Boundary value problem for fractional differential inclusions with integral boundary conditions

Arab Journal of Mathematical Sciences

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Received 17 November 2024
Revised 22 March 2025
Accepted 10 October 2025

Abstract

Purpose – In this paper, we investigate the existence of solutions for a class of Caputo fractional differential inclusions with two integral boundary conditions. The convex and nonconvex cases are separately considered. For the first case, an existence result is obtained by applying the Bohnenblust–Karlin's fixed point theorem. For the second case, a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler is used. Further, an illustrating example is presented.

Design/methodology/approach – In Section 2, we recall some basic concepts of the fractional calculus and the theory of multi-valued maps. Some well-known existence results are also recalled. In Section 3, we prove the existence result for the problem (1.1)–(1.3) when, in one case, the right-hand side is convex valued, and in the other case, nonconvex valued. The first result relies on the Bohnenblust–Karlin theorem, while the other is based upon a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler [37]. In Section 4, we propose an example to illustrate our results.

Findings – This paper is concerned with the existence of solutions for a certain class of fractional differential inclusion with integral boundary conditions. Thanks to the Bohnenblust–Karlin's fixed point theorem, an existence result is obtained.

Originality/value – Considering the particular case of single-valued second member, this result is then used to derive an existence result for a certain type of singular boundary value problems.

Keywords Integral conditions, Caputo fractional derivative, Fractional differential inclusions, Fixed point

Paper type Research article

1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There are applications in biology [1, 2], chaotic systems [3, 4], control [5–7], diffusion process [8–10], economics [11], electrochemistry [12], electromagnetism [13], epidemiology [14], photoelasticity [15], variational problems [16, 17] and other fields.

The theory of fractional calculus has also seen the emergence of new types of fractional differential in the modern era, we refer to the monographs of Hilfer [18], Jarad *et al.* [19, 20], Kilbas *et al.* [21], Podlubny [22], Zhou [23].

On the other hand, applied problems require the definitions of fractional derivatives allowing the utilization of physically interpretable initial data, which contain $y(0), y'(0)$, etc. . ., and the same requirements for boundary conditions. Caputo's fractional derivative satisfies these demands. It should be noted that there are several Caputo derivatives. Each of them is obtained by a modification of a previously defined derivative in order to maintain the so useful

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AMS Subject Classification — 26A33, 34A08, 35B40, 35L70

Arab Journal of Mathematical Sciences
Vol. 32 No. 2, 2026
pp. 161-177
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-11-2024-0171

and fundamental property of the classical derivative, namely that the derivative of any constant is the null function. Thus, we have the Caputo-Liouvile, the Caputo-Hadamard, the Caputo-Atangana-Baleanu derivatives and others. For more details concerning geometric, physical interpretations of fractional Caputo derivatives and their applications see Refs. [19–22, 24, 25] and references therein.

Realistic problems arising from economics, optimal control and so on, can be modeled as differential inclusions, so differential inclusions are widely investigated by many authors, see ([26–41]) and references therein.

In 2015, Rezapour *et al.* [42] studied the following singular one integral boundary-value problem:

$$^{C}D^{\alpha}x(t)+f(t,x(t))=0 \qquad t\in J=[0,1],\ n=[\alpha]+1; \tag{1.1}$$

$$x(0)=x'(0)=x^{(3)}(0)=\ldots=x^{(n-1)}(0)=0; \tag{1.2}$$

$$ax(1)=\frac{1}{\Gamma(p)}\int_0^1(1-s)^{p-1}x(s)ds,\ a\geqslant 1,\ \alpha\geqslant 3,\ p\geqslant 1, \tag{1.3}$$

where $f:[0,1]\times[0,+\infty[\mapsto[0,+\infty[$ has only one singularity at the point 0. The authors used the Banach contraction principle and the compressive form of Krasnosel'skii's cone fixed point theorem to prove successively an uniqueness and an existence results. In 2020, Kheyryan and Rezapour [43] investigated the Ulam - Hyers stability of this problem via its equivalent integral formulation. In this paper, motivated by the work of Rezapour [42], we wish to discuss the existence of solutions for the Boundary value problems (BVP for short), for the fractional differential inclusions of the form.

$$^{C}D^{\alpha}x(t)\in F(t,x(t)), \quad t\in J=[0,\ T],\ \alpha\geqslant 3,\ 1\leq T<+\infty; \tag{1.4}$$

$$x^{(j)}(0)=0, \qquad j\in\{0,1,\ldots,n-1\}\backslash\{j_1,j_2\},\ 0\leq j_1,\ j_2\leq n-1,\ n=[\alpha]+1; \tag{1.5}$$

$$a_k x^{(q_k)}(T)=\frac{1}{\Gamma(p_k)}\int_0^T(T-s)^{p_k-1}x(s)ds, \quad (a_k,p_k)\in\mathbb{R}\times\mathbb{N},\ 0<a_k,\ 1<p_k,\ k=\overline{1,2}, \tag{1.6}$$

