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Equivariant deformation cohomology and group actions on compatible Hom-Leibniz algebras

Rinkila Bhutia and Namita Behera
Department of Mathematics, Sikkim University, Gangtok, India

Abstract

Purpose – This paper introduces and studies compatible G-Hom-Leibniz algebras, namely compatible Hom-Leibniz algebras equipped with a finite group action. The aim is to develop their equivariant deformation cohomology theory and to analyze how this cohomology governs equivariant formal, finite-order and infinitesimal deformations. The paper also examines the role of G-Nijenhuis operators in generating trivial equivariant infinitesimal deformations.

Design/methodology/approach – We construct a graded Lie algebra whose Maurer–Cartan elements encode compatible G-Hom-Leibniz algebra structures. Using this graded Lie algebra, we define an equivariant cohomology theory with coefficients in the algebra itself and apply it to study several types of equivariant deformations. We further introduce the notion of a G-Nijenhuis operator and explore its connection with trivial infinitesimal deformations.

Findings – The proposed graded Lie algebra completely characterizes compatible G-HomLeibniz algebras. The resulting equivariant cohomology controls their deformation behaviour and provides criteria for extending finite-order deformations. We also show that trivial equivariant infinitesimal deformations correspond precisely to G-Nijenhuis operators.

Originality/value – This work provides the first systematic treatment of compatible Hom-Leibniz algebras under group actions and develops their equivariant deformation cohomology. The introduction of G-Nijenhuis operators and their deformation-theoretic interpretation offers new tools for further research in Hom-type algebras and related areas.

Keywords Hom-Leibniz algebra, Compatible structure, Maurer-Cartan element, Cohomology, Deformation, Group action

Paper type Research article

1. Introduction

Leibniz algebra is a non-commutative generalization of Lie algebra. It was introduced and called D-algebra in papers by A. M. Bloch and published in the 1960s to signify its relation with derivations. Later in 1993, J. L. Loday [1] introduced the same structure and called it Leibniz algebra. The cohomology theory of Leibniz algebra with coefficients in a bimodule has been studied in Ref. [2]. The concept of Hom algebras was introduced by Hartwig, Larsson and Silverstrov [3]. Makhlouf and Silverstrov [4] introduced the notion of Hom-Leibniz algebra, generalizing Hom-Lie algebras. Hom-algebra structures have been widely studied since then.

Algebraic deformation theory was introduced by Gerstenhaber for rings and algebra in a series of papers [5–9]. Subsequently, algebraic deformation theory has been studied for different kinds of algebras. To study the deformation theory of any algebra, one needs a suitable cohomology, known as the deformation cohomology, which controls the deformation. In Ref. [10], D. Balavoine studies the formal deformation of algebras using the theory of

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Maurer-Cartan elements in a graded Lie algebra. In particular, this approach is used to study the deformation of Leibniz algebras.

Equivariant cohomology, introduced by Borel in the late 1950s, is a cohomology theory for topological spaces equipped with group actions. Since then, equivariant techniques have found many applications in other areas such as algebraic geometry, representation theory and K-theory. In recent years, there have been many studies on equivariant cohomology, deformation theory and their relationships for various types of algebras and systems, for example, on Leibniz algebras [11], associative algebras [12], Lie triple systems [13], compatible Hom-associative algebras [14] etc. Here, we have defined a compatible Hom-Leibniz algebra to be a pair of Hom-Leibniz algebras such that the linear combination of their algebraic structure is also a Hom-Leibniz algebra. Using the Balavoine bracket, we define a graded Lie algebra whose Maurer-Cartan elements characterize the structure of compatible Hom-Leibniz algebras. We proceed to investigate the internal symmetry of the compatible Hom-Leibniz algebra by introducing a concept of group action on it. We then study the cohomology of a compatible Hom-Leibniz algebra in the equivariant context. To accomplish this, we introduce the equivariant cohomology group of a compatible Hom-Leibniz algebra endowed with a finite group action, inspired from studies such as those by Bredon [15], Elmendorf [16] and Illman [17]. "Equivariant deformation theory should classify deformations preserving symmetries of the deformed objects [12]." This is used to study equivariant infinitesimal deformation of a compatible Hom-Leibniz algebra. Furthermore, we establish the relationship between the Nijenhuis operator and the trivial infinitesimal deformation, all in the equivariant context. In this process, we prove some important aspects of the traditional algebraic deformation theory for a compatible Hom-Leibniz algebra in our equivariant setting.

The paper is structured as follows: Section 2 begins by outlining fundamental concepts of Hom-Leibniz algebra and explores group action on Hom-Leibniz algebras. It proceeds to examine the Balavoine bracket, some cohomological results and the differential graded Lie algebra responsible for governing Hom-Leibniz algebra deformations, all within the context of group actions. In Section 3, we define compatible Hom-Leibniz algebras alongside group actions, termed as compatible G-Hom-Leibniz algebras. We then construct a graded Lie algebra whose Maurer-Cartan elements characterize the structure of compatible G-Hom-Leibniz algebras. Section 4 delves into the formal deformation of compatible Hom-Leibniz algebras with group actions. Specifically, it examines how Nijenhuis operators generate trivial linear deformations and explores various facets of formal deformation theory.

The implications of this work are both theoretical and structural, contributing to the broader study of algebraic deformations and symmetry in non-associative algebras. By introducing compatible G-Hom-Leibniz algebras, this research provides a new framework for studying algebraic structures under group actions, which may have applications in mathematical physics, representation theory and category theory. The construction of a graded Lie algebra whose Maurer-Cartan elements characterize these algebras establishes a deeper connection between deformation theory and cohomology, potentially influencing further studies in homotopy algebra, operads and higher algebraic structures. Additionally, the examination of Nijenhuis operators and their role in trivial linear deformations enhances the understanding of stability and rigidity properties in these algebras. This framework may also lead to insights into deformation quantization, control theory and differential geometry, where algebraic deformations and symmetry play a crucial role.

