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Nehari manifold approach to singular fourth-order Leray–Lions equations with no-flux boundary conditions

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Abstract

Purpose – In this paper, we study a fourth-order singular problems of type Leray–Lions with singular weight and with no-flux boundary conditions.
Design/methodology/approach – We use the Nehari manifold to establish our result for our problem.
Findings – We prove that there exist at least two nontrivial positive weak solutions.
Originality/value – In particular, it investigates the problem in the presence of singular weights and variable exponents, a case that has not been extensively studied in the existing literature.

Keywords Leray–Lions type operator, Nehari manifold, Singular problem, Variational method, Weak solutions
Paper type Research article

1. Introduction

In this article, we study the following problem.

$$(PV)\begin{cases} \Delta(a(x,\Delta u)) + b(x)\dfrac{|u|^{p(x)-2}u}{|x|^{p(x)}} = \dfrac{|u|^{q(x)-2}u}{|x|^{\alpha(x)}} + \lambda\dfrac{|u|^{-1-s(x)}u}{|x|^{\beta(x)}} \text{ in } \Omega, \\ u = constant,\ \ \Delta u = 0, \text{ on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega}\frac{\partial}{\partial n}a(x,\Delta u)ds = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain containing the origin with sufficiently smooth boundary and $\Delta(a(x, \Delta u))$ is the fourth-order Leray–Lions operator, $b(.) \in C(\overline{\Omega})$ is a positive weight function and λ is a real parameter. The variable exponents $p(.), q(.), r(.)$, $\alpha(.), \beta(.) \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ on } \overline{\Omega}\}$.

The Leray–Lions-type operator is a more quiet general in the fourth-order, we can take for example $a(x, t) = h(x)|t|^{p(x)-2}t$, when $h(x) = 1$ and $t = \Delta u$ the above operator turns into the operator $\Delta^2_{p(\cdot)}u$ (the $p(x)$ – biharmonic operator), for mor details about Leray–Lions type operators one can see Refs. [1–4].

MSC Classication: 35D30, 35G30, 35J75, 46E35

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It should be noted that this type of problem is very active in several domains, such electrorheological fluids [5], elastic materials [6] and stationary thermo-rheological viscous flows [7].

In the last few years, various authors have studied this kind of problem; in the case of exponents constant, one can see Refs. [8, 9]. In 2020, Mokhtar, M. E. [10] studied the existence of solution by using variational approaches and utilizing the Pohozaev identity to show the nonexistence of solution for the following p-Laplace singular problem

$$(P_D)\begin{cases} -\Delta_p^2 = |u|^{q-1}u + \lambda \dfrac{|u|^{-1-s}}{|x|^{\beta}} & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

Moreover, many authors have studied problems similar to our problem; in the case of variable exponents, see for example [11–15]. In 2023, Repovš, D. D. and Saoudi, K. [16]. studied this $p(x)$-Laplace singular problem

$$(P_N)\begin{cases} \Delta_{p(x)} u = a(x)|u|^{q(x)-2}u + \lambda b(x)|u|^{-s(x)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

They used the Nehari manifold technique to demonstrate their results for the problem (P_N).

We start by giving the assumptions that we will consider for our problem.

(H_1) $0 < s^- < s^+ < 1 < 2 < p^- < p^+ < q^- \le q(x) < \frac{N-\alpha(x)}{N} p_2^\star(x)$.

where $h^- := \min_{x\in\overline{\Omega}} h(x)$, $h^+ := \max_{x\in\overline{\Omega}} h(x)$ and $p_2^\star(x) = \frac{Np(x)}{N-p(x)}$, for all $x \in \overline{\Omega}$.

(H_2) $Ns(x) < \beta(x) < N$.

(b) $b \in L^\infty(\Omega)$ and strictement positive in Ω.

(A_1) $a : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $a(x, 0) = 0$ for a.e. $x \in \overline{\Omega}$.

(A_2) there exists $c_1 > 0$ such that

$$c_1|t|^{p(x)} \le a(x,t)t \le p(x)A(x,t) \quad \text{for a.e.} x \in \Omega \quad \text{and all} \quad t \in \mathbb{R}.$$

such that $A(x,t) = \int_0^t a(x,s)ds$

(A_3) $[a(x, s) - a(x, t)](s - t) \ge 0$ holds for all $s, t \in \mathbb{R}$ and a.e. $x \in \Omega$ with equality if and only if $s = t$.

(A_4) There exist $0 < c_0 < c_1 \frac{q^- - 1}{p^+ - 1}$ and a non-negative function $d(x) \in L^{p'(x)}(\Omega)$, such that

$$|a(x,t)| \le c_0\left(d(x) + |t|^{p(x)-1}\right) \text{for a.e.} x \in \Omega$$

for all $t \in \mathbb{R}$ and $d(x)$ that verifies $|d|_{p'(x)} < \frac{1}{2}\min\left\{\frac{q^- c_1}{c_0} - \frac{1}{p^-}, \frac{q^- - p^+}{c_0}\right\}$.

(A_5) $A : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is homogeneous of degre $p(x)$, that is $A(x,tu) = t^{p(x)}A(x,u)$ $(t > 0)$ $\forall x \in \overline{\Omega}$, $u \in \mathbb{R}$.

