted 28 August 2024

Abstract

Purpose – The screen Cauchy–Riemann (SCR)-lightlike submanifold is an important class of submanifolds of semi-Riemannian manifolds. It contains various other classes of submanifolds as its sub-cases. It has been studied under various ambient space. The purpose of this research is to study the geometry of SCR-lightlike submanifolds of metallic semi-Riemannian manifolds.

Design/methodology/approach – The article is divided into five sections. The first section is introductory section which represents brief overview of the conducted research of this article. The second section outlines the key results that are utilized throughout the paper. In section three, the definition of SCR-lightlike submanifold is constructed with one non-trivial example. In section four and five, the important results on integrability, totally geodesic foliations and warped product are given.

Findings – The SCR-lightlike submanifold is introduced. One non-trivial example is constructed which helps to understand the given structure. The necessary and sufficient conditions for the integrability and to be totally geodesic for various distributions are obtained. The necessary and sufficient conditions for induced connection on totally umbilical SCR-lightlike submanifolds to be a metric connection are discussed. Various results are found on totally umbilical SCR-lightlike submanifolds. Finally, the existence of the warped product lightlike submanifold of the type $N^+ \times_{\lambda} N_T$ is studied.

Originality/value – SCR-lightlike submanifolds have been explored within ambient manifolds possessing various structures, such as Kaehler, Sasakian and Kenmotsu structures. In this article, we investigate this structure on submanifolds of metallic semi-Riemannian manifolds. This original and authentic research will aid researchers in advancing the study of semi-Riemannian manifolds.

Keywords Lightlike submanifolds, Totally umbilical lightlike submanifolds, Metallic semi-Riemannian manifolds, Warped product

Paper type Research paper

JEL Classification — Primary 53C15, Secondary 53C40, 53C50

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The first author is thankful to DST Government of India for providing financial support in terms of DST-FST label-I grant vide sanction number SR/FST/MS-I/2021/104(C).



Arab Journal of Mathematical
Sciences
Vol. 31 No. 2, 2025
pp. 179-196
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-11-2021-0286

1. Introduction

The study of lightlike submanifolds in semi-Riemannian manifolds presents a significantly greater level of complexity compared to study of submanifolds of Riemannian manifolds. In case of semi-Riemannian manifolds, the induced metric on submanifold has two cases, either degenerate or non-degenerate. If metric is non-degenerate, various structures of the ambient space can be induced uniquely and with same characteristic to ambient space. But this is not true for the case of degenerate metric. Also, if the submanifold has a degenerate metric then the intersection of the normal bundle and the tangent bundle is non-trivial. Due to this, we cannot study the geometry of degenerate or lightlike submanifolds by using tools of non-degenerate submanifold. The whole study of how to do calculus on lightlike submanifolds is known as lightlike geometry. The lightlike geometry is known as mathematical language of general theory of relativity. Due to this reason, it becomes a developing research topic in mathematics and physics.

In 1996, Duggal and Bejancu accumulated research on lightlike submanifolds in Ref. [1]. Later, various category of lightlike submanifolds, on the basis of action of (1,1) tensor on tangent bundle, of semi-Riemannian manifolds are studied.

The root $x = \frac{1+\sqrt{5}}{2}$ of the equation $x^2 - x - 1 = 0$ is known as a golden ratio [4]. This ratio has many applications in paintings, pictures, temples, fractals etc. Crasmeranue and Hretcanu [2] introduced golden manifold by defining the tensor ϕ on Riemannian manifold, such that $\phi^2 = \phi + 1$. The metallic means, the generalization of golden means, was introduced by Spinadel [3–5]. Later, on the basis of metallic structure, geometry of various lightlike submanifolds of golden and metallic semi-Riemannian manifold are introduced in [6–23] and [32–34]. In Ref. [24], the author introduced Cauchy–Riemann (CR)-submanifolds of Kaehler manifolds which contains invariant and totally-real submanifolds as its subcases. But it was observed that CR-lightlike submanifolds does not contain invariant and screen real lightlike submanifolds as its sub-cases. Therefore, a new class SCR-lightlike submanifolds has been introduced by Duggal and Sahin [25] which contains invariant lightlike submanifolds and screen-lightlike submanifolds as its sub-cases. Later, the concept of SCR-lightlike submanifolds has been studied in indefinite Sasakian manifolds with the name of contact SCR-lightlike submanifolds [26]. In Ref. [17], author introduced SCR-lightlike submanifolds and investigated the existence of warped product SCR-lightlike submanifolds of indefinite nearly Kahler. The metallic semi-Riemannian manifolds is an important class of semi-Riemannian manifolds, therefore, we introduce SCR-lightlike submanifolds of metallic semi-Riemannian manifolds which contains invariant lightlike submanifolds [18] and screen real lightlike submanifolds [27] of metallic semi-Riemannian manifolds as its special cases. The sections of this article are as follows:

In section 2, some basics of lightlike geometry and definition of metallic semi-Riemannian manifolds are recalled. In section 3, we introduce SCR-lightlike submanifold of metallic semi-Riemannian manifold. One example is also constructed. In section 4, we establish the necessary and sufficient conditions for the integrability of the distributions $Rad(TN)$, D' and D^\perp , and the foliations determined by the distribution $Rad(TN)$, D' and D^\perp to be totally geodesic, where $Rad(TN)$, D' and D^\perp represents radical distribution, invariant distribution and anti-invariant distribution of the tangent bundle TN . In section 5, the necessary and sufficient conditions for induced connection on totally umbilical SCR-lightlike submanifolds to be a metric connection are derived. Also, it is studied that the totally umbilical SCR-lightlike submanifolds with smooth vector fields $H^f \in \Gamma(\mu) \subseteq \Gamma(S(TM^\perp))$ cannot be the warped product lightlike submanifold of the type $N^\perp \times_{\lambda} N_T$.

2. Preliminaries

Suppose $(\tilde{N}^{m+n}, \tilde{g})$ be a semi-Riemannian manifold with constant index q ($1 \leq q \leq m+n-1$, $m, n \geq 1$), then the submanifold (N^m, g) is known as degenerate (lightlike) submanifold if the induced metric g is degenerate [28].