2. Preliminaries

In this section, we review the definitions and some basic facts on Hom-Leibniz algebra. See Refs. [11, 18]. We also recall the concept of group action on a Hom-Leibniz algebra, as given in Ref. [11].

Throughout the article, the linear maps are assumed to be over the field K of characteristic 0. We assume the group G to be finite.

Definition 2.1. *A Hom-Leibniz algebra is a vector space L together with a bilinear operation $[.,.]: L \otimes L \to L$ and a linear map $\alpha: L \to L$ such that $[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]]$, $\forall x, y, z \in L$.*

Moreover, if the Hom-Leibniz algebra given by the triple $(L, [.,.], \alpha)$ satisfies $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, we say $(L, [.,.], \alpha)$ is multiplicative.

Hereon we consider our Hom-Leibniz algebras to be multiplicative.

Definition 2.2. *A homomorphism between two Hom-Leibniz algebras $(L_1, [.,.]_1, \alpha_1)$ and $(L_2, [.,.]_2, \alpha_2)$ is a K-linear map $\phi: L_1 \to L_2$ satisfying*

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2 \quad \text{and} \quad \phi \circ \alpha_1 = \alpha_2 \circ \phi.$$

Definition 2.3. *Let $(L, [.,.], \alpha)$ be a Hom-Leibniz algebra. An L-bimodule is a vector space M together with two L-actions $m_L: L \otimes M \to M$, $m_R: M \otimes L \to M$ and a map $\beta \in End(M)$ such that for any $x, y \in L$ and $m \in M$ we have*

$$\beta(m_L(x, m)) = m_L(\alpha(x), \beta(m))$$

$$\beta(m_R(m, x)) = m_R(\beta(m), \alpha(x))$$

$$m_L(\alpha(x), m_L(y, m)) = m_L([x, y], \beta(m)) + m_L(\alpha(y), m_L(x, m))$$

$$m_L(\alpha(x), m_R(m, y)) = m_R(m_L(x, m), \alpha(y)) + m_R(\beta(m), [x, y])$$

$$m_R(\beta(m), [x, y]) = m_R(m_R(m, x), \alpha(y)) + m_L(\alpha(x), m_R(m, y)).$$

The following proposition is well-established.

Proposition 2.1. *Let $(L, [.,.], \alpha)$ be a Hom-Leibniz algebra and (M, m_L, m_R, β) its representation. Then $L \oplus M$ is a Hom-Leibniz algebra with the linear homomorphism $\alpha \oplus \beta: L \oplus M \to L \oplus M$ defined as $(\alpha \oplus \beta)(x, m) = (\alpha(x), \beta(m))$ and the Hom-Leibniz bracket defined as*

$$[(x, u), (y, v)]_{\ltimes} = \left([x, y], m_L^1(x, v) + m_R^1(u, y)\right) \quad \forall \quad x, y \in L \text{ and } u, v \in M.$$

This is known as the semi-direct product.

We now consider a finite group acting on Hom-Leibniz algebra.

Definition 2.4. *Let $(L, [.,.], \alpha)$ be a Hom-Leibniz algebra. We say that a group G acts on L if there exists a map $\phi: G \times L \to L$ such that $\forall\, g, g_1, g_2 \in G$, $x, y \in L$,*

(1) *$\phi_g: L \to L$ given by $x \to gx$ is K-linear*

(2) *$e.x = x$, where e is the identity in the group*

(3) $(g_1 g_2)x = g_1(g_2 x)$

(4) $g[x, y] = [gx, gy]$

(5) $\alpha(gx) = g\alpha(x)$.

When a group G acts on L, as given above, we say that L is a G-Hom-Leibniz algebra.

Definition 2.5. *Let $(L, [.,.], \alpha)$ be a G-Hom-Leibniz algebra. A G-bimodule over L is an L-bimodule given by a vector space M together with two L-actions $m_L: L\otimes M \to M$, $m_R: M\otimes L \to M$ and a map $\beta \in End(M)$ satisfying the conditions given in the definition* 2.3*, such that for any $x, y \in L$ and $m \in M$ and $g \in G$ we additionally have*

$$\beta g = g\beta$$

$$m_L(gx, gm) = gm_L(x, m)$$

$$m_R(gm, gx) = gm_R(m, x).$$

2.1 The Balavoine bracket

In his work [10], D. Balavoine introduced a Lie algebra structure on the cochain complex of a Leibniz algebra. In this subsection, we focus on the controlling algebra of Hom-Leibniz algebras within an equivariant framework, specifically a graded Lie algebra whose Maurer-Cartan elements are G-Hom-Leibniz algebras. We start by recalling key definitions and results from Refs. [10, 18 and 19].

Definition 2.6. *Let S_n denote the permutation group of n elements $1, 2, \ldots, n$. A permutation $\sigma \in S_n$ is called a (p, q)-shuffle if $p + q = n$ and $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(n)$. If $p = 0$ or n, we consider σ to be the identity permutation. Let $S_{(p,q)}$ denote the set of all (p, q)-shuffles in S_n.*

Definition 2.7. *Consider a differential graded Lie algebra $(\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k, [.,.], d)$. An element $x \in \mathfrak{g}^1$ is termed a Maurer-Cartan element of $\mathfrak{g}$ if it satisfies*

$$dx + \frac{1}{2}[x, x] = 0.$$

If $\mathfrak{g}$ is a vector space and $\alpha : \mathfrak{g} \to \mathfrak{g}$ a linear map such that the vector space operations are compatible with the map α, we say that $(\mathfrak{g}, \alpha)$ is a Hom-vector space.