Where $^{C}D^{\alpha}$ is the Caputo fractional derivative, $x^{(q)}=\frac{d^q}{dt^q}$. $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multi-valued map, j_1, j_2, q_1, q_2, p_1, p_2 are positive integers such that

$$0\leqslant q_1\neq q_2\leqslant q_0:=\max(q_1,q_2)<j_0:=\min(j_1,j_2)\leqslant j_1\neq j_2\leqslant\max(j_1,j_2)\leqslant[\alpha]. \tag{1.7}$$

It should be noted that if F is a single-valued function and

$$[0,\ T]=[0,\ 1],\ a_1=a_2,\ j_1=j_2=0,\ p_1=p_2,\ q_1=q_2=2,$$

we retrieve the formulation of Problem (1.1-1.3) investigated in Ref. [42].

In Section 2, we recall some basic concepts of the fractional calculus and the theory of multi-valued maps. Some well-known existence results are also recalled. In Section 3, we firstly use the Bohnenblust-Karlin's fixed point theorem to prove the existence result for the problem (1.4)-(1.6) in the case of convex valued right hand side. Secondly, we prove the existence result for the nonconvex valued right hand side by applying a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler [44]. In Section 4, we propose an example to illustrate our results.

Of course, approaches adopted in the present work are not the only possible. Other approaches as the end-point method, the nonlinear alternative of Leray-Schauder type and the nonlinear alternative for Kakutani maps, may also be considered under the appropriate conditions.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $J = [0, T]$. In the sequel, we will use the following real Banach spaces:

$C(J)$ is the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : \ t \in J\},$$

$L^1(J)$ is the Banach space of real functions x that are integrable on J with the norm

$$\|x\|_{L^1} = \int_J \Big|x(t)\Big| dt := \int_0^T |x(t)| dt.$$

Definition 2.1. *([21, 22]) If the function h is an element of* $L^1(J)$*, then its fractional integral order* $\alpha \in \mathbb{R}^+$ *is defined by*

$$I^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ *is the gamma function.*

Definition 2.2. *([21], [22]) For a function* $h : J \mapsto \mathbb{R}$ *the Caputo fractional derivative order* $\alpha \in \mathbb{R}^+$ *is defined by*

$$\left({}^C D^\alpha h \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}} \ ds,$$

where $n := [\alpha] + 1$ *and* $h^{(n)}$ *is the differential order n of h.*

Lemma 2.3. *[21].*

(1) *If* $v \in L^1(J)$, $1 \leqslant p \leqslant +\infty$, $\alpha > 0$, $\beta > 0$, *then*

$$I^\alpha I^\beta v(t) = I^{\alpha+\beta} v(t).$$

(2) *If* $\alpha > 0$, $\nu > -1$, *then*

$$I^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} t^{\alpha+\nu}$$

Lemma 2.4. *[21] Let the function* $h \in L^1(J)$*. Then, the general solution of the fractional differential equation*

$$ {}^{C}D^{\alpha}x(t) + h(t) = 0,\ (0 \leqslant t \leqslant T) $$

is given by

$$ x(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, $$

where c_0, c_1, . . ., c_{n-1} *are real constants.*

We now recall some useful concepts from the theory of multi-valued maps [45, 46]. If X is an infinite dimensional real Banach space with the norm $\|.\|_X$, then a multivalued map. $H : X \to \mathcal{P}(X)$

(1) is convex (closed) valued if $H(x)$ is convex (closed) for every $x \in X$.

(2) is bounded on bounded subsets if $H(\mathbb{B}) = \cup_{x\in\mathbb{B}} H(x)$ is bounded in X for any bounded subset $\mathbb{B}$ of X.

(3) is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $H(x_0)$ is a nonempty, closed in X and for each open set $\mathbb{B}$ of X containing $H(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $H(\mathcal{N}) \subseteq \mathbb{B}$.

(4) is said to be completely continuous if $H(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X.

(5) has a fixed point if there exists $x \in X$ such that $x \in H(x)$.

(6) is said to be Carathéodory if it is measurable with respect to $t \in J$ for each fixed $x \in X$ and u.s.c. on X for almost all $t \in J$.

Notice that if H is completely continuous with nonempty compact values, then H is u.s.c. if and only if H has a closed graph, that is:

$$ (x_n \to x,\ y_n \to y,\ y_n \in H(x_n)) \Rightarrow y \in H(x). $$

For each $y \in C(J, \mathbb{R})$, define the selection $S_{F,y}$ of F by

$$ S_{F,y} = \left\{ v \in L^1([b,T],\mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e } t \in [b,T] \right\}. $$

Let (X, d) be a metric space induced from the normed space $(X, \| \cdot \|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$ H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}. $$

For our purpose, we need to introduce the following subsets of $\mathcal{P}(X)$:

(1) $P_{cl}(X)$ (respectively $P_c(X)$ is the set of all closed (convex respectively) subsets of X.