If there exists a smooth distribution \mathcal{N} (known as null distribution) such that $\mathcal{N}_p = TN_p \cap TN_p^\perp$ for every $p \in N$ and with rank r then N is called the r -lightlike submanifold. The null distribution \mathcal{N} is also denoted by $Rad(TN)$. Also, on the basis of non-degenerate complementary subbundles $S(TN)$ (screen distribution) and $S(TN^\perp)$ (screen transversal distribution) of $Rad(TN)$ in TN and TN^\perp respectively,

we have the following decomposition (for detail see Ref. [28])

$$T\tilde{N}|_N = TN \oplus tr(TN) = [Rad(TN) \oplus ltr(TN)] \perp S(TN) \perp S(TN^\perp) \quad (2.1)$$

where

$$tr(TN) = ltr(TN) \perp S(TN^\perp) \quad (2.2)$$

Theorem 2.1. [28] *Let (\tilde{N}, \tilde{g}) be a semi-Riemannian manifold and $(N, g, S(TN), S(TN^\perp))$ be its r -lightlike submanifold. Then, for a coordinate neighborhood u of N , there exists a vector bundle $ltr(TN)$ and a basis of $\Gamma(ltr(TN)|_u)$ contains a smooth section $\{N_i\}$ of $S(TN^\perp)|_u$ such that, for any $i, j \in \{1, 2, \dots, r\}$, where $\{\xi_i\}$ ($1 \leq i \leq r$)*

$$\tilde{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}_{ij}(N_i, N_j) = 0, \quad (2.3)$$

is a lightlike basis of $\Gamma(Rad(TN))$.

For any $X_1, X_2 \in \Gamma(TN)$ and $W \in \Gamma(tr(TN))$, the Gauss and Weingarten formula are

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h(X_1, X_2), \quad (2.4)$$

$$\tilde{\nabla}_{X_1} W = -A_W X_1 + \nabla_{X_1}^\perp W, \quad (2.5)$$

where $\{\nabla_{X_1} X_2, A_U X_1\}$ and $\{h(X_1, X_2), \nabla_{X_1}^\perp W\}$ belong to $\Gamma(TN)$ and $\Gamma(tr(TN))$ respectively, and ∇ is induced connection on N . Further (2.4) and (2.5) reduce to

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h^l(X_1, X_2) + h^s(X_1, X_2), \quad (2.6)$$

$$\tilde{\nabla}_{X_1} N = -A_N X_1 + \nabla_{X_1}^l(N) + D^s(X_1, N), \quad N \in \Gamma(ltr(TN)), \quad (2.7)$$

$$\tilde{\nabla}_{X_1} W = -A_W X_1 + \nabla_{X_1}^s(W) + D^l(X_1, W), \quad W \in \Gamma(S(TN^\perp)). \quad (2.8)$$

For the lightlike submanifold N of \tilde{N} , (2.4), (2.6) and (2.5), (2.7), (2.8) are known as Gauss equations and Weingarten equations respectively.

Also, we have the following equations [28]:

$$\tilde{g}(h^s(X_1, X_2), W) + \tilde{g}(X_1, D^l(X_2, W)) = g(A_W X_1, X_2), \quad (2.9)$$

$$\tilde{g}(h^l(X_1, X_2), \xi) + \tilde{g}(X_1, h^l(X_2, \xi)) = -g(X_1, \nabla_{X_2} \xi), \quad (2.10)$$

for any $\xi \in \Gamma(Rad(TN))$, $X_1, X_2 \in \Gamma(TN)$ and $W \in \Gamma(S(TN^\perp))$.

Since the induced connection from the ambient connection is not always a Levi-Civita connection. Then, for any $X_1, X_2, Z \in \Gamma(TN)$ and $U, V \in \Gamma(tr(TN))$, we have following formulas

$$(\nabla_{X_1} g)(X_2, Z) = \tilde{g}(h^l(X_1, X_2), Z) + \tilde{g}(h^l(X_1, Z), X_2) \tag{2.11}$$

Let us consider S a projection map on $S(TN)$ from TN . Then, we have the following equations, for any $X_1, X_2 \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$:

$$\nabla_{X_1} S Y = \nabla_{X_1}^* S X_2 + h^*(X_1, S X_2) \tag{2.12}$$

$$\nabla_{X_1} \xi = -A_{\xi}^* X_1 + \nabla_{X_1}^t(\xi), \tag{2.13}$$

where $\{h^*(X_1, P Y), \nabla_{X_1}^*(\xi)\}$ and $\{\nabla_{X_1}^* S Y, A_{\xi}^* X_1\}$ belong to $\Gamma(Rad(TN))$ and $\Gamma(S(TN))$, respectively.

See pg. 196–198 [28], for detail explanation of equations (2.4)-(2.13).

Definition 2.1. [18] A smooth semi-Riemannian manifold (\tilde{N}, \tilde{g}) with (1,1) tensor field \tilde{P} such that

$$\tilde{P}^2 = p\tilde{P} + qI, \tag{2.14}$$

and \tilde{g} is \tilde{P} -compatible, i.e.,

$$\tilde{g}(\tilde{P}X_1, X_2) = \tilde{g}(X_1, \tilde{P}X_2) \tag{2.15}$$

is said to be a metallic semi-Riemannian manifold.

From (2.14) in (2.15), we get

$$\tilde{g}(\tilde{P}X_1, \tilde{P}X_2) = p\tilde{g}(\tilde{P}X_1, X_2) + q\tilde{g}(X_1, X_2), \tag{2.16}$$

for any $X_1, X_2 \in \Gamma(TN)$ [2,8].

Also, if $(\tilde{\nabla}_{X_1} \tilde{P})X_2 = 0$ then \tilde{P} is called locally metallic structure. Throughout the paper, we assume that $(\tilde{\nabla}_{X_1} \tilde{P})X_2 = 0$.

3. SCR-lightlike submanifolds

Definition 3.1. A screen Cauchy-Riemann (SCR)-lightlike submanifold of a metallic semi-Riemannian manifold is a lightlike submanifold $(N, g, S(TN))$ which satisfies the following conditions:

(i) $\tilde{P}(Rad(TN)) = Rad(TN)$,

i.e., $Rad(TN)$ is invariant with respect to \tilde{P} .

(ii) There exist non-null distribution D and D^\perp of $S(TN)$ such that

$\tilde{P}(D) = D$, $\tilde{P}(D^\perp) \subset S(TN^\perp)$ and $D \cap D^\perp = \{0\}$, where D^\perp is orthogonal complementary to D in $S(TN)$.

Here, N is proper if, $D \neq \{0\}$ and $D^\perp \neq \{0\}$.

From above definition, $ltr(TN)$ is invariant w.r.t \tilde{P} and

$$TN = D' \perp D,$$

where $D' = Rad(TN)^\perp D$.

Suppose μ be the orthogonal complementary part to $\tilde{P}D^\perp$ in $S(TN^\perp)$, then

$$tr(TN) = ltr(TN) \perp \mu \perp \tilde{P}(D^\perp)$$

Example 3.1. Suppose $(\tilde{N} = \mathbb{R}_1^7, \tilde{g}, \tilde{P})$ be a 7-dimensional metallic semi-Euclidean space with semi-Euclidean metric \tilde{g} and sign $(- + + + + -)$.