For each $n \geq 1$, we denote $\mathbb{C}^n_\alpha(\mathfrak{g}, \mathfrak{g}) = \{f \in Hom(\otimes^n \mathfrak{g}, \mathfrak{g}) | \alpha \circ f = f \circ \alpha^{\otimes n}\}$ and set $\mathbb{C}^*_\alpha(\mathfrak{g}, \mathfrak{g}) = \oplus_{n\in\mathbb{N}}\mathbb{C}^n_\alpha(\mathfrak{g}, \mathfrak{g})$.

For $P \in \mathbb{C}^{p+1}_\alpha(\mathfrak{g}, \mathfrak{g})$, $Q \in \mathbb{C}^{q+1}_\alpha(\mathfrak{g}, \mathfrak{g})$ we the **Balavoine bracket** is defined as

$$[P, Q]_B = P \circ Q - (-1)^{pq} Q \circ P$$

where $P \circ Q \in \mathbb{C}^{p+q+1}_\alpha$ is defined as

$$(P \circ Q)(x_1, x_2, \ldots, x_{p+q+1}) = \sum_{k=1}^{p+1} (-1)^{(k-1)q} P \circ_k Q,$$

and

$$P o_k Q(x_1, x_2, \ldots, x_{p+q+1}) =$$

$$\sum_{\sigma\in S(k-1,q)} (-1)^\sigma P\left(\alpha^p(x_{\sigma(1)}), \ldots, \alpha^p(x_{\sigma(k-1)}), Q(x_{\sigma(k)}, \ldots, x_{\sigma(k+q-1)}, x_{k+q}), \alpha^p(x_{k+q+1}), \ldots, \alpha^p(x_{p+q+1})\right).$$

In [18] it is shown that the graded vector space $\mathbb{C}_{\alpha}^{*}(\mathfrak{g},\mathfrak{g})$ equipped with the Balavoine bracket given above is a graded Lie algebra.

Definition 2.8. Suppose on the Hom-vector space $(\mathfrak{g},\alpha)$ a group G acts such that $\alpha:\mathfrak{g}\to\mathfrak{g}$ is equivariant in the sense that $g\alpha=\alpha g$, we say that the Hom-vector space $\mathfrak{g}$ is a G-Hom-vector space.

Now if $\mathfrak{g}$ is a G-Hom-vector space we define

$$\mathbb{C}_{\alpha,G}^{n}(\mathfrak{g},\mathfrak{g})=\left\{f\in\mathbb{C}_{\alpha}^{n}(\mathfrak{g},\mathfrak{g}):gf=fg,\ \forall g\in G\right\}.$$

Note that for $P\in\mathbb{C}_{G,\alpha}^{p+1}(\mathfrak{g},\mathfrak{g}), Q\in\mathbb{C}_{G,\alpha}^{q+1}(\mathfrak{g},\mathfrak{g}), [P,Q]_B\in\mathbb{C}_{G,\alpha}^{p+q+1}(\mathfrak{g},\mathfrak{g})$, i.e the Balavoine bracket is G-invariant. Hence, we have the following theorem:

Theorem 2.1. $(\mathbb{C}_{G,\alpha}^{*}(\mathfrak{g},\mathfrak{g}),[.,.]_B)$ forms a graded Lie algebra.

In particular, for $\pi\in\mathbb{C}_{G,\alpha}^{2}(\mathfrak{g},\mathfrak{g})$, $[\pi,\pi]_B\in\mathbb{C}_{G,\alpha}^{3}(\mathfrak{g},\mathfrak{g})$ such that

$$[\pi,\pi]_B=2(\pi(\pi(x,y),\alpha(z))-\pi(\alpha(x),\pi(y,z))+\pi(\alpha(y),\pi(x,z)))\text{ and}$$
$$g[\pi,\pi]_B(x,y,z)=[\pi,\pi]_B(gx,gy,gz),\ \forall g\in G.$$

Thus we have the following corollary.

Corollary 2.1. π defines a G-Hom-Leibniz algebra structure on $\mathfrak{g}$ iff π is a Maurer-Cartan element of the graded Lie algebra $(\mathbb{C}_{G,\alpha}^{*}(\mathfrak{g},\mathfrak{g}),[.,.]_B)$.

Theorem 2.2. Let $(\mathfrak{g},\pi,\alpha)$ be a G-Hom-Leibniz algebra. Then $(\mathbb{C}_{G,\alpha}^{*}(\mathfrak{g},\mathfrak{g}),[.,.],d_\pi)$ becomes a differential graded Lie algebra (dgLa), where $d_\pi:=[\pi,.]_B$.
Further, given $\pi'\in\mathbb{C}_{G,\alpha}^{2}(\mathfrak{g},\mathfrak{g})$, $\pi+\pi'$ defines a Leibniz algebra structure on $\mathfrak{g}$ iff π' is a Maurer-Cartan element of the dgLa $(\mathbb{C}_{G,\alpha}^{*}(\mathfrak{g},\mathfrak{g}),[.,.],d_\pi)$.

3. Compatible Hom-Leibniz algebra with a group action

In this section, we consider a pair of Hom-Leibniz algebras such that their linear combination is also a Hom-Leibniz algebra. Such a pair is referred to as a compatible Hom-Leibniz algebra, a notion introduced in Ref. [20]. We then consider these compatible Hom-Leibniz algebras along with a group action, with the intention of studying the behaviour of their deformation under the presence of group action.