Under the above hypotheses, we give our main result.

Theorem 1. *Suppose that* $(A_1)-(A_5)$, (H_1), (H_2) *and* (b) *hold. Then there exists a positive constants* $\lambda_\star$ *such that the problem* (PV) *has at least two positive solutions for each* $\lambda \in (0, \lambda_\star)$.

The structure of this article is as follows: Section 2 presents key background results concerning variable exponent Lebesgue and Sobolev spaces, along with a compact embedding theorem of the Sobolev–Hardy type. Section 3 introduces the essential lemmas required for our analysis. We then establish that the energy functional achieves its minimum on both N_λ^+ and N_λ^-. We conclude with the proof of our main result.

2. Preliminaries

In this part, we review several fundamental properties and definitions of Lebesgue and Sobolev spaces with variable exponents. For further details, readers are encouraged to consult the comprehensive studies in Refs. [17–20].

In this study, we denote by $M(\Omega)$ the set of all real-valued functions that are measurable with respect to the Lebesgue measure on Ω, and we suppose that:

(H) p is log-Hölder continuous, satisfying $1 < p^- \leq p^+ < \infty$.

That is, there exists $k > 0$ such that

$$|p(x)-p(y)| \leq \frac{k}{-\log|x-y|} \text{ for all } x, y \in \Omega, \text{ with } 0 < |x-y| \leq \frac{1}{2}.$$

Let $b \in M(\Omega)$ satisfies the condition (b), we define the space:

$$L_{b(x)}^{p(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega b(x)|u(x)|^{p(x)} dx < \infty \right\}$$

equipped with the norm,

$$|u|_{L_{b(x)}^{p(x)}(\Omega)} = |u|_{(p(x),b(x))} = \inf\left\{ \gamma > 0 : \int_\Omega b(x)\left|\frac{u(x)}{\gamma}\right|^{p(x)} dx \leq 1 \right\}$$

$\left(L_{b(x)}^{p(x)}(\Omega), |\cdot|_{L_{b(x)}^{p(x)}}\right)$ is a Banach space. For more details on this norm, (see for example [20], Theorem 2.5 and Corollary 2.7).

Additionally, we can establish the following inequality,

$$\left|\int_\Omega u(x)v(x)\,dx\right| \leq 2|u|_{p(x)}|v|_{p'(x)}, \tag{1}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, (see Ref. [20], Theorem 2.1).

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Taking into account the log-Hölder countinuity of the exponent $p(x)$, we have

$$W_0^{1,p(x)}(\Omega) = \left\{ u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial\Omega \right\}.$$

As a consequence of the Poincaré inequality, $\|u\|_{W^{1,p(x)}(\Omega)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\|_{W_0^{1,p(x)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = |\nabla u|_{p(x)},$$

and which makes $W_0^{1,p(x)}(\Omega)$ a separable and reflexive Banach space (see Ref. [21], Proposition 2.1).

Under condition (H), we can see that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(x)}(\Omega)$ (see Refs. [22, 17]).

Theorem 2. (see ([22]) and ([19], Theorem 2.3)). If $q \in C_+(\overline{\Omega})$ *and* $q(x) < p_2^*(x)$ *for all* $x \in \overline{\Omega}$. *Then* $W^{2,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$. *, where*

$$p_2^*(x) = \begin{cases} \dfrac{Np(x)}{N-2p(x)} & if \quad p(x) < \dfrac{N}{2}; \\ +\infty & if \quad p(x) \geq \dfrac{N}{2}, \end{cases}$$

The following lemma is very important.

Lemma 1. Let us consider $r(x), \gamma(x) \in C_+(\overline{\Omega})$ *and*

$$r(x) < \frac{N-\gamma(x)}{N} p_2^\star(x) \quad \forall x \in \overline{\Omega}. \tag{2}$$

Then $W^{2,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$.

Proof. We notice $|x|^{-\gamma(x)} \in L^{\frac{N-\epsilon}{\gamma(x)}}(\Omega)$ such that $\epsilon > 0$ is a constant small enough (see Ref. [21]). Let $u \in W^{2,p(x)}(\Omega)$. Set $h(x) = \left(\frac{N-\epsilon}{\gamma(x)}\right)' r(x) = \frac{(N-\epsilon)r(x)}{N-\epsilon-\gamma(x)}$.

Then (2) implies $h(x) < p_2^*(x)$ and by Theorem 2.3 (in Ref. [19]) we have $W^{2,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{h(x)}(\Omega)$, then for $u \in W^{2,p(x)}(\Omega)$ we have $|u(x)|^{r(x)} \in L^{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}}(\Omega)$ and, from inequality 1, from inequality, one can see that

$$\int_\Omega \frac{|u|^{r(x)}}{|x|^{\gamma(x)}} dx \leq 2 \left| |x|^{-\gamma(x)} \right|_{\frac{N-\epsilon}{\gamma(x)}} \left| |u|^{r(x)} \right|_{\frac{(N-\epsilon)}{N-\gamma(x)-\epsilon}} < \infty.$$

This proves $W^{2,p(x)}(\Omega) \subset L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$.