Let us define, for the standard coordinate system $(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$ of \mathbb{R}_1^7 ,

$$\tilde{P}(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = (\sigma y_1, \sigma y_2, (p - \sigma)y_3, \sigma y_4, \sigma y_5, \sigma y_6, (p - \sigma)y_7).$$

Then, \tilde{P} is a metallic structure.

Suppose N is a submanifold of \tilde{N} such that

$$\begin{aligned} y_1 = y_2 = u_1, & \quad y_3 = \frac{\sigma}{\sqrt{q}}u_2, \\ y_4 = u_2, & \quad y_5 = -\sin \sigma u_3, \\ y_6 = \cos \sigma u_3 & \quad . \end{aligned}$$

from above system, we have following tangent vectors:

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, & Z_2 &= \frac{\sigma}{\sqrt{q}} \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4}, \\ Z_3 &= -\sin \sigma \frac{\partial}{\partial y_5} + \cos \sigma \frac{\partial}{\partial y_6}. \end{aligned}$$

thus,

$$\begin{aligned} Rad(TN) &= span\{Z_1 = \xi\}, & S(TN) &= span\{Z_2, Z_3\}, \\ ltr(TN) &= span\left\{N = -\frac{1}{2}\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)\right\}, \\ W &= -\sqrt{q} \frac{\partial}{\partial y_3} + \sigma \frac{\partial}{\partial y_4} \in S(TN^\perp), \end{aligned}$$

where $\tilde{P}\xi = \sigma\xi$, $\tilde{P}(N) = \sigma N$, $\tilde{P}(Z_2) \in W$ and $\tilde{P}(Z_3) = \sigma Z_3$. Here, $D' = D^\perp Rad(TN) = span\{Z_1, Z_3\}$ and $D^\perp = span\{Z_2\}$. Hence, N is SCR-lightlike submanifold.

Let Q_1, Q_2 and R be the projection maps on $Rad(TN), D$ and D^\perp , respectively.

For any $X \in \Gamma(TN)$, we have

$$X = Q_1X + Q_2X + RX \tag{3.1}$$

or

$$X = QX + RX, \tag{3.2}$$

where Q is projection map on D' such that $QX = Q_1X + Q_2X$.

Applying \tilde{P} on (3.2), we obtain

$$\tilde{P}X = \tilde{P}Q_1X + \tilde{P}Q_2X + \tilde{P}RX. \tag{3.3}$$

We write above equation as

$$\tilde{P}X = TX + wX, \tag{3.4}$$

where TX and wX are the tangential and transversal components of $\tilde{P}X$, respectively.

For any $U \in tr(TN)$, we assume

$$\tilde{P}U = BU + CU, \tag{3.5}$$

where BU and CU are tangential and transversal components of $\tilde{P}U$, respectively.

For any $X, Y \in \Gamma(TN)$, expanding the expression $(\tilde{V}_X\tilde{P})Y = 0$, by using (3.3)-(3.5), we obtain

$$\begin{aligned} -A_{TQ_1Y}^*X + \nabla_X^*TQ_1Y + h^l(X, TQ_1Y) + h^l(X, TQ_1Y) + \nabla_X TQ_2Y + h^s(X, TQ_2Y) \\ + h^s(X, TQ_2Y) - A_{wRY}X + D^l(X, wRY) + \nabla_X^*wRY = \tilde{P}Q_1\nabla_XY + \tilde{P}Q_2\nabla_XY + \tilde{P}R\nabla_XY \\ + Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y). \end{aligned} \tag{3.6}$$

By comparing components of $Rad(TN)$, D , D^\perp , $ltr(TN)$, and $S(TN^\perp)$, we get the following equations

$$\nabla_X^*TQ_1Y + Q_1\nabla_X TQ_2Y = Q_1A_{wRY}X + TQ_1\nabla_XY, \tag{3.7}$$

$$Q_2\nabla_X TQ_2Y = Q_2A_{TQ_1Y}^*X + Q_2A_{wRY}X + TQ_2\nabla_XY, \tag{3.8}$$

$$R\nabla_X TQ_2Y = RA_{TQ_1Y}^*X + RA_{wRY}X + Bh^s(X, Y), \tag{3.9}$$

$$h^l(X, TQ_1Y) + h^l(X, TQ_2Y) + D^l(X, wRY) = Ch^l(X, Y), \tag{3.10}$$

and

$$h^s(X, TQ_1Y) + h^s(X, TQ_2Y) + \nabla_X^*wRY = wR\nabla_XY + Ch^s(X, Y). \tag{3.11}$$

Lemma 3.1. *Let $(N, g, S(TN))$ be a SCR-lightlike submanifold of a metallic semi-Riemannian manifold (\tilde{N}, \tilde{P}) . Then we have*

$$(\nabla_X T)Y = A_{wY}X + Bh^s(X, Y) \tag{3.12}$$

and

$$(\nabla_X^t w)Y = -h(X, TY) + D^j(X, wY) + Ch^l(X, Y) + Ch^s(X, Y), \quad (3.13)$$

where $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$ and $(\nabla_X^t w)Y = \nabla_X^t Y - w\nabla_X Y$.

Proof. For any $X, Y \in \Gamma(TN)$, we have

$$\tilde{\nabla}_X \tilde{P}Y = \tilde{P}\tilde{\nabla}_X Y.$$

Expanding above equation by using (3.4), (3.5) and Gauss and Weingarten formula, we obtain

$$\tilde{\nabla}_X(TY + wY) = \tilde{P}\left(\nabla_X Y + h^l(X, Y) + h^s(X, Y)\right),$$

this reduces to

$$\begin{aligned} \nabla_X TY + h(X, TY) - A_{wY}X + \nabla_X^s Y + D^j(X, wY) &= T\nabla_X Y + w\nabla_X Y + \\ Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y) \end{aligned}$$

By comparing tangential and transversal components in above equation, we obtain

$$(\nabla_X T)Y = A_{wY}X + Bh^s(X, Y) \quad (3.15)$$

and

$$(\nabla_X^t w)Y = -h(X, TY) + D^j(X, wY) + Ch^l(X, Y) + Ch^s(X, Y). \quad (3.16)$$

□

4. Integrability and totally geodesic foliations

In this section, we find necessary and sufficient conditions for the distributions $Rad(TN)$, D and D' to be integrable and to define totally geodesic foliations.