Definition 3.1. A compatible Hom-Leibniz algebra is a quadruple $(L,[.,.],\{.,.\},\alpha)$ such that $(L,[.,.],\alpha)$ and $(L,\{.,.\},\alpha)$ are Hom-Leibniz algebras such that $\forall\ x,y,z\in L$

$$[\alpha(x),\{y,z\}]+\{\alpha(x),[y,z]\}=[\{x,y\},\alpha(z)]+\{[x,y],\alpha(z)\}+[\alpha(y),\{x,z\}]+\{\alpha(y),[x,z]\}. \tag{1}$$

We recall the following characterization of a compatible Hom-Leibniz algebra, established in Proposition 3.1 of [20].

Proposition 3.1. A quadruple $(L,[.,.],\{.,.\},\alpha)$ is a compatible Hom-Leibniz algebra iff $(L,[.,.],\alpha)$ and $(L,\{.,.\},\alpha)$ are Hom-Leibniz algebras such that for any k_1,k_2 in K, the bilinear operation

$$[[x,y]] = k_1[x,y] + k_2\{x,y\}, \quad \forall x,y \in L$$

together with the K-linear map α: $L \to L$ defines a Hom-Leibniz algebra structure on L.

Definition 3.2. *A homomorphism between two compatible Hom-Leibniz algebras $(L_1, [.,.]_1, \{.,.\}_1, \alpha)$ and $(L_2, [.,.]_2, \{.,.\}_2, \alpha)$ is a K-linear map ϕ: $L_1 \to L_2$ satisfying*

$$\phi([x,y]_1) = [\phi(x),\phi(y)]_2, \quad \phi(\{x,y\}_1) = \{\phi(x),\phi(y)\}_2 \text{ and } \phi \circ \alpha_1 = \alpha_2 \circ \phi.$$

We now define a bimodule for the compatible structure defined earlier.

Definition 3.3. *Let $(L, [., .], \{., .\}, \alpha)$ be a compatible Hom-Leibniz algebra. A compatible L-bimodule is a vector space M together with four L-actions*

$$m_L^1 : L \otimes M \to M, \qquad m_R^1 : M \otimes L \to M$$

$$m_L^2 : L \otimes M \to M, \qquad m_R^2 : M \otimes L \to M$$

and a linear map β: $M \to M$ such that.

(1) (M, m_L^1, m_R^1, β) *is a bimodule over* $(L, [., .], \alpha)$.

(2) (M, m_L^2, m_R^2, β) *is a bimodule over* $(L, \{., .\}, \alpha)$.

(3) *the following compatibility conditions hold for all $x, y \in L$, $m \in M$*

$$LLM: \quad m_L^1\big(\alpha(x), m_L^2(y,m)\big) + m_L^2\big(\alpha(x), m_L^1(y,m)\big) = m_L^1(\{x,y\},\beta(m)) + m_L^2([x,y],\beta(m)) + m_L^1\big(\alpha(y), m_L^2(x,m)\big) + m_L^2\big(\alpha(y), m_L^1(x,m)\big)$$

$$LML: \quad m_L^1\big(\alpha(x), m_R^2(m,y)\big) + m_L^2\big(\alpha(x), m_R^1(m,y)\big) = m_R^1\big(m_L^2(x,m),\alpha(y)\big) + m_R^2\big(m_L^1(x,m),\alpha(y)\big) + m_R^1(\beta(m),\{x,y\}) + m_R^2(\beta(m),[x,y])$$

$$MLL: \quad m_R^1(\beta(m),\{x,y\}) + m_R^2(\beta(m),[x,y]) = m_R^1\big(m_R^2(m,x),\alpha(y)\big) + m_R^2\big(m_R^1(m,x),\alpha(y)\big) + m_L^1\big(\alpha(x), m_R^2(m,y)\big) + m_L^2\big(\alpha(x), m_R^1(m,y)\big)$$

We also say that $(M, m_L^1, m_R^1, m_L^2, m_R^2, \beta)$ is a representation of the compatible Hom-Leibniz algebra $(L, [., .], \{., .\}, \alpha)$.

Note: Any compatible Hom-Leibniz algebra $(L, [., .], \{., .\}, \alpha)$ is a compatible L-bimodule in which $m_L^1 = m_R^1 = [.,.]$ and $m_L^2 = m_R^2 = \{.,.\}$.

The following result can be proved just like the standard case.

Proposition 3.2. *Let $(L, [., .], \{., .\}, \alpha)$ be a compatible Hom-Leibniz algebra and $(M, m_L^1, m_R^1, m_L^2, m_R^2, \beta)$ its representation. Then $L \oplus M$ is a compatible Hom-Leibniz algebra with the linear homomorphism $\alpha \oplus \beta$ and the compatible Hom-Leibniz brackets defined as*

$$[(x,u),(y,v)]_{\ltimes} = \big([x,y], m_L^1(x,v) + m_R^1(u,y)\big) \text{ and}$$

$$\{(x,u),(y,v)\}_{\ltimes} = \big(\{x,y\}, m_L^2(x,v) + m_R^2(u,y)\big) \ \forall \ x,y \in L \text{ and } u,v \in M.$$

We now consider compatible Hom-Leibniz algebras and compatible bimodules along with a finite group action.

Definition 3.4. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible Hom-Leibniz algebra. We say that a group G acts on L if*

(1) $(L, [.,.], \alpha)$ *and* $(L, \{.,.\}, \alpha)$ *are G-Hom-Leibniz algebras.*

(2) $\alpha g = g\alpha$.
We call such an L, a compatible G-Hom-Leibniz algebra.