Now let $(u_n) \subset W^{2,p(x)}(\Omega)$ and $u_n \rightharpoonup 0$ in $W^{2,p(x)}(\Omega)$. Using Theorem 2, we find that $u_n \to 0$ in $L^{h(x)}(\Omega)$, and from this, we get $\left| |u_n|^{r(x)} \right|_{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}} \to 0$. Then

$$\int_\Omega \frac{|u|^{r(x)}}{|x|^{\gamma(x)}} dx \leq 2 \left| |x|^{-\gamma(x)} \right|_{\frac{N-\epsilon}{\gamma(x)}} \left| |u|^{r(x)} \right|_{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}} \to 0.$$

that implies $|u_n|_{(r(x),|x|^{-\gamma(x)})} \to 0$. Therefore, we have $W^{2,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$.

Remark 1. when $r(x) = \gamma(x) = p(x)$ $\forall x \in \overline{\Omega}$. *Then* $W^{2,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}_{|x|^{-p(x)}}(\Omega)$. □

This result indicates the existence of a positive constant note C_p such that,

$$\int_{\Omega} \frac{|u|^{p(x)}}{|x|^{p(x)}} dx \leq C_p \|u\|^{p^{\pm}}_{W^{2,p(x)}(\Omega)}, \tag{3}$$

where $\pm = +$ if $\|u\|_{W^{2,p(x)}(\Omega)} > 1$, and $\pm = -$ if $\|u\|_{W^{2,p(x)}(\Omega)} \leq 1$.

Throughout this paper, we opted for the norm:

$$\|u\|_b = \inf\left\{\mu > 0 : \int_{\Omega} \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + \frac{b(x)}{|x|^{p(x)}} \left|\frac{u(x)}{\mu}\right|^{p(x)} \right) dx \leq 1 \right\},$$

where $b(x)$ satisfies (b), this norm represents on both $W^{2,p(x)}(\Omega)$ and $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and we can see by relation (3) that is equivalent to $\|\cdot\|_{W^{2,p(x)}(\Omega)}$. We will search for the solutions of problem (PV) in this space.

$$Y = \left\{ u \in W^{2,p(x)}(\Omega) \ : u|_{\partial\Omega} \equiv constant \right\}.$$

$\left(Y, \|\cdot\|_{W^{2,p(x)}(\Omega)}\right)$ is a reflexive and separable Banach space ([23]). As a result, we take into account the modular $\rho : Y \to \mathbb{R}$ such that

$$\rho(u) = \int_{\Omega} \left[|\Delta u|^{p(x)} + b(x) \frac{|u|^{p(x)}}{|x|^{p(x)}} \right] dx.$$

The following inequalities have a significant connection to the norm $\|.\|_b$, (see for example [23], Proposition 1). If $u \in W^{2,p(x)}(\Omega)$ then:

$$\|u\|_b \geq 1 \Rightarrow \|u\|_b^{p^-} \leq \rho(u) \leq \|u\|_b^{p^+} \tag{4}$$

$$\|u\|_b \leq 1 \Rightarrow \|u\|_b^{p^+} \leq \rho(u) \leq \|u\|_b^{p^-} \tag{5}$$

We define the functional $\mathsf{F} : Y \to \mathbb{R}$ such that

$$\mathsf{F}(u) = \int_{\Omega} \left[A(x, \Delta u) + \frac{b(x)}{p(x)} \frac{|u|^{p(x)}}{|x|^{p(x)}} \right] dx.$$

To see the properties of this functional, refer to ([1], Proposition 3; 5 and Theorem 3.2).

3. Some necessary lemmas

Definition 1. $u \in Y$ *is a weak solution of problem* (PV) *if*

$$\int_{\Omega} a(x, \Delta u)\Delta \upsilon dx \qquad + \int_{\Omega} b(x) \frac{|u|^{p(x)-2}}{|x|^{p(x)}} u\upsilon dx - \lambda \int_{\Omega} \frac{|u|^{-1-s(x)}}{|x|^{\beta(x)}} u\upsilon dx - \int_{\Omega} \frac{|u|^{q(x)-2}}{|x|^{\alpha(x)}} u\upsilon dx = 0,$$

for all $\upsilon \in Y$.

For the problem (PV), the energy functional $T : Y \to \mathbb{R}$ is defined as

$$T(u) = J_1(u) - \lambda I_1(u),$$

where

$$J_1(u) = \mathsf{F}(u) - \int_\Omega \frac{1}{q(x)} \frac{|u|^{q(x)}}{|x|^{\alpha(x)}} dx \text{and} \, I_1(u) = \int_\Omega \frac{1}{1-s(x)} \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}} dx.$$

The weakly lower semi-continuous function T has a Gâteaux derivative that is defined by

$$\langle T'(u), v \rangle = \int_\Omega a(x, \Delta u)\Delta v dx + \int_\Omega b(x) \frac{|u|^{p(x)-2}}{|x|^{p(x)}} uv dx - \lambda \int_\Omega \frac{|u|^{-1-s(x)}}{|x|^{\beta(x)}} uv dx - \int_\Omega \frac{|u|^{q(x)-2}}{|x|^{\alpha(x)}} uv dx,$$

for all $v \in Y$.

Remark 2. T is not in $C^1(Y, \mathbb{R})$ because of the singular term in I_1.