(2) $P_{cp}(X)$ is the set of all compact subsets of X.

(3) $P_{cl,c}(X)$ (respectively $P_{cp,c}(X)$) is the set of all nonempty, closed and convex (nonempty, compact and convex respectively) subsets of X.

Definition 2.5. A multivalued operator N: $X \to P_{cl}(X)$ *is called*

(1) γ*-Lipschitz if and only if there exists* $\gamma > 0$ *such that*

$$ H_d(N(x), N(y)) \leq \gamma d(x, y), \text{ for each } x, y \in X, $$

(2) *a contraction if and only if it is* γ*-Lipschitz with* $\gamma < 1$.

Lemma 2.6. *(Covitz and Nadler [44]) Let (X, d) be a complete metric space. If $N: X \to P_{cl}(X)$ is a contraction, then $Fix(N) \neq \emptyset$, where $Fix(N)$ is the set of all fixed points for N.*

Lemma 2.7. *(Bohnenblust-Karlin [47]) Let $D \in P_{cl,c}(X)$ and let $H: D \to P_{cl,c}(D)$ be u.s.c. such that $\overline{H(D)}$ is compact. Then, H has a fixed point.*

Lemma 2.8. *([48]) Let I be a compact real interval, ϕ a linear continuous map from $L^1(I)$ into $C(I)$. Let also $F: I \times \mathbb{R} \to P_{cl,c}(\mathbb{R})$ be a multi-valued map satisfying the following assumptions:*

(1) *F is Carathéodory.*

(2) *For each fixed $x \in \mathbb{R}$, $S_{F,\ x} := \left\{ f \in L^1(I) :\ f(t) \in F(t,x) \text{ for a.e. } t \in I \right\}$ is nonempty.*

Then, the operator $\phi \circ S_F$ defined from $C(I)$ into $P_{cl,c}(C(I))$ by $(\phi \circ S_F)(x) = \phi(S_{F,x})$ is a closed graph operator in $C(I) \times C(I)$.

For more details on multivalued maps see the books of Aubin and Cellina [49], Aubin and Frankowska [50], Deimling [45], Castaing and Valadier [51].

3. Existence of solutions

3.1 Auxiliary lemmas

We will use the following auxiliary scalar functions:

$$u(a,p,j,q) = \Gamma(j+1)\frac{a\Gamma(p+j+1)T^{j-q} - \Gamma(j-q+1)T^{j+p}}{\Gamma(p+j+1)\Gamma(j-q+1)},\ q \leq j. \tag{3.1}$$

$$v(s,a,p,q) = \frac{a\Gamma(\alpha+p)(T-s)^{\alpha-q-1} - \Gamma(\alpha-q)(T-s)^{\alpha+p-1}}{\Gamma(\alpha-q)\Gamma(\alpha+p)}, \quad (0 \leqslant s \leqslant T), \tag{3.2}$$

$$A = u(a_1,p_1,j_1,q_1)u(a_2,p_2,j_2,q_2) - u(a_1,p_1,j_2,q_1)u(a_2,p_2,j_1,q_2), \tag{3.3}$$

$$g_1(s) = \frac{u(a_2,p_2,j_2,q_2)v(s,a_1,p_1,q_1) - u(a_1,p_1,j_2,q_1)v(s,a_2,p_2,q_2)}{A}, \tag{3.4}$$

$$g_2(s) = \frac{u(a_1,p_1,j_1,q_1)v(s,a_2,p_2,q_2) - u(a_2,p_2,j_1,q_2)v(s,a_1,p_1,q_1)}{A}. \tag{3.5}$$

There is no difficulty to check that

$$u(a_1,p_1,j_1,q_1)g_1(s) + u(a_1,p_1,j_2,q_1)g_2(s) = v(s,a_1,p_1,q_1), \quad (s \in J) \tag{3.6}$$

and

$$u(a_2,p_2,j_1,q_2)g_1(s) + u(a_2,p_2,j_2,q_2)g_2(s) = v(s,a_2,p_2,q_2), \quad (s \in J). \tag{3.7}$$

Lemma 3.1. *Let $h \in L^1(J)$, $A \neq 0$ and assume that conditions 1.7 are satisfied. Then, the solution of the equation*

$$\left(D_C^\alpha x\right)(t) + h(t) = 0, \quad t \in J = [0,T],\ \alpha \geqslant 3 \tag{3.8}$$

subject to the conditions

$$x^{(j)}(0)=0,\quad j\in\{0,1,\dots,n-1\}\backslash\{j_1,j_2\},\ 0\le j_1,\ j_2\le n-1,\ n=[\alpha]+1 \tag{3.9}$$

and

$$a_k x^{(q_k)}(T)=I^{p_k}x(T),\qquad (a_k,p_k)\in\mathbb{R}\times\mathbb{N},\ 0<a_k,\ 1\leqslant p_k,\ k=\overline{1,2} \tag{3.10}$$

is given by

$$x(t)=\int_0^T G(t,s)h(s)ds, \tag{3.11}$$

where

$$G(t,s)=\begin{cases}\dfrac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+t^{j_1}g_1(s)+t^{j_2}g_2(s); & 0\leqslant s<t\\ t^{j_1}g_1(s)+t^{j_2}g_2(s); & t\leqslant s\leqslant T\end{cases}. \tag{3.12}$$

Proof According to Lemma 2.4, the solution of the problem (3.8)-(3.10) has the form

$$x(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$

where, c_0, c_1, …, c_{n-1} are real constants.