Example 3.1. *Any G-Hom-vector space* (L, α) *with the trivial brackets* $[x, y] = \{x, y\} = 0, \forall x, y \in L$ *is a compatible Hom-Leibniz algebra with an action of G.*

Example 3.2. *The authors in Ref. [21] classifies compatible Leibniz algebras of dimensions 2 and 3. On each of these L, consider the hom map* α *to be the identity map and the trivial group action* $G \times L \to L$ *given by* $g.x = x \ \forall \ g \in G$ and $x \in L$, *gives a compatible G-Hom-Leibniz algebra.*

Definition 3.5. *A G-homomorphism between two compatible G-Hom-Leibniz algebras* $(L_1, [.,.]_1, \{.,.\}_1, \alpha_1)$ *and* $(L_2, [.,.]_2, \{.,.\}_2, \alpha_2)$ *is a K-linear map* $\phi: L_1 \to L_2$ *satisfying*

$$\phi([x,y]_1) = [\phi(x), \phi(y)]_2$$

$$\phi(\{x,y\}_1) = \{\phi(x), \phi(y)\}_2$$

$$\phi \circ \alpha_1 = \alpha_2 \circ \phi \quad \text{and}$$

$$g\phi = \phi g.$$

Definition 3.6. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra. A compatible G-bimodule over L is a vector space M together with four L-actions*

$$m_L^1 : L \otimes M \to M, \qquad m_R^1 : M \otimes L \to M$$

$$m_L^2 : L \otimes M \to M, \qquad m_R^2 : M \otimes L \to M$$

and a map $\beta \in End(M)$ such that for any $x, y \in L$, $m \in M$ and $g \in G$ we have.

(1) (M, m_L^1, m_R^1, β) *is a G-bimodule over* $(L, [.,.], \alpha)$.

(2) (M, m_L^2, m_R^2, β) *is a G-bimodule over* $(L, \{.,.\}, \alpha)$.

(3) *the following compatibility conditions hold for all* $x, y \in L$, $m \in M$

$$LLM: \quad m_L^1\big(\alpha(x), m_L^2(y,m)\big) + m_L^2\big(\alpha(x), m_L^1(y,m)\big) = m_L^1(\{x,y\}, \beta(m)) + m_L^2([x,y], \beta(m)) + m_L^1\big(\alpha(y), m_L^2(x,m)\big) + m_L^2\big(\alpha(y), m_L^1(x,m)\big)$$

$$LML: \quad m_L^1\big(\alpha(x), m_R^2(m,y)\big) + m_L^2\big(\alpha(x), m_R^1(m,y)\big) = m_R^1\big(m_L^2(x,m), \alpha(y)\big) + m_R^2\big(m_L^1(x,m), \alpha(y)\big) + m_R^1(\beta(m), \{x,y\}) + m_R^2(\beta(m), [x,y])$$

$$MLL: \quad m_R^1(\beta(m), \{x,y\}) + m_R^2(\beta(m), [x,y]) = m_R^1\big(m_R^2(m,x), \alpha(y)\big) + m_R^2\big(m_R^1(m,x), \alpha(y)\big) + m_L^1\big(\alpha(x), m_R^2(m,y)\big) + m_L^2\big(\alpha(x), m_R^1(m,y)\big).$$

We also say that $(M, m_L^1, m_R^1, m_L^2, m_R^2, \beta)$ is an equivariant representation of the compatible G-Hom-Leibniz algebra $(L[.,.], \{.,.\}, \alpha)$.

Note: Any compatible G-Hom-Leibniz algebra $(L, [.,.], \{.,.\}, \alpha)$ is a compatible G-bimodule of L in which $m_L^1 = m_R^1 = [.,.]$ and $m_L^2 = m_R^2 = \{.,.\}$.

The following is the equivariant version of a result seen earlier. The proof is similar.

Proposition 3.3. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra and* $(M, m_L^1, m_R^1, m_L^2, m_R^2, \beta)$ its equivariant representation. Then $L \oplus M$ is a compatible G-Hom-Leibniz algebra with the G-action defined as $G \times (L \oplus M) \to (L \oplus M), g.(x, u) \mapsto (gx, gy)$, linear homomorphism $\alpha \oplus \beta$ and the compatible Hom-Leibniz brackets defined as

$$[(x,u),(y,v)]_{\ltimes} = \left([x,y], m_L^1(x,v) + m_R^1(u,y)\right) \quad \text{and}$$

$$\{(x,u),(y,v)\}_{\ltimes} = \left(\{x,y\}, m_L^2(x,v) + m_R^2(u,y)\right) \quad \forall \quad x,y \in L \text{ and } u,v \in M.$$

3.1 Maurer-Cartan characterization of compatible G*-Hom-Leibniz algebra*

In this section, we recall the concept of a bi-differential graded Lie algebra (b-dgLa) and discuss some relevant results, as given in Ref. [22]. We then define the bi-differential graded Lie algebra whose Maurer-Cartan elements characterize the compatible G-Hom-Leibniz algebra.

Definition 3.7. *[22] Let* $(\mathfrak{g}, [.,.], \delta_1)$ *and* $(\mathfrak{g}, [.,.], \delta_2)$ *be two differential graded Lie algebras. We call* $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ *a bi-differential graded Lie algebra (b-dgLa) if* δ_1 *and* δ_2 *satisfy*

$$\delta_1\delta_2 + \delta_2\delta_1 = 0.$$

Proposition 3.4. *[22] Let* $(\mathfrak{g}, [.,.], \delta_1)$ *and* $(\mathfrak{g}, [.,.], \delta_2)$ *be two differential graded Lie algebras. Then* $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ *is a bi-differential graded Lie algebra iff for any* k_1 *and* $k_2 \in K$, $(\mathfrak{g}, [.,.], \delta_{k_1k_2})$ *is a differential graded Lie algebra, where* $\delta_{k_1k_2} = k_1\delta_1 + k_2\delta_2$.