For some problems, like (PV), T may not be bounded below on Y, but it does possess a bounded below on N_λ, such that

$$N_\lambda = \left\{ u \in Y / \{0\} :< T'_\lambda(u), u >= 0 \right\}.$$

$u \in N_\lambda$ if and only if

$$\int_\Omega a(x, \Delta u)\Delta u dx + \int_\Omega b(x) \frac{|u|^{p(x)}}{|x|^{p(x)}} dx - \int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}} dx - \lambda \int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}} dx = 0.$$

We see that all solutions to the problem (PV) are included in N_λ.

Defined $\phi_u(t) = T(tu)$,

As a result, it makes sense to divide N_λ into the following three parts.

$$\phi_u(t) = \int_\Omega t^{p(x)} A(x, \Delta u) dx + \int_\Omega \frac{t^{p(x)} b(x)}{p(x)} \frac{|u|^{p(x)}}{|x|^{p(x)}} dx - \int_\Omega \frac{t^{q(x)}}{q(x)} \frac{|u|^{q(x)}}{|x|^{\alpha(x)}} dx - \lambda \int_\Omega \frac{t^{1-s(x)}}{1-s(x)} \frac{|u|^{2-s(x)}}{|x|^{\beta(x)}} dx.$$

$$\phi'_u(t) = \int_\Omega p(x) t^{p(x)-1} A(x, \Delta u) dx + \int_\Omega b(x) t^{p(x)-1} \frac{|u|^{p(x)}}{|x|^{p(x)}} dx - \int_\Omega t^{q(x)-1} \frac{|u|^{q(x)}}{|x|^{\alpha(x)}} dx - \lambda \int_\Omega t^{-s(x)} \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}} dx$$

$$\phi''_u(t) = \int_\Omega p(x)(p(x)-1) t^{p(x)-2} A(x, \Delta u) dx + \int_\Omega (p(x)-1) b(x) \frac{|u|^{p(x)}}{|x|^{p(x)}} dx - \int_\Omega (q(x)-1) t^{q(x)-2} \frac{|u|^{q(x)}}{|x|^{\alpha(x)}} dx + \lambda \int_\Omega s(x) t^{-1-s(x)} \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}} dx.$$

$$N^+_\lambda := \left\{ u \in N_\lambda : \Phi''_u(1) > 0 \right\} = \left\{ tu \in V \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) > 0 \right\},$$
$$N^-_\lambda := \left\{ u \in N_\lambda : \Phi''_u(1) < 0 \right\} = \left\{ tu \in V \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) < 0 \right\},$$
$$N^0_\lambda := \left\{ u \in N_\lambda : \Phi''_u(1) = 0 \right\} = \left\{ tu \in V \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) = 0 \right\}.$$

Lemma 2. T is bounded below on N_λ and is coercive.

Proof. Lemma 1 and (A_2) provide us, if $u \in N_\lambda$ such that $\|u\|_b > 1$.

$$
\begin{aligned}
T(u) &= \int_\Omega A(x,\Delta u)dx + \int_\Omega \frac{b(x)}{p(x)}\frac{|u|^{p(x)}}{|x|^{p(x)}}dx - \lambda\int_\Omega \frac{1}{1-s(x)}\frac{|u|^{2-s(x)}}{|x|^{\beta(x)}}dx \\
&\quad - \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx \\
&\geq \int_\Omega A(x,\Delta u)dx + \int_\Omega \frac{b(x)}{p(x)}\frac{|u|^{p(x)}}{|x|^{p(x)}}dx - \frac{1}{q^-}\left(\int_\Omega a(x,\Delta u)\Delta u dx\right. \\
&\quad \left. + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx - \lambda\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx\right) \\
&\geq \left(\frac{1}{p^+}-\frac{1}{q^+}\right)\int_\Omega a(x,\Delta u)\Delta u dx + \left(\frac{1}{p^-}-\frac{1}{q^+}\right)\int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx \\
&\quad - \lambda\left(\frac{1}{1-s^+}-\frac{1}{q^-}\right)\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx \\
&\geq C_1\left(\frac{1}{p^+}-\frac{1}{q^+}\right)\|u\|_b^{p^-} - \lambda C_s\left(\frac{1}{1-s^+}-\frac{1}{q^-}\right)\|u\|_b^{1-s^+}.
\end{aligned}
$$

With $C_1 = \min\{1, c_1\}$, since $1 - s^+ < p^-$, then $T(u) \to +\infty$ when $\|u\|_b \to +\infty$. □

Lemma 3. Given u as a local minimum of T on subsets N_λ^+ or N_λ^-, where u is not an element of N_λ^0, it follows that u represents a critical point of T.

Proof. Assume that, under the given restriction, u is a local minimizer of T.

$$I_\lambda(u) := \langle T'(u), u\rangle = 0. \tag{6}$$

Thus, we use the theory of Lagrange multipliers to prove that there exists $\mu \in \mathbb{R}$ that satisfies

$$T'(u) = \mu I_\lambda'(u)$$

Consequently,

$$\langle T'(u), u\rangle = \mu\langle I_\lambda'(u), u\rangle = \mu\Phi_u''(1) = 0.$$

From this $u \notin N_\lambda^0$, it follows that $\Phi_u''(1) \neq 0$. As a result, $\mu = 0$. □

Lemma 4. There is λ_0 such that, $N_\lambda^\pm \neq \emptyset$ and $N_\lambda^0 = \emptyset$, for every $0 < \lambda < \lambda_0$.