Using initial conditions at $t=0$, one obtains that $c_k=0$ for $k\notin\{j_1,j_2\}$ and then,

$$x(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds+c_{j_1}t^{j_1}+c_{j_2}t^{j_2},\quad (0\leqslant t\leqslant T).$$

Using integral conditions $a_k x^{(q_k)}(T)=I^{p_k}(T),\quad k=\overline{1,2}$ and after some transformations, we get that constants c_{j_1} and c_{j_2} satisfy the following linear system:

$$(S):\ \begin{cases} c_{j_1}u(a_1,p_1,j_1,q_1)+c_{j_2}u(a_1,p_1,j_2,q_1)=\displaystyle\int_0^T v(s,a_1,p_1,q_1)h(s)ds\\ c_{j_1}u(a_2,p_2,j_1,q_2)+c_{j_2}u(a_2,p_2,j_2,q_2)=\displaystyle\int_0^T v(s,a_2,p_2,q_2)h(s)ds\end{cases} \tag{3.13}$$

System (S) admits a unique solution (c_{j_1},c_{j_2}) given by

$$\begin{cases} c_{j_1}=\dfrac{1}{A}\displaystyle\int_0^T (u(a_2,p_2,j_2,q_2)v(s,a_1,p_1,q_1)-u(a_1,p_1,j_2,q_1)v(s,a_2,p_2,q_2))h(s)ds\\ c_{j_2}=\dfrac{1}{A}\displaystyle\int_0^T (u(a_1,p_1,j_1,q_1)v(s,a_2,p_2,q_2)-u(a_2,p_2,j_1,q_2)v(s,a_1,p_1,q_1))h(s)ds\end{cases}. \tag{3.14}$$

Remarking that

$$(c_{j_1}, c_{j_2}) = \left(\int_0^T g_1(s)h(s)ds,\ \int_0^T g_2(s)h(s)ds\right)$$

and replacing in the expression of $x(t)$, we get

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_{j_1}t^{j_1} + c_{j_2}t^{j_2} \\ &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + t^{j_1}\int_0^T g_1(s)h(s)ds + t^{j_2}\int_0^T g_2(s)h(s)ds \\ &= \int_0^T G(t,s)h(s)ds. \end{aligned}$$

Conversely, assume that function $x(t),\ \ t\in[0,\ \ T]$ admits the representation given by (3.11), (3.12). Remarking that

$$x(t) = -I^{\alpha}_{0^+}h(t) + t^{j_1}\int_0^T g_1(s)h(s)ds + t^{j_2}\int_0^T g_2(s)h(s)ds,$$

it is not difficult to check that $(D^{\alpha}_{C}x)(t) + h(t) = 0, \quad t\in J = [0, T]$ and that initial conditions (3.9) are all satisfied. Let us proof the condition (3.10) for $k = 1$ (the case $k = 2$ is demonstrated in a similar way). Direct calculations give us,

$$\begin{aligned} a_1x^{(q_1)}(T) &= \int_0^T \frac{-a_1(T-s)^{\alpha-q_1-1}}{\Gamma(\alpha-q_1)}h(s)ds + \int_0^T \frac{a_1\Gamma(j_1+1)T^{j_1-q_1}}{\Gamma(j_1-q_1+1)}g_1(s)h(s)ds \\ &+ \int_0^T \frac{a_1\Gamma(j_2+1)T^{j_2-q_1}}{\Gamma(j_2-q_1+1)}g_2(s)h(s)ds, \end{aligned}$$

$$\begin{aligned} I^{p_1}_{0^+}x(T) &= \int_0^T \frac{-(T-s)^{\alpha+p_1-1}}{\Gamma(\alpha+p_1)}h(s)ds + \int_0^T \frac{\Gamma(j_1+1)T^{j_1+p_1}}{\Gamma(p_1+j_1+1)}g_1(s)h(s)ds \\ &+ \int_0^T \frac{\Gamma(j_2+1)T^{j_2+p_1}}{\Gamma(p_1+j_2+1)}g_2(s)h(s)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} I^{p_1}_{b^+}x(T) - a_1x^{(q_1)}(T) &= \int_0^T \frac{a_1\Gamma(\alpha+p_1)(T-s)^{\alpha-q_1-1} - \Gamma(\alpha-q_1)(T-s)^{\alpha+p_1-1}}{\Gamma(\alpha-q_1)\Gamma(\alpha+p_1)}h(s)ds \\ &- \int_0^T \Gamma(j_1+1)\frac{a_1\Gamma(p_1+j_1+1)T^{j_1-q_1} - \Gamma(j_1-q_1+1)T^{j_1+p_1}}{\Gamma(p_1+j_1+1)\Gamma(j_1-q_1+1)}g_1(s)h(s)ds \\ &- \int_0^T \Gamma(j_2+1)\frac{a_1\Gamma(p_1+j_2+1)T^{j_2-q_1} - \Gamma(j_2-q_1+1)T^{j_2+p_1}}{\Gamma(p_1+j_2+1)\Gamma(j_2-q_1+1)}g_2(s)h(s)ds \\ &= \int_0^T v(s,a_1,p_1,q_1)h(s)ds \\ &- \int_0^T [u(a_1,p_1,j_1,q_1)g_1(s) + u(a_1,p_1,j_2,q_1)g_2(s)]h(s)ds. \end{aligned}$$