Definition 3.8. *Let* $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ *be a b-dgLa. A pair* $(\pi_1, \pi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ *is called a Maurer-Cartan element of the b-dgLa* $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ *if* π_1 *and* π_2 *are Maurer-Cartan elements of the dgLas* $(\mathfrak{g}, [.,.], \delta_1)$ *and* $(\mathfrak{g}, [.,.], \delta_2)$ *respectively, and*

$$\delta_2\pi_1 + \delta_1\pi_2 + [\pi_1, \pi_2] = 0.$$

Proposition 3.5. *A pair* $(\pi_1, \pi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ *is a Maurer-Cartan element of the b-dgLa* $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ *iff for any* $k_1, k_2 \in K$, $k_1\pi_1 + k_2\pi_2$ *is a Maurer-Cartan element of the dgLa* $(\mathfrak{g}, [.,.], \delta_{k_1k_2})$.

Theorem 3.1. *Let* (L, α) *be a Hom-vector space with a G-action. Let* $\pi_1, \pi_2 \in \mathbb{C}^2_{G,\alpha}(L, L)$. *Then* $(L, \pi_1, \pi_2, \alpha)$ *is a compatible G-Hom-Leibniz algebra iff* (π_1, π_2) *is a Maurer-Cartan element of the b-dgLa* $(\mathbb{C}^*_{G,\alpha}(L, L), [.,.]_B, \delta_1 = 0, \delta_2 = 0)$.

Proof. $(L, \pi_1, \pi_2, \alpha)$ is a compatible G-Hom-Leibniz algebra gives (L, π_1, α) and (L, π_2, α) are G-Hom-Leibniz algebras. Hence we get $[\pi_1, \pi_1]_B = [\pi_2, \pi_2]_B = 0$.

Further, $\forall x, y, z \in L$ we have the compatibility condition,

$$\begin{aligned}\pi_1(\alpha(x), \pi_2(y,z)) + \pi_2(\alpha(x), \pi_1(y,z)) \quad &= \pi_1(\pi_2(x,y), \alpha(z)) + \pi_2(\pi_1(x,y), \alpha(z)) + \\ &\pi_1(\alpha(y), \pi_2(x,z)) + \pi_2(\alpha(y), \pi_1(x,z))\end{aligned} \tag{2}$$

We note that, $[\pi_1, \pi_2]_B = \pi_1 \circ \pi_2 + \pi_2 \circ \pi_1$, where

$$\begin{aligned}\pi_1 \circ \pi_2(x,y,z) &= (\pi_1 \circ_1 \pi_2 - \pi_1 \circ_2 \pi_2)(x,y,z)\\ &= \pi_1(\pi_2(x,y),\alpha(z)) - \pi_1(\alpha(x),\pi_2(y,z)) + \pi_1(\alpha(y),\pi_2(x,z))\end{aligned}$$

and

$$\begin{aligned}\pi_2 \circ \pi_1(x,y,z) &= (\pi_2 \circ_1 \pi_1 - \pi_2 \circ_2 \pi_1)(x,y,z)\\ &= \pi_2(\pi_1(x,y),\alpha(z)) - \pi_2(\alpha(x),\pi_1(y,z)) + \pi_2(\alpha(y),\pi_1(x,z))\end{aligned}$$

i.e.,

$$\begin{aligned}[\pi_1,\pi_2]_B(x,y,z) &= \pi_1(\pi_2(x,y),\alpha(z)) - \pi_1(\alpha(x),\pi_2(y,z)) + \pi_1(\alpha(y),\pi_2(x,z))\\ &\quad + \pi_2(\pi_1(x,y),\alpha(z)) - \pi_2(\alpha(x),\pi_1(y,z)) + \pi_2(\alpha(y),\pi_1(x,z)).\end{aligned}$$

In above we see that, $RHS = 0$ is the compatibility condition on $(L, \pi_1, \pi_2, \alpha)$. Thus we get $[\pi_1, \pi_2] = 0$. Hence if $(L, \pi_1, \pi_2, \alpha)$ is a compatible G-Hom-Leibniz algebra, then $\delta_2\pi_1 + \delta_1\pi_2 + [\pi_1, \pi_2] = 0$. i.e. (π_1, π_2) is a Maurer-Cartan element of L.

Conversely if (π_1, π_2) is a Maurer-Cartan element of the b-dgLa $(\mathbb{C}^*_{G,\alpha}(L,L), [.,.]_B, 0, 0)$, then π_1 and π_2 are Maurer-Cartan elements of $(\mathbb{C}^*_{G,\alpha}(L,L), [.,.]_B, 0)$ and $(\mathbb{C}^*_{G,\alpha}(L,L), [.,.]_B, 0)$ respectively, and $[\pi_1,\pi_2]_B = 0$. Thus we get that (L, π_1, α) and (L, π_2, α) are G-Hom-Leibniz algebras. Further, the condition $[\pi_1,\pi_2]_B = 0$ gives the compatibility condition for π_1 and π_2. Thus $(L, \pi_1, \pi_2, \alpha)$ is a compatible G-Hom-Leibniz algebra.