Proof. Initially, applying Lemma 3, we can infer that $N_\lambda^\pm \neq \emptyset$ for $\lambda \in (0, \lambda_0)$. Now since $u \in N_\lambda$, we deduce the following

$$\int_\Omega a(x,\Delta u)\Delta u dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx - \int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx - \lambda\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx = 0. \tag{7}$$

Assuming $u \in N_\lambda^0$ with $\|u\|_b > 1$. By combining the above equality with (6), under the conditions (A_1) and (A_4), we get

$$
\begin{aligned}
0 \quad &=< I'_\lambda(u), u > \geq (p^- - 1)\int_\Omega p(x)A(x,\Delta u)dx + (p^- - 1)\int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx \\
&\quad - (q^+ - 1)\int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx + s^+\left(\int_\Omega a(x,\Delta u)\Delta u dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx\right. \\
&\quad \left. - \int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx\right) \\
&\geq (p^- - 1 + s^+)\int_\Omega a(x,\Delta u)\Delta u dx + (p^- - 1 + s^+)\int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx \\
&\quad - (q^+ + s^+ - 1)\int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx \\
&\geq (p^- - 1 + s^+)\left[c_1\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx\right] \\
&\quad - (q^+ + s^+ - 1)\int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx \\
&\geq C_2(p^- - 1 + s^+)\left(\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx\right) \\
&\quad - (q^+ + s^- - 1)\int_\Omega \frac{|u|^{q(x)}}{|x|^{\alpha(x)}}dx
\end{aligned}
$$

such that $C_2 = \min\{c_1, 1\}$. From lemma 1 and (4), we get

$$
C_2(p^- - 1 + s^+)\|u\|_b^{p^-} - C_q(q^+ + s^- - 1)\|u\|_b^{q^-} \leq 0
$$

Hence

$$
\|u\|_b \geq \left(\frac{C_2}{C_q(q^+ + s^- - 1)}\right)^{\frac{1}{q^+ - p^-}}. \tag{8}
$$

Since $u \in N_\lambda$ and since $u \in N_\lambda^0$, we get

$$
\begin{aligned}
0 \quad &\leq (p^+ - 1)\left[\int_\Omega p(x)A(x,\Delta u)dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx\right] + \lambda s^+\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx \\
&\quad - (q^- - 1)\left(\int_\Omega a(x,\Delta u)\Delta u dx + \int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx - \lambda\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx\right) \\
&\leq (c_0(p^+ - 1) - (q^- - 1)c_1)\int_\Omega |\Delta u|^{p(x)}dx + (p^+ - q^-)\int_\Omega b(x)\frac{|u|^{p(x)}}{|x|^{p(x)}}dx \\
&\quad + c_0(p^+ - 1)p^+\int_\Omega d(x)\Delta u dx + \lambda(q^- - 1 + s^+)\int_\Omega \frac{|u|^{1-s(x)}}{|x|^{\beta(x)}}dx.
\end{aligned}
$$

Since $\|u\|_b > 1$, by lemma 1 and (4) we get

$$0 \leq -C_3\|u\|_b^{p^-} + 2p^+(p^+-1)c_0|d|_{p'(x)}\|u\|_b^{p^+} + \lambda C_s(q^- - 1 + s^+)\|u\|_b^{1-s^-}.$$

such that $C_3 = \min\{(q^- - 1)c_1 - c_0(p^+ - 1), q^- - p^+\}$, hence

$$\|u\|_b \leq \left(\frac{\lambda C_s(q^- + s^+ - 1)}{C_3 - 2p^+(p^+-1)c_0|d|_{p'(x)}}\right)^{\frac{1}{p^- + s^- - 1}}. \tag{9}$$

Using the above inequality, we find

$$\left(\frac{C_2}{C_q(q^+ + s^- - 1)}\right)^{\frac{1}{q^+ - p^-}} \leq \left(\frac{\lambda C_s(q^- + s^+ - 1)}{C_3 - 2p^+(p^+-1)c_0|d|_{p'(x)}}\right)^{\frac{1}{p^- + s^- - 1}}. \tag{10}$$

we get

$$\lambda \geq \left(\frac{C_2}{C_q(q^+ + s^- - 1)}\right)^{\frac{p^- + s^- - 1}{q^+ - p^-}} \frac{C_3 - 2p^+(p^+-1)c_0|d|_{p'(x)}}{C_s(q^- + s^+ - 1)} = \lambda_0.$$

Then, if λ small enough, we obtain $\|u\|_b < 1$, which is impossible. Therefore, $N_\lambda^0 = \varnothing$ for all $\lambda \in (0, \lambda_0)$. Hence, this proof is complete. □

Now let us show that the energy functional T in N_λ^+ has a minimum. Furthermore, we will demonstrate that this minimum corresponds to a solution for the problem (PV).