Using 3.6, we get finally,

$$I_{0^+}^{p_1}x(T) - a_1 x^{(q_1)}(T) = 0.$$

$\square$

Remark 3.2. *We have* $A \neq 0$ *for example if we have simultaneously*

$$\begin{cases} a_1 > \max\left(\dfrac{\Gamma(j_1-q_1+1)T^{j_1+p_1}}{\Gamma(p_1+j_1+1)T^{j_1-q_1}};\ \dfrac{\Gamma(j_2-q_1+1)T^{j_2+p_1}}{\Gamma(p_1+j_2+1)T^{j_2-q_1}}\right) \\[2ex] a_2 < \max\left(\dfrac{\Gamma(j_2-q_2+1)T^{j_2+p_2}}{\Gamma(p_2+j_2+1)T^{j_2-q_2}};\ \dfrac{\Gamma(j_1-q_2+1)T^{j_1+p_2}}{\Gamma(p_2+j_1+1)T^{j_1-q_2}}\right) \\[2ex] a_2 > \min\left(\dfrac{\Gamma(j_2-q_2+1)T^{j_2+p_2}}{\Gamma(p_2+j_2+1)T^{j_2-q_2}};\ \dfrac{\Gamma(j_1-q_2+1)T^{j_1+p_2}}{\Gamma(p_2+j_1+1)T^{j_1-q_2}}\right) \end{cases}$$

Indeed, in this case either

$$u(a_1,p_1,j_1,q_1)\ u(a_2,p_2,j_2,q_2) > 0, \quad u(a_1,p_1,j_2,q_1)\ u(a_2,p_2,j_1,q_2) < 0$$

or,

$$u(a_1,p_1,j_1,q_1)u(a_2,p_2,j_2,q_2) > 0, \quad u(a_1,p_1,j_2,q_1)u(a_2,p_2,j_1,q_2) < 0.$$

Lemma 3.3. *Suppose that* $A \neq 0$ *and let*

$$\Delta := \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\Gamma(j_1+1)+\Gamma(j_2+1))}{|A|} \times \frac{\left(a_2T^{\alpha-q_2-1}+T^{\alpha+p_2-1}\right)\left(a_1T^{\alpha-q_1-1}+T^{\alpha+p_1-1}\right)}{\Gamma(j_0-q_0+1)\Gamma(\alpha-q_0).}.$$

Then,

$$\left|G(t,s)\right| < \Delta, \quad \forall (t,s) \in J^2. \tag{3.15}$$

Proof. It is a direct calculation that uses the expression of each $g_i(s)$, $i = 1, 2$ and the obvious inequalities

$$q_0 := \max(q_1,q_2) < j_0 := \min(j_1,j_2) \leqslant [\alpha] = n-1,$$

$$0 \leq \left|u(a,p,j,q)\right| \leqslant \Gamma(j+1)\,\frac{aT^{\alpha-q-1}+T^{\alpha+p-1}}{\Gamma(j-q+1)},$$

$$0 \leqslant \left|v(s,a,p,q)\right| \leqslant \frac{aT^{\alpha-q-1}+T^{\alpha+p-1}}{\Gamma(\alpha-q)}, \quad 0 \leq s \leq T,$$

$$\min(\Gamma(\alpha-q_1);\ \Gamma(\alpha-q_2)) > \Gamma(\alpha-q_0).$$

□

3.2 The convex case

In this section, we present an existence result of solutions for the problem (1.4)-(1.7) under the condition that $F(t, s)$ is a convex multi-valued map. Let us start by defining what we mean by a solution of the problem.

Definition 3.4. *A* $n = ([\alpha]+1)$ *-times continuously differentiable function* $y : J \mapsto \mathbb{R}$ *is said to be a solution of the problem* (1.4)-(1.7) *if it satisfies the equation* $({}^{C}D^{\alpha}x)\ (t) = F(t, x(t))$ *on* J *together with the conditions* (1.5) *and* (1.6).