□

Theorem 3.2. *[22] Let (π_1, π_2) be a Maurer-Cartan element of the b-dgLa $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$. Define $d_1 := \delta_1 + [\pi_1, \_]$ and $d_2 := \delta_2 + [\pi_2, \_]$. Then $(\mathfrak{g}, [.,.], d_1, d_2)$ is a b-dgLa.*
Further, for any $\tilde{\pi}_1, \tilde{\pi}_2 \in \mathfrak{g}_1$, $(\pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a Maurer-Cartan element of the b-dgLa $(\mathfrak{g}, [.,.], \delta_1, \delta_2)$ iff $(\tilde{\pi}_1, \tilde{\pi}_2)$ is a Maurer-Cartan element of the b-dgLa $(\mathfrak{g}, [.,.], d_1, d_2)$.

Let $(L, \pi_1, \pi_2, \alpha)$ be a compatible G-Hom-Leibniz algebra. From theorems 3.1 and 3.2, we conclude the following important results:

Theorem 3.3. *$(\mathbb{C}^*_{G,\alpha}(L,L), [.,.], d_1, d_2)$ is a b-dgLa where $d_1 := [\pi_1, \_]_B$ and $d_2 := [\pi_2, \_]_B$.*

Theorem 3.4. *For any $\tilde{\pi}_1, \tilde{\pi}_2 \in \mathbb{C}^2_{G,\alpha}(L,L)$, $(L, \pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a compatible G-Hom-Leibniz algebra iff $(\pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a Maurer Cartan element of the b-dgLa $(\mathbb{C}^*_{G,\alpha}(L,L), [.,.]_B, d_1, d_2)$.*

3.2 Equivariant cohomology of compatible G*-Hom-Leibniz algebra with coefficients in itself*

Let $(L, [.,.], \{.,.\}, \alpha)$ be a compatible G-Hom-Leibniz algebra with $\pi_1(x, y) = [x, y]$ and $\pi_2(x, y) = \{x, y\}$. By theorem 3.1, (π_1, π_2) is a Maurer-Cartan element of the b-dgLa $(\mathbb{C}^*_{G,\alpha}(L,L), [.,.]_B, 0, 0)$.

We define the n-cochains for $n \geq 1$ as

$$LC^n_{G,\alpha}(L,L) := \mathbb{C}^n_{G,\alpha}(L,L) \oplus \mathbb{C}^n_{G,\alpha}(L,L) \ldots \oplus \mathbb{C}^n_{G,\alpha}(L,L), \ \text{n times}$$

and $d^n : LC^n_{G,\alpha}(L,L) \to LC^{n+1}_{G,\alpha}(L,L)$ by

$$d^1 f = ([\pi_1, f]_B, [\pi_2, f]_B), \ \forall f \in LC^1_{G,\alpha}(L, L)$$

$$d^n(f_1, f_2, \dots, f_n) = (-1)^{n-1}\big([\pi_1, f_1]_B, \dots, [\pi_2, f_{i-1}]_B + [\pi_1, f_i]_B, \dots, [\pi_2, f_n]_B\big),$$

where$(f_1, f_2, \dots, f_n) \in LC^n_{G,\alpha}(L, L)$ and $2 \le i \le n$.

Note: The G-invariance of the Balavoine bracket gives that if $(f_1, f_2, \dots, f_n) \in LC^n_{G,\alpha}(L, L)$ then $d^n(f_1, f_2, \dots, f_n) \in LC^{n+1}_{G,\alpha}(L, L)$.

Now d defined as above gives the following theorem.

Theorem 3.5. We have $d^{n+1} \circ d^n = 0$.

Proof. We first note that since (π_1, π_2) is a Maurer-Cartan element of $(\mathbb{C}^*_{G,\alpha}(L, L), [.,.]_B, 0, 0)$ we have $[\pi_1, \pi_1] = 0$, $[\pi_1, \pi_2] = 0$ and $[\pi_2, \pi_2] = 0$.

For convenience we put $[\pi_1, f_i]_B = g_i$ and $[\pi_2, f_i]_B = h_i$. Then for any $(f_1, \dots, f_n) \in LC^n_{G,\alpha}(L, L)$, $2 \le i \le n$ we have,

$$\begin{aligned}
& d^{n+1} d^n (f_1, f_2, \dots, f_n) \\
= \; & (-1)^{n-1} d^{n+1}(g_1, \dots, h_{i-1} + g_i, \dots, h_n) \quad (2 \le i \le n) && (3) \\
= \; & -\big([\pi_1, g_1]_B, [\pi_2, g_1]_B + [\pi_1, h_1]_B + [\pi_1, g_2]_B, \cdots \\
& [\pi_2, h_{i-2}]_B + [\pi_2, g_{i-1}]_B + [\pi_1, h_{i-1}]_B + [\pi_1, g_i]_B, \dots, \\
& [\pi_2, h_{n-1}]_B + [\pi_2, g_n]_B\big) + [\pi_1, h_n]_B, [\pi_2, h_n]_B\big) \quad (3 \le i \le n) && (4) \\
= \; & -\Big(\frac{1}{2}[[\pi_1, \pi_1]_B, f_1]_B, [[\pi_1, \pi_2]_B, f_1]_B + \frac{1}{2}[[\pi_1, \pi_1]_B, f_2]_B, \cdots \\
& \frac{1}{2}[[\pi_2, \pi_2]_B, f_{i-2}]_B + [[\pi_1, \pi_2]_B, f_{i-1}]_B + \frac{1}{2}[[\pi_1, \pi_1]_B, f_i]_B, \dots, \\
& \frac{1}{2}[[\pi_2, \pi_2]_B, f_{n-1}]_B + [[\pi_1, \pi_2]_B, f_n]_B, \frac{1}{2}[[\pi_2, \pi_2]_B, f_n]_B\Big) && (5) \\
= \; & (0, 0, \dots, 0).
\end{aligned}$$

□

Hence we have a cochain complex, $(LC^*_{G,\alpha}(L, L), d^*) = (\oplus_{n \in \mathbb{N}} LC^n_{G,\alpha}(L, L), d^*)$. We call this the equivariant cochain complex of $(L, [.,.], \{.,.\}, \alpha)$. We have the following definition.