Theorem 4.1. *Let $(N, g, S(TN))$ be a SCR-lightlike submanifold of a metallic semi-Riemannian manifold (N, P) . Then,*

(i) *the distribution $Rad(TN)$ is integrable if and only if,*

$$Q_2 A_{TQ_1 \xi_2}^* \xi_1 = Q_2 A_{TQ_1 \xi_1}^* \xi_2 \text{ and } h^s(\xi_1, TQ_1 \xi_2) = h^s(\xi_2, TQ_1 \xi_1).$$

(ii) *the distribution $Rad(TN)$ defines totally geodesic foliations if and only if,*

$$ph^l(\xi_1, Y) = h^l(\xi_1, TY) + D^j(\xi_1, wY).$$

Proof. (i) Let $\xi_1, \xi_2 \in Rad(TN)$, from (3.7), we obtain

$$Q_2 A_{TQ_1 \xi_1}^* \xi_2 = TQ_2 \nabla_{\xi_1} \xi_2 \quad (4.1)$$

Interchanging role of ξ_1 and ξ_2 in (4.1), we obtain

$$Q_2 A_{TQ_1 \xi_2}^* \xi_1 = TQ_2 \nabla_{\xi_2} \xi_1 \tag{4.2}$$

Subtracting (4.2) from (4.1), we obtain

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$$Q_2 A_{TQ_1 \xi_2}^* \xi_1 - Q_2 A_{TQ_1 \xi_1}^* \xi_2 = TQ_2 [\xi_1, \xi_2]. \tag{4.3}$$

Similarly, from (3.11), we obtain

$$h^s(\xi_1, TQ_1 \xi_2) = wR \nabla_{\xi_1} \xi_2 + Ch^s(\xi_1, \xi_2) \tag{4.4}$$

Interchanging role of ξ_1 and ξ_2 in (4.4) and subtract resulting equation from (4.4), we obtain

$$h^s(\xi_1, TQ_1 \xi_2) - h^s(\xi_2, TQ_1 \xi_1) = wR[\xi_1, \xi_2] \tag{4.5}$$

From (4.3) and (4.5) we obtain, $Rad(TN)$ is integrable if and only if,

$$Q_2 A_{TQ_1 \xi_2}^* \xi_1 = Q_2 A_{TQ_1 \xi_1}^* \xi_2 \text{ and } h^s(\xi_1, TQ_1 \xi_2) = h^s(\xi_2, TQ_1 \xi_1).$$

(ii) For any $\xi_1, \xi_2 \in Rad(TN)$ and $Y \in \Gamma(S(TN))$, from (2.4), we have

$$\tilde{g}(\nabla_{\xi_1} \xi_2, Y) = \tilde{g}(\tilde{\nabla}_{\xi_1} \xi_2, Y) \tag{4.6}$$

Since $\tilde{\nabla}$ is metric connection, using (2.16) in above equation, we obtain

$$\tilde{g}(\xi_2, \tilde{\nabla}_{\xi_1} Y) = -\frac{1}{q} \tilde{g}(\tilde{P} \xi_2, \tilde{P} \tilde{\nabla}_{\xi_1} Y) + \frac{p}{q} \tilde{g}(\tilde{P} \xi_2, \tilde{\nabla}_{\xi_1} Y). \tag{4.7}$$

Using $\tilde{P} \tilde{\nabla}_{\xi_1} Y = \tilde{\nabla}_{\xi_1} \tilde{P} Y$ in (4.7), (4.7) reduces to

$$\tilde{g}(\xi_2, \tilde{\nabla}_{\xi_1} Y) = -\frac{1}{q} \tilde{g}(\tilde{P} \xi_2, \tilde{\nabla}_{\xi_1} (TY + wY)) + \frac{p}{q} \tilde{g}(\tilde{P} \xi_2, \tilde{\nabla}_{\xi_1} Y). \tag{4.8}$$

Using (2.6) and (2.8), we get □

$$-q \tilde{g}(\xi_2, \tilde{\nabla}_{\xi_1} Y) = \tilde{g}(\tilde{P} \xi_2, h^l(\xi_1, TY) + D^l(\xi_1, wY) - ph^l(\xi_1, Y)). \tag{4.9}$$

Since $Rad(TN)$ defines totally geodesic foliation if and only if $\nabla_{\xi_1} \xi_2 \in Rad(TN)$, from (4.9) and (4.6), we obtain $\nabla_{\xi_1} \xi_2 \in Rad(TN)$ if and only if

$$ph^l(\xi_1, Y) = h^l(\xi_1, TY) + D^l(\xi_1, wY).$$

Theorem 4.2. Let $(N, g, S(TN))$ be a SCR-lightlike submanifold of metallic semi-Riemannian manifold (\tilde{N}, \tilde{P}) . Then,

(i) the distribution D' is integrable if and only if,

$$h^s(X, \tilde{P}Y) = h^s(Y, \tilde{P}X)$$

(ii) the distribution D' defines totally geodesic foliations if and only if,

$$h^s(X, \tilde{P}Y) - ph^s(X, Y) \in \Gamma(\mu).$$

Proof.

(i) For any $X, Y \in \Gamma(D')$, from (3.11), we obtain

$$h^s(X, TQ_1Y) + h^s(X, TQ_2Y) = wR\nabla_X Y + Ch^s(X, Y). \quad (4.10)$$

Interchanging role of X and Y in (4.10), we get

$$h^s(Y, TQ_1X) + h^s(Y, TQ_2X) = wR\nabla_Y X + Ch^s(Y, X). \quad (4.11)$$

Subtracting (4.10) from (4.11), we obtain

$$h^s(X, TQ_1Y) + h^s(X, TQ_2Y) - h^s(Y, TQ_1X) - h^s(Y, TQ_2X) = wR[X, Y] \quad (4.12)$$

Above equation can be written as

$$h^s(X, \tilde{P}Y) - h^s(Y, \tilde{P}X) = wR[X, Y]. \quad (4.13)$$

From (4.12), D' is integrable if and only if

$$h^s(X, \tilde{P}Y) = h^s(Y, \tilde{P}X)$$

(ii) For any $X, Y \in \Gamma(D')$, the distribution D' defines totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D')$.

For any $Z \in \Gamma(D^\perp)$, from (2.6), we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z).$$

Using (2.16) in above equation, we get

$$qg(\nabla_X Y, Z) = q\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\tilde{P}\tilde{\nabla}_X Y, \tilde{P}Z) - p\tilde{g}(\tilde{\nabla}_X Y, \tilde{P}Z). \quad (4.14)$$

Using (2.6) in (4.14), we get

$$qg(\nabla_X Y, Z) = \tilde{g}(h^s(X, \tilde{P}Y), \tilde{P}Z) - p\tilde{g}(h^s(X, Y), \tilde{P}Z). \quad (4.15)$$

Above equation reduces to

$$qg(\nabla_X Y, Z) = \tilde{g}(h^s(X, \tilde{P}Y) - ph^s(X, Y), \tilde{P}Z) \quad (4.16)$$

From (4.16), we obtain $\nabla_X Y \in \Gamma(TN)$ if and only if $h^s(X, \tilde{P}Y) - ph^s(X, Y) \in \Gamma(\mu)$. □

Theorem 4.3. *Let $(N, g, S(TN))$ be a SCR-lightlike submanifold of metallic semi-Riemannian manifold \tilde{N} . Then,*

(i) *the distribution D^\perp is integrable if and only if,*

$$TQA_{wRZ_1}Z_2 = TQA_{wRZ_2}Z_1.$$

(ii) *the distribution D^\perp defines totally geodesic foliation if and only if, $B\nabla_{Z_1}^s \tilde{P}Z_2 = -A_{\tilde{P}Z_2}Z_1$. and $TQ_1A_{\tilde{P}Z_2}Z_1 = pQ_1A_{\tilde{P}Z_2}Z_1$.*

Proof.