Theorem 3.5. *Let* $A \neq 0$ *and assume the following hypothesis hold:*

(H1) $F : [0, T] \times \mathbb{R} \to P_{cl,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(H2) *For each* $r > 0$, *there exists a positive function* $\rho_r \in L^1(J)$ *such that*

$$(t, x) \in J \times \mathbb{R} \text{ and } \ |x| \leq r \Rightarrow \|F(t,\ x)\| := \sup\{|u| : u(t) \in F(t, x)\} \leq \rho_r(t).$$

If

$$\mu := \liminf_{r\to+\infty}\left(\frac{\int_0^T \rho_r(t)}{r}\right) < \frac{1}{\Delta}, \tag{3.16}$$

then the problem (1.4)-(1.6) *has at least one solution on* J.

Proof. We begin by transform this problem into a fixed point one. For this, consider the multi-valued operator,

$$N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$$

defined by

$$N(y) = \left\{ h \in C(J, \mathbb{R}) :\ h(t) = \int_0^T G(t, s) f(s) ds,\ f \in S_{F,y} \right\},$$

where the function $G(t, s)$ is given by (3.12). For $r > 0$ let

$$D_r := \left\{ x \in J : \|x\|_\infty \leq r \right\}$$

According to Lemma 3.1, fixed points of N are the solutions of the problem (1.4)-(1.6).

Step 1: *We prove that there exists* $r_0 > 0$ *such that* $N(D_{r_0}) \subseteq D_{r_0}$.

If it is not true, we get

$$\forall r > 0,\ N(D_r) = \bigcup_{x \in D_r} N(x) \nsubseteq D_r.$$

This is equivalent to

$$\forall r > 0,\ \exists x_r \in D_r,\ \exists h_r \in N(x_r) : \left\|h_r\right\|_\infty > r$$

with

$$h_r(t) = \int_0^T G(t,s)f(s)ds,\ f \in S_{F,\ x_r}.$$

According to (H2), there exists a positive function $\rho_r \in L^1(J)$ such that

$$|f(s)| \leq \rho_r(s),\ \forall s \in J.$$

Thus,

$$r < \|h_r\|_\infty \leq \Delta \int_0^T \rho_r(s)ds.$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we obtain that

$$\Delta \liminf_{r \to +\infty} \left(\frac{\int_0^T \rho_r(t)}{r} \right) \geq 1$$

which contradicts 3.16. Hence $N(D_{r_0}) \subseteq D_{r_0}$ for a certain $r_0 > 0$.

Let N_0 be the restriction of N to closed and convex subset D_{r_0}. Clearly, each fixed point for N_0 is also a fixed point for N. So, we shall follow the scheme proposed in Ref. [35] to prove that N_0 satisfies all assumptions of the theorem 2.7.

Step 2: *We prove that $N_0(x)$ is convex for each $x \in D_{r_0}$.*

Let h_1, h_2 belong to $N_0(x)$. There exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in J$ and for each $i = 1, 2$, we have

$$h_i(t) = \int_0^T G(t,s)f_i(s)ds.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \int_0^T G(t,s)(\lambda f_i + (1-\lambda)f_2)(s)ds.$$

Since $S_{F,x}$ is convex (F has convex values), we have
$\lambda h_1 + (1-\lambda)h_2 \in N_0(y)$.

Step 3: *We prove that $\overline{N_0(D_{r_0})}$ is a compact subset of D_{r_0}.*

According to the Step 1, $N_0(D_{r_0})$ is a subset of D_{r_0} and hence is uniformly bounded. Thus, it suffices to prove that $N_0(D_{r_0})$ is equicontinuous. Let $x \in D_{r_0}$ and $h \in N_0(x)$. There exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^T G(t,s)f(s)ds.$$

Thus, for $t_1, t_2 \in J$: $t_1 < t_2$, and applying (H2), we get

$$\begin{aligned}|h(t_2)-h(t_1)| &\leq \int_0^T |G(t_1,s)-G(t_2,s)||f(s)|ds\\ &\leq \int_0^T |G(t_1,s)-G(t_2,s)|\rho_{r_0}(s)ds\end{aligned}$$

Since $\rho_{r_0}\in L^1(J)$ and $G(t,s)$ is continuous with respect to $t\in J$, the right hand side of the above inequality tends to zero independently of $x\in D_{r_0}$ and $h\in N(x)$, whenever $t_1\rightarrow t_2$. This implies that $N_0(D_{r_0})$ is equicontinuous and that $\overline{N_0(D_{r_0})}$ is a compact subset of D_{r_0}.