Theorem 3. There is $u_0 \in N_\lambda^+$ for any $\lambda \in (0, \lambda_0)$, which satisfies

$$T(u_0) = \inf_{u \in N_\lambda^+} T(u).$$

Proof. Consider $\lambda \in (0, \lambda_0)$, it follows that T is bounded below on N_λ and, consequently on N_λ^+. Hence, we can establish the existence of a sequence $\{u_n\} \subset N_\lambda^+$, satisfying $T(u_n) \to \inf_{u \in N_\lambda^+} T(u)$ as $n \to \infty$. T is coercive on N_λ and $\{u_n\}$ is bounded in V; by utilizing lemma 1, we deduce the following

$$u_n \to u_0 \text{ in } L^{1-s(x)}_{|x|^{-\beta(x)}}(\Omega) \text{ and } u_n \to u_0 \text{ in } L^{q(x)}_{|x|^{-\alpha(x)}}(\Omega).$$

First we will show that $\inf_{u \in N_\lambda^+} T(u) > 0$

Let $u_0 \in N_\lambda^+$, then $\phi''_{u_0}(1) > 0$, which gives

$$(p^+-1)\int_\Omega p(x)A(x, \Delta u_0)dx \quad + (p^+-1)\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx - (q^- - 1)\int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx + \lambda s^+ \int_\Omega \frac{|u_0|^{1-s(x)}}{|x|^{\beta(x)}}dx > 0. \tag{11}$$

We can write

$$T(u_0) \le \int_\Omega A(x,\Delta u_0)dx + \frac{1}{p^-}\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx) - \frac{1}{q^+}\int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx - \lambda\frac{1}{1-s^-}\int_\Omega \frac{|u_0|^{1-s(x)}}{|x|^{\beta(x)}}dx. \tag{12}$$

We multiply (6) with s^+, we obtain

$$s^+\left(\int_\Omega a(x,\Delta u_0)\Delta u_0 dx + \int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx\right) - s^+\int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx - \lambda s^+\int_\Omega \frac{|u_0|^{1-s(x)}}{|x|^{\beta(x)}}dx = 0, \tag{13}$$

invoking (13) and (11), we obtain

$$\int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx < \frac{1}{q^- + s^+ - 1}\left((p^+-1)\int_\Omega p(x)A(x,\Delta u_0)dx + s^+\int_\Omega a(x,\Delta u_0)\Delta u_0 dx + (p^+ - 1 + s^+)\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx\right). \tag{14}$$

However, we multiply (6) with $-\frac{1}{1-s^+}$, and from (12), we get

$$\begin{aligned} T(u_0) &\le \int_\Omega A(x,\Delta u_0)dx - \frac{1}{1-s^+}\int_\Omega a(x,\Delta u_0)\Delta u_0 dx \\ &+ \frac{1-s^+-p^-}{p^-(1-s^+)}\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx - \left(\frac{1}{q^+} - \frac{1}{1-s^+}\right)\int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx, \end{aligned}$$

Replacing the inequality (14) in the previous inequality, we get

$$\begin{aligned} T(u_0) &\le \int_\Omega A(x,\Delta u_0)dx - \frac{1}{1-s^+}\int_\Omega a(x,\Delta u_0)\Delta u_0 dx \\ &+ \frac{1-s^+-p^-}{p^-(1-s^+)}\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx + \frac{1}{q^+(1-s^+)}\left((p^+-1)\int_\Omega p(x)A(x,\Delta u_0)dx\right. \\ &\left. + s^+\int_\Omega a(x,\Delta u_0)\Delta u_0 dx + (p^+ - 1 + s^+)\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx\right) \\ &\le \left(\frac{1}{p^-} + \frac{p^+-1}{q^+(1-s^+)}\right)\int_\Omega p(x)A(x,\Delta u_0)dx - \left(\frac{1}{1-s^+} + \frac{s^+}{q^+(s^+-1)}\right)\times \\ &\int_\Omega a(x,\Delta u_0)\Delta u_0 dx + \left(\frac{1-s^+-p^-}{p^-(1-s^+)} + \frac{p^+-1+s^+}{q^+(1-s^+)}\right)\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx \end{aligned}$$

and by the conditions A_2 and A_4, we have

$$\begin{aligned} T(u_0) \leq & c_0\left(\frac{1}{p^-}+\frac{1}{q^+(1-s^+)}\right)\int_\Omega |\Delta u|^{p(x)}dx - c_1\left(\frac{1}{1-s^+}+\frac{s^+}{q^+(s^+-1)}\right)\times \\ & \int_\Omega |\Delta u|^{p(x)}dx + \left(\frac{1-s^+-p^-}{p^-(1-s^+)}+\frac{p^+-1+s^+}{q^+(1-s^+)}\right)\int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx \\ & + c_0\left(1+\frac{1}{q^+(1-s^+)}\right)\int_\Omega d(x)\Delta u_0 dx \\ \leq & -C_4\|u_0\|_b^{p^-} + 2c_0C|d|_{p'(x)}\|u_0\|_b^{p^-} \end{aligned}$$

such that $0 < C_4 = \min\left\{\frac{c_0}{p^-}\left(\frac{1}{p^-}+\frac{1}{q^+(1-s^+)}\right) - c_1\left(\frac{1}{1-s^+}+\frac{s^+}{q^+(s^+-1)}\right), \frac{1-s^+-p^-}{p^-(1-s^+)}+\frac{p^+-1+s^+}{q^+(1-s^+)}\right\}$, then by the definition of the function d and the above inequality, we get

$$T(u_0) \leq \left(2c_0|d|_{p'(x)} - C_3\right)\|u\|_b^{p^-} < 0.$$

Assume that $u_n \nrightarrow u_0$ strongly in Y. Then

$$\int_\Omega |\Delta u_0|^{p(x)}dx \leq \liminf_{n\to\infty}\int_\Omega |\Delta u_n|^{p(x)}dx.$$