(i) For any $Z_1, Z_2 \in \Gamma(D^\perp)$, from (3.7) and (3.8), we obtain

$$-QA_{wRY}Z_1 = TQ\nabla_{Z_1}Z_2, \quad (4.17)$$

Interchanging Z_1 and Z_2 , we obtain

$$-QA_{wRX}Z_2 = TQ\nabla_{Z_2}Z_1, \quad (4.18)$$

Subtracting (4.18) from (4.17), we obtain

$$-QA_{wRX}Z_2 + QA_{wRY}Z_1 = TQ[Z_1, Z_2] \quad (4.19)$$

From (4.19), we obtain D^\perp is integrable if and only if

$$QA_{wRZ_1}Z_2 = QA_{wRZ_2}Z_1.$$

(ii) For any $Z_1, Z_2 \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$, from (2.6), we have

$$g(\nabla_{Z_1}Z_2, X) = \tilde{g}(\tilde{\nabla}_{Z_1}Z_2, X). \tag{4.20}$$

Using (2.16) in (4.20), we obtain

$$qg(\nabla_{Z_1}Z_2, X) = \tilde{g}(\tilde{P}\tilde{\nabla}_{Z_1}Z_2, \tilde{P}X) - p\tilde{g}(\tilde{\nabla}_{Z_1}Z_2, \tilde{P}X). \tag{4.21}$$

Since $\tilde{\nabla}_X\tilde{P}Y = \tilde{P}\tilde{\nabla}_XY$, above equation reduces to

$$qg(\nabla_{Z_1}Z_2, X) = \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{P}Z_2, \tilde{P}X) - p\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{P}Z_2, X). \tag{4.22}$$

Expanding above equation using (2.6) and (2.8) in (4.22), we obtain

$$qg(\nabla_{Z_1}Z_2, X) = \tilde{g}(B\nabla_{Z_1}^{\tilde{P}}\tilde{P}Z_2, X) + p\tilde{g}(A_{\tilde{P}Z_2}Z_1, X). \tag{4.23}$$

On the other hand, for any $N \in \text{ltr}(TN)$, we have

$$\tilde{g}(\nabla_{Z_1}Z_2, N) = \tilde{g}(\tilde{\nabla}_{Z_1}Z_2, N) = \frac{1}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_{Z_1}Z_2, \tilde{P}N) - \frac{p}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_{Z_1}Z_2, N). \tag{4.24}$$

Using (2.16) in (4.24), we obtain

$$\tilde{g}(\nabla_{Z_1}Z_2, N) = \frac{1}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_{Z_1}Z_2, \tilde{P}N) - \frac{p}{q}\tilde{g}(\tilde{P}\tilde{\nabla}_{Z_1}Z_2, N). \tag{4.25}$$

Further, expanding (4.25) by using (2.8), we obtain

$$\tilde{g}(\nabla_{Z_1}Z_2, N) = -\frac{1}{q}\tilde{g}(A_{\tilde{P}Z_2}Z_1, \tilde{P}N) + \frac{p}{q}\tilde{g}(A_{\tilde{P}Z_2}Z_1, N). \tag{4.26}$$

Since the distribution D^\perp defines totally geodesic foliation if and only if $\nabla_{Z_1}Z_2 \in \Gamma(D^\perp)$, from (4.23) and (4.26), we get, $\nabla_{Z_1}Z_2 \in \Gamma(D^\perp)$ if and only if $B\nabla_{Z_1}^{\tilde{P}}\tilde{P}Z_2 = -A_{\tilde{P}Z_2}Z_1$. and $TQ_1 A_{\tilde{P}Z_2}Z_1 = pQ_1A_{\tilde{P}Z_2}Z_1$.

□

5. Totally umbilical SCR-lightlike submanifolds

Definition 5.1. [1] A lightlike submanifolds $(N, g, S(TN))$ of a semi-Riemannian manifold (\tilde{N}, \tilde{g}) is said to be totally umbilical in \tilde{N} if there is smooth transversal vector field $\mathcal{H} \in \Gamma(\text{tr}(TN))$ on \tilde{N} called the transversal curvature vector field of \tilde{N} , such that, for all $X_1, X_2 \in \Gamma(TN)$

$$h(X_1, X_2) = \mathcal{H}\tilde{g}(X_1, X_2). \tag{5.1}$$

Alternatively, N is totally umbilical, if and only if on each coordinate neighborhood \mathcal{U} there exist a smooth vector fields $\mathcal{H}^l \in \Gamma(\text{tr}(TN))$ and $\mathcal{H}^s \in \Gamma(S(TN^\perp))$ such that

$$h^l(X_1, X_2) = \mathcal{H}^l\tilde{g}(X_1, X_2), \tag{5.2}$$

$$h^s(X_1, X_2) = \mathcal{H}^s\tilde{g}(X_1, X_2), \tag{5.3}$$

$$D^l(X_1, W) = 0, \tag{5.4}$$

for any $X_1, X_2 \in \Gamma(TN)$ and $W \in S(TN^\perp)$.

Theorem 5.1. Let $(N, g, S(TN))$ be a totally umbilical r -lightlike SCR submanifold of metallic semi-Riemannian manifold \tilde{N} , then the induced connection ∇ is metric connection if and only if, $\tilde{g}(X, TQ_2Y) = p\tilde{g}(X, Y)$.

Proof. The induced connection ∇ is metric connection if and only if, for any $X \in \Gamma(TN)$ and $\xi \in \Gamma(\text{Rad}(TN))$, $\nabla_X\xi \in \text{Rad}(TN)$.

Suppose N is totally umbilical lightlike submanifold and $Y \in S(TN)$, from (2.16), we get

$$g(\nabla_X\xi, Y) = \frac{1}{q}g(\tilde{P}\tilde{\nabla}_X\xi, \tilde{P}Y) - \frac{p}{q}g(\tilde{P}\tilde{\nabla}_X\xi, Y). \tag{5.5}$$

Since $\tilde{\nabla}$ is metric connection, (5.5) reduces to

$$g(\nabla_X\xi, Y) = -\frac{1}{q}\tilde{g}(\tilde{P}\xi, \tilde{\nabla}_X(TY + wY)) + \frac{p}{q}\tilde{g}(\tilde{P}\xi, \tilde{\nabla}_X Y). \tag{5.6}$$

Using (5.2) and (5.3) in (5.6), we get

$$g(\nabla_X\xi, Y) = \tilde{g}(H^l, \tilde{P}\xi) \left(-\frac{1}{q}\tilde{g}(X, TQ_2Y) + \frac{p}{q}\tilde{g}(X, Y) \right) - \frac{1}{q}\tilde{g}(\xi, D^l(X, wY)). \tag{5.7}$$

Since $D^l(X\tilde{P}Q_2Y) = 0$ and $\tilde{g}(H^l, \tilde{P}\xi)$ is not identically zero, from (5.7) $g(\nabla_X\xi, Y) = 0$ if and only if, $\tilde{g}(X, TQ_2Y) = p\tilde{g}(X, Y)$.