Step 4: *We prove that* N_0 *has a closed graph.*

We need to prove that

$$\left(x_n\rightarrow x_*,\ h_n\in N_0(x_n),\ h_n\rightarrow h_*\right)\rightarrow h_*\in N_0\left(x_*\right).$$

We have

$$h_n\in N_0(x_n)\Leftrightarrow\left(\exists f_n\in S_{F,x_n}:\ h_n(t)=\int_0^T G(t,s)f_n(s)ds,\ \forall t\in J\right). \tag{3.17}$$

Consider the linear operator ϕ: $L^1(J)\rightarrow C(J)$ given by

$$f\mapsto(\phi(f))(t)=\int_0^T G(t,s)f(s)ds.$$

It is easy to see that

$$\|\phi(f)\|_\infty=\sup_{t\in J}|(\phi(f))(t)|\leq\Delta\|f\|_{L^1}.$$

This means that ϕ is bounded and continuous. By Lemma 2.8, $\phi\circ S_F$ has a closed graph. Since, $h_n=\phi(f_n)\in\phi(S_{F,\ x_n})=\phi\circ S_F(x_n)$, we deduce that

$$h_*=\lim_{n\rightarrow+\infty}h_n=\lim_{n\rightarrow+\infty}\phi(f_n)\in\phi(S_{F,\ x^*})=\phi\circ S_F(x^*).$$

This is equivalent to,

$$\exists f_*\in S_{F,\ x^*}:\ h_*(t)=\left(\phi\left(f_*\right)\right)(t)=\int_0^T G(t,s)f_*(s)ds,\quad\forall t\in J.$$

In other words, N_0 has a closed graph.

Summarizing, we get that the multi-valued N_0 has a closed graph, bounded, convex and closed values which are also compact since $\overline{N(D_{r_0})}$ is so. Hence, by Lemma 2.8, N_0 is upper semicontinuous. Applying the theorem 2.7, we deduce that N_0 has a fixed point x in D_r which is a solution of the problem (1.4)-(1.6). ☐

3.3 The nonconvex case

We present now a result for the problem (1.4)-(1.5) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler ([44]).

Theorem 3.6. Assume that the following hypotheses hold:

(H3) $F: J\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(.,u): J\to\mathcal{P}_{cp}(R)$ *is measurable for each* $u\in\mathbb{R}$.

(H4) *For each* $y\in C(J)$, *there exists a constant* $M_y>0$ *depending only of* y *such that*

$|w|\leq M_y$ *for each* $w\in\bigcup_{t\,\in\,J}F(t,y(t))$.

(H5) *There exists* $l\in L^1(J,\mathbb{R}^+)$, *such that*

$$H_d(F(t,u),F(t,\overline{u}))\leq l(t)|u-\overline{u}| \text{ for every } u,\overline{u}\in\mathbb{R},$$

If

$$\Delta\|l\|_{L^1(J)}<1, \tag{3.18}$$

then the problem (1.4)-(5) *has at least one solution on* J.

Remark 3.7. *By (H3), we can see that* $S_{F,x}$ *is nonempty for each* $x\in L^1(J,\mathbb{R})$, *so* F *has a measurable selection (see Ref. [51], Theorem III.6)*

Proof. We shall show that N satisfies the assumptions of Lemma (2.6). The proof will be given in two steps.

Step1: $N(x)\in\mathcal{P}_{cl}(C(J,\mathbb{R}))$ *for each* $x\in C(J,\mathbb{R})$.

Let $(h_n)_{n\geq 0}\subset N(x)$ be such that $h_n\to\overline{h}$ in $C(J,\mathbb{R})$. Then $\overline{h}\in C(J,\mathbb{R})$ and for each natural number n there exists a null-measure subset J_n of $J=[0,T]$ together with a function $v_n\in L^1(J)$ such that

$$t\in J\backslash J_n\Rightarrow v_n(t)\in F(t,x(t)) \quad\text{and}\quad h_n(t)=\int_0^T G(t,s)v_n(s)ds.$$

The subset $\overline{J}=\bigcup_n J_n$ is null-measure and $v_n(t)\in F(t,x(t))$ for each natural n and each $t\in J\backslash\overline{J}$.

Using the compactness of each $F(t,x(t))$, $t\in J$, we may pass to a subsequence in necessary to get that $\{v_n(t)\}_n$ converges to a certain function $\overline{v}(t):=\lim_{n\to+\infty}v_n(t),\ t\in J$.

Clearly, $\overline{v}(t)\in F(t,x(t)),\ \forall t\in J\backslash\overline{J}$.

By the hypothesis (H4),

$$\sup\{|\overline{v}(t)|,\ |v_1(t)|,\ |v_2(t)|,\ \cdots\}\leq M_x,\ \forall t\in J.$$

This implies that $\overline{v}\in L^1(J)$.

Thus, by the dominated convergence Lebesgue theorem, we have for each $t\in J$

$$\begin{aligned}\overline{h}(t) &= \lim_{n\to+\infty}h_n(t)=\lim_{n\to+\infty}\int_0^T G(t,s)v_n(s)ds\\ &= \int_0^T G(t,s)\left(\lim_{n\to+\infty}v_n(s)\right)ds=\int_0^T G(t,s)\overline{v}(s)ds.\end{aligned}$$

So, $\overline{x}\in N(x)$ and thus, $N(x)\in\mathcal{P}_{cl}(C(J,\mathbb{R}))$ for each $x\in C(J,\mathbb{R})$.