We use the compact embedding to derive the following

$$\begin{aligned} \int_\Omega b(x)\frac{|u_0|^{p(x)}}{|x|^{p(x)}}dx &= \liminf_{n\to\infty}\int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx \\ \int_\Omega \frac{|u_0|^{q(x)}}{|x|^{\alpha(x)}}dx &= \liminf_{n\to\infty}\int_\Omega \frac{|u_n|^{q(x)}}{|x|^{\alpha(x)}}dx \\ \int_\Omega \frac{|u_0|^{1-s(x)}}{|x|^{\beta(x)}}dx &= \liminf_{n\to\infty}\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx. \end{aligned}$$

Now, by (6) and Theorem 2.3 in Ref. [24].

$$\begin{aligned} T(u_n) \geq & \int_\Omega A(x,\Delta u_n)dx - \frac{1}{q^-}\int_\Omega a(x,\Delta u_n)\Delta u_n dx + \left(\frac{1}{p^+}-\frac{1}{q^-}\right)\int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx \\ & + \lambda\left(\frac{1}{q^+}-\frac{1}{1-s^+}\right)\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx, \end{aligned}$$

and by (A_2), we get

$$\begin{aligned}T(u_n) \quad &\geq \frac{q^- - p^+}{p^+ q^-}\int_\Omega a(x,\Delta u_n)\Delta u_n dx + \frac{q^- - p^+}{q^- p^+}\int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx \\ &\quad + \lambda \frac{1 - s^+ - q^+}{q^+(1-s^+)}\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx \\ &\geq C_5 \frac{q^- - p^+}{q^- p^+}\left(\int_\Omega |\Delta u_n|^{p(x)}dx + \int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx\right) \\ &\quad + \lambda \frac{1 - s^+ - q^+}{q^+(1-s^+)}\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx\end{aligned}$$

which $C_5 = \min\{c_1, 1\}$, now passing $n \to \infty$, we obtain

$$\begin{aligned}\lim_{n\to\infty} T(u_n) \geq \quad & C_5 \frac{q^- - p^+}{q^- p^+}\left(\lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p(x)}dx + \int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx\right) \\ &+ \lambda\left(\frac{1}{q^+} - \frac{1}{1-s^+}\right)\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx.\end{aligned}$$

using Theorem 2.3 in Ref. [24], we get

$$\inf_{u\in N_\lambda^+} T(u_n) > \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\|u_0\|_b^{p^-} + \lambda C_s\left(\frac{1}{q^+} - \frac{1}{1-s^+}\right)\|u_0\|_b^{1-s^+} > 0.$$

Since $p^- > 1 - s^+$, which give a contradiction. Therefore, $u_n \to u_0$ in Y and $\inf_{u\in N_\lambda^+} T(u_n) = T(u_0)$

Second, we shall now prove that the functional energy T in N_λ^- is at a minimum. □

Theorem 4. for every $\lambda \in (0, \lambda_0)$, there exists $u_1 \in N_\lambda^-$ satisfying

$$T(u_1) = \inf_{u\in N_\lambda^-} T(u).$$

Proof. Consider that $\lambda \in (0, \lambda_0)$; thus, it follows T, which is bounded below on N_λ^-. Consequently, there exists $\{u_n\} \subset N_\lambda^-$, with $T(u_n) \to \inf_{u\in N_\lambda^-} T(u)$ as $n \to \infty$. T is coercive in N_λ, $\{u_n\}$ is bounded in V; therefore, we can assume that using $u_n \to u_1$ in Y by lemma 1, we obtain

$$u_n \to u_1 \text{ in } L^{1-s(x)}_{|x|^{-\beta(x)}}(\Omega),$$

$$u_n \to u_1 \text{ in } L^{q(x)}_{|x|^{-\alpha(x)}}(\Omega).$$

Currently, prove that $u_n \to u_1$ in Y. First, we will show that $\inf_{u\in N_\lambda^-} T(u) > 0$

Let $u_1 \in N_\lambda^-$, then from (6), we give

$$\int_{\Omega} a(x,\Delta u_1)\Delta u_1 dx + \int_{\Omega} b(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}dx - \int_{\Omega}\frac{|u_1|^{q(x)}}{|x|^{\alpha(x)}}dx - \lambda\int_{\Omega}\frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx = 0. \quad (15)$$

Then

$$\begin{aligned} T(u) \geq & \int_{\Omega} A(x,\Delta u_1)dx + \frac{1}{p^+}\int_{\Omega} a(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}dx - \frac{1}{q^-}\left(\int_{\Omega} a(x,\Delta u_1)\Delta u_1 dx \right.\\ & \left. + \int_{\Omega} b(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}dx - \lambda\int_{\Omega}\frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx\right) - \frac{\lambda}{1-s^+}\int_{\Omega}\frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx \quad (16)\\ \geq & C_5\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\left(\int_{\Omega}|\Delta u_n|^{p(x)}dx + \int_{\Omega} b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}dx\right) \end{aligned}$$

$$\begin{aligned} & + \lambda\left(\frac{1}{q^+}-\frac{1}{1-s^+}\right)\int_{\Omega}\frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx \\ \geq & C_4\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\|u_1\|_b^{p^-} + \lambda C_s\left(\frac{1}{q^-}-\frac{\lambda}{1-s^+}\right)\|u_1\|_b^{1-s^+}. \end{aligned} \quad (17)$$