This implies, the induced connection ∇ is metric connection if and only if, $\tilde{g}(X, TQ_2Y) = p\tilde{g}(X, Y)$. □

Theorem 5.2. Let N be a totally umbilical lightlike submanifold of metallic semi-Riemannian manifold. Then

(i) $Rad(TN)$ is always integrable.

(ii) $Rad(TN)$ always defines totally geodesic foliation.

Proof. (i) Let $\xi_1, \xi_2 \in Rad(TN)$ and $X \in \Gamma(S(TN))$, from (2.16), we get

$$\tilde{g}([\xi_1, \xi_2], X) = \frac{1}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, \tilde{P}X) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}X) - \frac{1}{q}\tilde{g}(\tilde{\nabla}_{\xi_2}\tilde{P}\xi_1, \tilde{P}X) + \frac{p}{q}\tilde{g}(\tilde{\nabla}_{\xi_2}\xi_1, \tilde{P}X). \tag{5.8}$$

Above equation reduces to

$$\begin{aligned} \tilde{g}([\xi_1, \xi_2], X) &= -\frac{1}{q}\tilde{g}(\tilde{P}\xi_2, \tilde{\nabla}_{\xi_1}\tilde{P}X) + \frac{p}{q}\tilde{g}(\xi_2, \tilde{\nabla}_{\xi_1}\tilde{P}X) + \frac{1}{q}\tilde{g}(\tilde{P}\xi_2, \tilde{\nabla}_{\xi_1}\tilde{P}X) \\ &\quad - \frac{p}{q}\tilde{g}(\xi_2, \tilde{\nabla}_{\xi_1}\tilde{P}X). \end{aligned} \tag{5.9}$$

Using (2.6) and (2.8) in above equation, we get

$$\begin{aligned} \tilde{g}([\xi_1, \xi_2], X) &= -\frac{1}{q}\tilde{g}(\tilde{P}\xi_2, D'(\xi_1, wX)) + \frac{p}{q}\tilde{g}(\xi_2, D'(\xi_1, wX)) + \frac{1}{q}\tilde{g}(\tilde{P}\xi_2, D'(\xi_1, wX)) \\ &\quad - \frac{p}{q}\tilde{g}(\xi_2, D'(\xi_1, wX)) - \frac{1}{q}\tilde{g}(\tilde{P}\xi_2, h'(\xi_1, TX)) + \frac{p}{q}\tilde{g}(\xi_2, h'(\xi_1, TX)) \\ &\quad + \frac{1}{q}\tilde{g}(\tilde{P}\xi_2, h'(\xi_1, TX)) - \frac{p}{q}\tilde{g}(\xi_2, h'(\xi_1, TX)). \end{aligned} \tag{5.10}$$

Since N is totally umbilical, $D'(\xi, wX) = 0$ and $h'(\xi, wX) = 0$. From (5.10), we obtain

$$\tilde{g}([\xi_1, \xi_2], X) = 0. \tag{5.11}$$

Therefore, $[\xi_1, \xi_2] \in Rad(TN)$ implies $Rad(TN)$ is integrable.

(ii) Let $\xi_1, \xi_2 \in Rad(TN)$ and $X \in \Gamma(S(TN))$, from (2.16), we obtain

$$\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, X) = \frac{1}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, TX) + \frac{1}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, wX) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, TX) - \frac{p}{q}\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, TX), \tag{5.12}$$

this implies

$$\begin{aligned} \tilde{g}(\nabla_{\xi_1}\xi_2, X) &= \frac{1}{q}\tilde{g}(h^l(\xi_1, TX), \tilde{P}\xi_2) + \frac{1}{q}\tilde{g}(D^l(\xi_1, wX), \tilde{P}\xi_2) - \frac{p}{q}\tilde{g}(h^l(\xi_1, TX), \xi_2) \\ &\quad - \frac{p}{q}\tilde{g}(D^l(\xi_1, wX), \xi_2). \end{aligned} \tag{5.13}$$

192 Since N is totally umbilical, $h(X, \tilde{P}\xi_1) = H\tilde{g}(X, \tilde{P}\xi_1) = 0$, $h(X, \tilde{P}\xi_1) = H\tilde{g}(X, \xi_1) = 0$ and $D^l(\xi, wX) = 0$. From (5.13), we get

$$\tilde{g}(\nabla_{\xi_1}\xi_2, X) = 0.$$

Therefore $\nabla_{\xi_1}\xi_2 \in \text{Rad}(TN)$. □

Theorem 5.3. *Let N be a totally umbilical lightlike submanifold of metallic semi-Riemannian manifold. Then*

(i) D' is integrable.

(ii) D' defines totally geodesic foliation if and only if, $H^s \in \Gamma(\mu)$.

Proof. (i) From Theorem 4.2 (i), D' is integrable if and only if,

$$h^s(X, \tilde{P}Y) = h^s(\tilde{P}X, Y).$$

Since N is totally umbilical,

$$h^s(X, \tilde{P}Y) = H^s\tilde{g}(X, \tilde{P}Y) \tag{5.14}$$

and

$$h^s(\tilde{P}X, Y) = H^s\tilde{g}(\tilde{P}X, Y). \tag{5.15}$$

Since $\tilde{g}(X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y)$, from (5.14) and (5.15), we obtain

$$h^s(X, \tilde{P}Y) = h^s(\tilde{P}X, Y) = H^s\tilde{g}(\tilde{P}X, Y) = h^s(\tilde{P}X, Y). \tag{5.16}$$

Therefore, D' is always integrable.

(ii) From Theorem 4.2 (ii), D' defines totally geodesic foliation if and only if, $h^s(X, \tilde{P}Y) - ph^s(X, Y) \in \Gamma(\mu)$.

Since N is totally umbilical, D' defines totally geodesic foliation if and only if,

$$H^s(\tilde{g}(X, \tilde{P}Y) - p\tilde{g}(X, Y)) \in \Gamma(\mu).$$

This implies, D' defines totally geodesic foliation if and only if, $H^s \in \Gamma(\mu)$. □

Theorem 5.4. *Let N be a totally umbilical lightlike submanifold of metallic semi-Riemannian manifold. Then, for any Z_1 and $Z_2 \in D^\perp$*

(i) D^\perp is integrable if and only if,

$$TQA_{wRZ_1}Z_2 = TQA_{wRZ_2}Z_1.$$

(ii) D^\perp defines totally geodesic foliation if and only, if, $H^\perp = 0$.