Step2: *There exists* $\gamma<1$ *such that* $H_d(N(x),N(\overline{x}))<\gamma\|x-\overline{x}\|_{L^1}$ *for each* $x,\overline{x}\in C(J,\mathbb{R})$.

Let $x,\overline{x}\in C(J,\mathbb{R})$ and $h_1\in N(x)$. There exists $v_1\in F(t,x(t))$ such that for each $t\in J$

$$h_1(t)=\int_0^T G(t,s)v_1(s)ds.$$

From (H5) it follows that

$$H_d(F(t,x(t)),F(t,\overline{x}(t)) \leq l(t)|x(t)-\overline{x}(t)|$$

Hence, there exists $w \in F(t,\overline{x}(t))$ such that

$$|v_1(t)-w| \leq l(t)|x(t)-\overline{x}(t)|, \ t \in J.$$

Consider $U: J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t)=\{w \in \mathbb{R}: |v_1(t)-w| \leq l(t)|x(t)-\overline{x}(t)|\}$$

Since the multivalued operator $V(t)=U(t)\cap F(t,\overline{x}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t,\overline{x}(t))$ and for each $t \in J$,

$$|v_1(t)-v_2(t)| \leq l(t)|x(t)-\overline{x}(t)|, \ t \in J.$$

Let us define for each $t \in J$

$$h_2(t)=\int_0^T G(t,s)v_2(s)ds.$$

Then, $h_2 \in N(\overline{x})$ and for each $t \in J$

$$\begin{aligned}
\|h_1-h_2\|_\infty &\leq \int_0^T |G(t,s)|l(s)|v_1(s)-v_2(s)|ds \\
&\leq \int_0^T \Delta l(s)|v_1(s)-v_2(s)|ds \\
&\leq \Delta\|l\|_{L^1(J)}\|y-\overline{y}\|_\infty.
\end{aligned}$$

Thus

$$\|h_1-h_2\|_\infty \leq \Delta\|l\|_{L^1(J)}\|x-\overline{x}\|_\infty$$

We obtain an analogous relation by interchanging the roles of x and $\overline{x}$. Consequently, we can affirm that

$$H_d(N(x),N(\overline{x})) \leq \Delta\|l\|_{L^1(J)}\|x-\overline{x}\|_\infty.$$

By (3.18), N is a contraction. Applying Lemma(2.6), we get that N has a fixed point x which is solution to (1.4)-(1.5). The proof is complete. ☐

4. Example

In this section, we give an example illustrating our main results. Consider the boundary fractional differential value problem

$$\left({}^{C}D^{\frac{70}{8}}x\right)(t) \in F(t,x(t)), \qquad t \in J=[0,1] \tag{4.1}$$

$$x^{(j)}(0)=0, \qquad j \notin \{4,8\} \tag{4.2}$$

$$ax^{(q)}(1) = \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} x(s)ds, \qquad (a,p,q) \in \{(1,5,0);(3,7,2)\} \tag{4.3}$$

where

$$F(t,x) = \begin{cases} \frac{(1-\sigma)|x|}{2\Delta t^{\sigma}}.[0,\ 1] := \left[0,\ \frac{(1-\sigma)|x|}{2\Delta t^{\sigma}}\right], & if\ t \neq 0 \\ \{0\}, & if\ t = 0 \end{cases},\ 0 < \sigma < 1.$$

In this case,

$$\alpha = \frac{70}{8},\ \ n := [\alpha] + 1 = 9,\ (j_1, j_2) = (4,8),\ (q_1,q_2) = (0,2),\ (a_1,a_2) = (1,3),\ (p_1,p_2) = (5,7)$$

and $A{:} = \det(S) = 131, 99$. Recalling formula (3.15), we obtain that $\Delta \simeq 5, 3979$.

Clearly, F is Carathéodory with closed and convex values. For each x fixed in $\mathbb{R}$ function f_x defined on [0, 1] by

$$f_x(t) = \begin{cases} \frac{(1-\sigma)|x|}{2\Delta t^{\sigma}}, & if\ t \neq 0 \\ 0, & if\ t = 0 \end{cases}$$

is an element of $S_{F,\ x}$. Moreover,

$$\forall r > 0,\ (t,x) \in [0,\ 1] \times \mathbb{R} \wedge |x| \leq r \Rightarrow \|F(t,x)\| \leq \rho_r(t) := f_r(t).$$

Finally, it follows from inequality

$$\left(\int_0^1 \frac{\rho_r(t)dt}{r}\right) = \int_0^1 \frac{(1-\sigma)}{2\Delta t^{\sigma}} dt = \frac{1}{2\Delta}$$

that

$$\liminf_{r \to +\infty} \left(\int_0^1 \frac{\rho_r(t)dt}{r}\right) < \frac{1}{\Delta}.$$

Since all the conditions of Theorem(3.5) are satisfied, the problem (1.4)-(1.6) has at least one solution x on [0,1].

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