As we have $p^- > 1-\beta^+$, we can make the following choice.

$$\lambda < \min\left\{\frac{C_5(q^- - p^+)(1-s^+)}{C_s p^+(q^+ + s^+ - 1)}, \lambda 0\right\}$$

We obtain $T(u_1) > 0$. Moreover, since $N_\lambda^+ \cap N_\lambda^- = \varnothing$ and $\inf_{u\in N_\lambda^+} T(u) < 0$, then $u_1 \in N_\lambda^-$. The same, if $u_1 \in N_\lambda^-$; hence, there is t_1, satisfying $t_1 u_1 \in N_\lambda^-$ and so $T(t_1 u_1) \leq T(u_1)$.

$$\begin{aligned} I'_\lambda(u_1) = & \int_{\Omega} A(x,\Delta u_1)dx + \int_{\Omega} p(x)b(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}dx - \int_{\Omega} q(x)\frac{|u_1|^{q(x)}}{|x|^{\alpha(x)}}dx \\ & - \lambda\int_{\Omega}(1-s(x))\frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx < 0, \end{aligned}$$

we get

$$\begin{aligned} I'_\lambda(t_1 u_1) \leq & t_1^{p^+}\int_{\Omega} A(x,\Delta u_1)dx + t_1^{p^+}p^+\int_{\Omega} b(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}\Bigg)dx - t_1^{q^-}q^-\int_{\Omega}\frac{|u_1|^{q(x)}}{|x|^{\alpha(x)}}dx - \\ & \lambda t_1^{1-s^+}(1-s^+)\int_{\Omega}\frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx. \end{aligned}$$

Since $1-s^+ < p^+ < q^-$, $I'_\lambda(t_1 u_1) < 0$; so by definition of N_λ^-, we have $t_1 u_1 \in N_\lambda^-$

Now, let us assume that $u_n \nrightarrow u_1$ in Y, we know that

$$\int_\Omega A(x, \Delta u_1)dx \leq \liminf_{n\to\infty} \int_\Omega A(x, \Delta u_n)dx.$$

which gives us

$$\begin{aligned} T(t_1 u_1) \quad &\leq \int_\Omega A(x, \Delta(t_1 u_1))dx + \frac{t_1^{p^+}}{p^+}\int_\Omega b(x)\frac{|u_1|^{p(x)}}{|x|^{p(x)}}dx - \frac{t_1^{q^-}}{q^-}\int_\Omega \frac{|u_1|^{q(x)}}{|x|^{\alpha(x)}}dx \\ &- \lambda \frac{t_1^{1-s^+}}{1-s^+}\int_\Omega \frac{|u_1|^{1-s(x)}}{|x|^{\beta(x)}}dx. \\ &\leq \lim_{n\to\infty}\left[t_1^{p^+}\int_\Omega A(x, \Delta(t_1 u_n))dx + \frac{t_1^{p^+}}{p^+}\int_\Omega b(x)\frac{|u_n|^{p(x)}}{|x|^{p(x)}}\right)dx \\ &- \frac{t_1^{q^-}}{q^-}\int_\Omega \frac{|u_n|^{q(x)}}{|x|^{\alpha(x)}}dx - \lambda\frac{t_1^{1-s^+}}{1-s^+}\int_\Omega \frac{|u_n|^{1-s(x)}}{|x|^{\beta(x)}}dx.\Bigg] \\ &\leq \lim_{n\to\infty} T(t_1 u_n) \leq \lim_{n\to\infty} T(u_n) = \inf_{u\in N_\lambda^-} T(u), \end{aligned}$$

This contradicts with $t_1 u_1 \in N_\lambda^-$; hence, $u_n \to u_1$ in Y and $T(u_1) = \inf_{u\in N_\lambda^-} T(u)$. □

4. Proof of theorem 1

Proof. Invoking Theorems 3 and 4, we can conclude that for every $\lambda \in (0, \lambda_0)$, there exist $u_0 \in N_\lambda^+$ and $u_1 \in N_\lambda^-$ such that

$$T(u_0) = \inf_{u\in N_\lambda^+} T(u) \text{and} T(u_1) = \inf_{u\in N_\lambda^-} T(u).$$

On the one hand, considering the properties of the function T, we can assume that u_0 and u_1 are non-negative, and by applying Lemma 2, we can show that they are critical points of T on the set Y, which makes them weak solutions of the problem (PV). Using the Harnack inequality and the results of Zhang-Liu [25], we can then demonstrate that u_0 and u_1 are non-negative solutions of (PV).

Since $N_\lambda^+ \cap N_\lambda^- = \emptyset$, we conclude that u_0, u_1 must be distinct. Thus, we have demonstrated that the solutions u_0 and u_1 obtained from Theorem 3 and Theorem 4 are indeed different. Consequently, Theorem 1 has now been completely proved. □

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