Proof. The Proof is same as [Theorem 4.3](#). □

The notion of warped product was defined by Bishop and O' Niell in Ref. [29] as a generalization of the notion of the Cartesian product. Further, various types of warped product submanifolds have been studied in Riemannian manifolds with certain metric structures. Later, the warped product lightlike submanifolds of a semi-Riemannian manifold have been studied by Sahin [22]. In the [Theorem 5.6](#), we investigate the existence of warped product lightlike submanifolds of the type $N^\perp \times_\lambda N_T$ in metallic semi-Riemannian manifold \tilde{N} when $H^\perp \in \Gamma(\mu)$.

Definition 5.2. [29] *Let $B \times F$ be a product manifold of Riemannian manifolds B and F with Riemannian metric g_B and g_F , respectively, and $\pi: B \times F \rightarrow B$ and $\eta: F \times B \rightarrow F$ be projection maps. The manifold $B \times F$ equipped with Riemannian metric g such that*

$$g = g_B + \lambda^2 g_F$$

that is, for any $X \in TN$ at (p, q) ,

$$\|X\|^2 = \|\pi_*(X)\|^2 + \lambda^2 \|\eta_*(X)\|^2,$$

is known as warped product $N = B \times_\lambda F$.

If λ (Known as warping function) is constant then the warped product manifold is trivial. For differentiable function λ on N , the gradient $\nabla\lambda$ is defined by

$$g(\nabla\lambda, Z) = Z\lambda, \quad \forall Z \in \Gamma(TN).$$

Theorem 5.5. [29] *Let $N = B \times_\lambda F$ be a warped product manifolds. If $X_1, X_2 \in T(B)$ and $Y_1, Y_2 \in T(F)$, then*

$$\nabla_{X_1} X_2 \in T(B), \tag{5.17}$$

$$\nabla_{X_1} Y_1 = \nabla_{Y_1} X_1 = \left(\frac{X_1 \lambda}{\lambda} \right) Y_1, \tag{5.18}$$

$$\nabla_{Y_1} Y_2 = -\frac{g(Y_1, Y_2)}{\lambda} \nabla\lambda. \tag{5.19}$$

Theorem 5.6. *Let $(N, g, S(TN))$ be totally umbilical SCR-lightlike submanifold of metallic semi-Riemannian manifold \tilde{N} with $H^\perp \in \Gamma(\mu)$ Then, there does not exist any non-trivial lightlike warped product of the type $N^\perp \times_\lambda N_T$*

Proof. Let $X \in \Gamma(D_T)$ and $Z \in \Gamma(D^\perp)$. Then, from (5.18), we obtain

$$\nabla_X Z = \nabla_Z X = (Z \ln \lambda)X \tag{5.20}$$

this implies

$$g(\nabla_X Z, X) = Z \ln \lambda g(X, X). \tag{5.21}$$

Using (2.16) in (5.21), we get

$$Z \ln \lambda g(X, X) = Z \ln \lambda \left\{ \frac{1}{q} g(PX, PX) - \frac{p}{q} g(PX, X) \right\}. \tag{5.22}$$

Above equation can be re-written as

$$Z \ln \lambda g(X, X) = \frac{1}{q} g(Z \ln \lambda PX, PX) - \frac{p}{q} g(Z \ln \lambda PX, X). \tag{5.23}$$

Again, using (5.18) in (5.23), we get

$$Z \ln \lambda g(X, X) = \frac{1}{q} g(\nabla_{PX} Z, PX) - \frac{p}{q} g(\nabla_{PX} Z, X). \tag{5.24}$$

Using (2.6) in equation in (5.24), we obtain

$$Z \ln \lambda g(X, X) = \frac{1}{q} \tilde{g} \left(\tilde{\nabla}_{PX} Z - h^l(PX, Z) - h^s(PX, Z), PX \right) \tag{5.25}$$

$$- \frac{p}{q} \tilde{g} \left(\tilde{\nabla}_{PX} Z - h^l(PX, Z) - h^s(PX, Z), X \right) \tag{5.26}$$

Since N is totally umbilical, $h^l g(PX, Z) = H^l g(PX, Z)$ and $h^s g(PX, Z) = H^s g(PX, Z)$. Therefore, equation (5.25) reduces to

$$Z \ln \lambda g(X, X) = \frac{1}{q} \tilde{g} \left(\tilde{\nabla}_{PX} Z, PX \right) - \frac{p}{q} \tilde{g} \left(\tilde{\nabla}_{PX} Z, X \right). \tag{5.27}$$

Since the $\tilde{\nabla}$ is Levi-Civita connection, (5.27) can be re-written as

$$Z \ln \lambda g(X, X) = -\frac{1}{q} \tilde{g} \left(\tilde{\nabla}_{PX} PX, Z \right) + \frac{p}{q} \tilde{g} \left(\tilde{\nabla}_{PX} X, Z \right). \tag{5.28}$$

Above equation reduces to

$$Z \ln \lambda g(X, X) = -\frac{1}{q} g(\nabla_{PX} PX, Z) + \frac{b}{q} g(\nabla_{PX} X, Z). \quad (5.29)$$

From theorem 5.3 if $H^f \in \Gamma(\mu)$ then $\nabla_{PX} X$ and $\nabla_{PX} X$ belong to D . Therefore, $g(\nabla_{PX} PX, Z) = 0$ and $g(\nabla_{PX} X, Z) = 0$. From (5.29), we get $(Z \ln \lambda)g(X, X) = 0$.

For $Y \in D \subset D'$, we have

$$(Z \ln \lambda)g(Y, Y) = 0. \quad (5.30)$$

Since D is non-degenerate distribution, we get $Z \ln \lambda = 0$ this leads to $\lambda = 0$ which is a contradiction.

Hence, there does not exist any non-trivial lightlike warped product of type $N^{\perp} \times_{\lambda} N_T$. \square

References

1. Duggal KL, Bejancu A. Lightlike submanifolds of semi-riemannian manifolds and applications. Dordrecht: Kluwer Academic; 1996.
2. Crasmareanu M, Hretcanu CE. Golden differential geometry. Chaos, Solitons and Fractals. 2008; 38(5): 1229-38. doi: [10.1016/j.chaos.2008.04.007](https://doi.org/10.1016/j.chaos.2008.04.007).
3. de Spinadel VW. The family of metallic means. Vis Math. 1999.
4. de Spinadel VW. The metallic means family and multifractal spectra. Nonlinear Anal Ser B: Real World Appl. 1999; 36(6): 721-45. doi: [10.1016/s0362-546x\(98\)00123-0](https://doi.org/10.1016/s0362-546x(98)00123-0).
5. de Spinadel VW. The metallic means family and forbidden symmetries. Int. Math. J. 2002; 02: 279-88.
6. Acet BE. Lightlike hypersurfaces of metallic semi-Riemannian manifolds. Int. J. Geom. Methods Mod. Phys. 2008; 15(12): 185-201. doi: [10.1142/s0219887818502018](https://doi.org/10.1142/s0219887818502018).
7. Akyol MA. Remarks on metallic maps between metallic Riemannian manifolds and constancy of certain maps. Honam Math J. 2019; 41: 343-56.
8. Erdoğan FE. Transversal lightlike submanifolds of metallic semi-Riemannian manifolds Turk. J of Math. 2018; 42(6): 3133-48. doi: [10.3906/mat-1804-88](https://doi.org/10.3906/mat-1804-88).
9. Erdoğan FE, Yıldırım C. On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. Politeknik Dergisi. 2018; 21: 967-70. doi: [10.2339/politeknik.389629](https://doi.org/10.2339/politeknik.389629).
10. Erdoğan FE, Perktes SY, Acet BE, Blaga AM. Screen transversal lightlike submanifolds of metallic semi-Riemannian manifold. J. Geom. Phys. 2019; 119: 111-20.
11. Gezer A, Cengiz N, Salimov A. On integrability of golden Riemannian structures. Turkish J Math. 2013; 37: 693-703.
12. Hretcanu CE, Blaga AM. Submanifolds in metallic Riemannian manifolds. Diff Geom Dynm Syst. 2018; 20: 83-97.
13. Hretcanu CE, Blaga AM. Slant and semi-slant submanifolds in metallic Riemannian manifolds. J of Func Spac. 2018; 2864263: 13. doi: [10.1155/2018/2864263](https://doi.org/10.1155/2018/2864263),
14. Hretcanu CE, Crasmareanu M. On some invariant submanifolds in Riemannian manifold with golden structure. An. Ştiint Univ Al Cuza Iaşi Mat. 2007; 53: 199-211.
15. Kaur R, Shanker G, Yadav A, Ali A. A note on semi-slant lightlike submanifolds of an indefinite Kenmotsu manifold. Honam Math Jour. 2021; 43: 152-66.
16. Kaur R, Shanker G, Yadav A. Semi-invariant lightlike submanifolds of a metallicsemi-Riemannian manifold, DGDS. 2021; 23: 90-104.
17. Kumar S. Geometry of warped product lightlike submanifolds of indefinite nearly Kaehler manifolds. J geometry. 2018; 109(1): 01-18. doi: [10.1007/s00022-018-0425-3](https://doi.org/10.1007/s00022-018-0425-3).

18. Perktes SY, Erdoğan FE, Acet BE. Lightlike submanifolds of metallic semi-Riemannian manifold. *Filomat*. 2013; 34(6): 1781-94. doi: [10.2298/fil2006781y](https://doi.org/10.2298/fil2006781y).
19. Poyraz NÖ, Yaşar E. Lightlike hypersurfaces of a Golden semi- Riemannian manifold. *Mediterr J. Math.* 2019; 14(5): 01-20. doi: [10.1007/s00009-017-0999-2](https://doi.org/10.1007/s00009-017-0999-2).
20. Poyraz NÖ, Yaşar E. Lightlike submanifolds of golden semi-Riemannian manifolds. *J Geom. Phy.* 2019; 141: 92-104.
21. Poyraz NÖ. Golden GCR – lightlike submanifolds of golden semi-Riemannian manifolds. *Mediterr. J. Math.* 2021; 17(5): 01-16. doi: [10.1007/s00009-020-01567-7](https://doi.org/10.1007/s00009-020-01567-7).
22. Şahin B. Warped product lightlike submanifolds. *Sarajevo Journal of Mathematics*. 2005; 14(2): 251-60. doi: [10.5644/sjm.01.2.10](https://doi.org/10.5644/sjm.01.2.10).
23. Şahin B, Akyol MA. Golden maps between Golden Riemannian manifolds and constancy of certain maps. *Math. Commun.* 2014; 19: 333-42.
24. Bejancu A. CR-Submanifolds of Kahler manifolds. *Transaction Amer Math Soc.* 1979; 250(0): 333-45. doi: [10.1090/s0002-9947-1979-0530059-6](https://doi.org/10.1090/s0002-9947-1979-0530059-6).
25. Duggal KL, Sahin B. Screen Cauchy Riemann lightlike submanifolds. *Acta. Math. Hung.* 2005; 106(1-2): 137-65. doi: [10.1007/s10474-005-0011-7](https://doi.org/10.1007/s10474-005-0011-7).
26. Duggal KL, Sahin B. Lightlike submanifolds of indefinite Sasakian manifolds. *Int J Math Math Sci.* 2007; 2007: 057585-21. doi: [10.1155/2007/57585](https://doi.org/10.1155/2007/57585).
27. Shanker G., Yadav A. An investigation on the existence of warped product irrotational screen-real lightlike submanifolds of metallic semi-Riemannian manifolds. *Arab Jour Math Sc.* 2021; 27: 01-10.
28. Duggal KL, Sahin B. *Differential geometry of lightlike submanifolds.* Birkhäuser Verlag AG. Berlin. 2010.
29. Bishop RL, O'Neill B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* 1969; 145(0): 1-49. doi: [10.1090/s0002-9947-1969-0251664-4](https://doi.org/10.1090/s0002-9947-1969-0251664-4).

Further reading

30. Bahadır O, Siddiqui AN, Gülbahar M, Alkhalidi AH. Main curvatures identities on lightlike hypersurfaces of statistical manifolds and their characterizations. *Mathematics.* 2022; 10(13): 2290. doi: [10.3390/math10132290](https://doi.org/10.3390/math10132290).
31. Chen BY. *Differential geometry of warped product manifolds and submanifolds.* Hackensack: World Scientific; 2017.
32. Crasmareanu M, Hretcanu CE. Metallic structures on Riemannian manifolds. *Revista de la union matematica argentina.* 2013; 54: 15-27.
33. Zaidi A, Shanker G, Yadav A. Conformal Anti-Invariant Riemannian Maps from or To Sasakian Manifolds. *Lobachevskii Journal of Mathematics.* 2023; 44(4): 1518-1527. doi: [10.1134/S1995080223040297](https://doi.org/10.1134/S1995080223040297).
34. Shanker G, Yadav A. A study on the geometry of totally umbilical (TU) screen-transversal (ST) lightlike submanifolds of metallic semi-Riemannian manifolds. *AIP Conference Proceedings.* 2022; 2435(1). doi: [10.1063/5.0085626](https://doi.org/10.1063/5.0085626).

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