Definition 3.9. Let $(L, [.,.], \{.,.\}, \alpha)$ be a compatible G-Hom-Leibniz algebra. The cohomology of the equivariant cochain complex $(LC^*_{G,\alpha}(L, L), d^*)$ is called the equivariant cohomology of the compatible G-Hom-Leibniz algebra $(L, [.,.], \{.,.\}, \alpha)$. We denote the n^{th} equivariant cohomology group of $(L, [.,.], \{.,.\}, \alpha)$ over itself by $H^n_{G,\alpha}(L, L)$.

4. Equivariant one-parameter formal deformation of compatible Hom-Leibniz algebra

In this section, we introduce the formal deformation of a compatible Hom-Leibniz algebra along with a group action. In particular, we see how Nijenhuis operators generate trivial linear deformations. We also study some aspects of formal deformation theory. We begin by defining an equivariant formal one-parameter deformation of a compatible G-Hom-Leibniz algebra.

Definition 4.1. Let $(L, [.,.], \{.,.\}, \alpha)$ be a compatible G-Hom-Leibniz algebra. An equivariant formal one-parameter deformation of L is a pair of $K[[t]]$-linear maps

$$\mu_t : L[[t]] \otimes L[[t]] \to L[[t]] \text{ and}$$

$$m_t : L[[t]] \otimes L[[t]] \to L[[t]] \text{ such that :}$$

(1) $\mu_t(a,b) = \sum_{i=0}^{\infty}\mu_i(a,b)t^i$, $m_t(a,b) = \sum_{i=0}^{\infty}m_i(a,b)t^i$
for all $a, b \in L$, *where* μ_i, m_i: $L \otimes L \to L$ *are K-linear and* $\mu_0(a, b) = [a, b]$ *and* $m_0(a, b) = \{a, b\}$.

(2) *For any t,* $(L[[t]], \mu_t, m_t, \alpha)$ *is a compatible G-Hom-Leibniz algebra.*

Note that (b) is equivalent to.

- $(L[[t]], \mu_t, \alpha)$ is a G-Hom-Leibniz algebra.
- $(L[[t]], m_t, \alpha)$ is a G-Hom-Leibniz algebra.
- $\mu_t(\alpha(x), m_t(y, z)) + m_t(\alpha(x), \mu_t(y, z)) = \mu_t(m_t(x, y), \alpha(z)) + m_t(\mu_t(x, y), \alpha(z)) +$
- $\mu_t(\alpha(y), m_t(x, z)) + m_t(\alpha(y), \mu_t(x, z))$

Equivalently, by theorem 3.1, we have that (L, μ_t, m_t, α) is a compatible G-Hom-Leibniz algebra $\Leftrightarrow$ (μ_t, m_t) is a Maurer Cartan element of.

$(C^{*}_{G,\alpha}(L,L), [.,.]_B, 0, 0) \Leftrightarrow$

(1) $[\mu_t, \mu_t]_B = 0$

(2) $[m_t, m_t]_B = 0$

(3) $[\mu_t, m_t]_B = 0.$

We now define the equivalence of two deformations.

Definition 4.2. *Two equivariant one-parameter formal deformations* (L, μ_t, m_t, α) *and* $(L, \mu'_t, m'_t, \alpha)$ *of a compatible G-Hom-Leibniz algebra* $(L, [., .], \{., .\}, \alpha)$ *are said to be equivalent if there exists K-linear maps* $\phi'_i s$ *from* $L \to L$ *such that*

$$\phi_t = \sum_{i=0}^{\infty} \phi_i t^i : (L, \mu_t, m_t, \alpha) \to \left(L, \mu'_t, m'_t, \alpha\right)$$

is a compatible G-Hom-Leibniz algebra homomorphism.

4.1 Extension of finite order deformation and obstruction

Here, we define a finite order equivariant deformation of a compatible G-Hom-Leibniz algebra L. Subsequently, we study the extensibility of the deformation of a finite order.

Definition 4.3. *An equivariant one parameter formal deformation of a compatible G-Hom-Leibniz algebra L of order n is a pair of* $K[[t]]$

$$\mu_t: L[[t]] \otimes L[[t]] \to L[[t]] \text{ and}$$

$$m_t : L[[t]] \otimes L[[t]] \to L[[t]] \text{ such that :}$$

(1) $\mu_t(a,b) = \sum_{i=0}^{n}\mu_i(a,b)t^i$, $m_t(a,b) = \sum_{i=0}^{n}m_i(a,b)t^i$
for all $a, b \in L$, *where* μ_i, m_i: $L \otimes L \to L$ *are K-linear and* $\mu_0(a, b) = [a, b]$ *and* $m_0(a, b) = \{a, b\}$.

(2) *For any t,* $(L[[t]], \mu_t, m_t, \alpha)$ *is a compatible G-Hom-Leibniz algebra taken modulo* t^{n+1}.

We say that an equivariant deformation of order n, given by (μ_t, m_t), of a compatible G-Hom-Leibniz algebra is extendable to a deformation of order $n+1$ if there exists an element $(\mu_{n+1}, m_{n+1}) \in LC^2_{G,\alpha}(L,L)$ such that $\mu'_t = \mu_t + \mu_{n+1}t^{n+1}$ and $m'_t = m_t + m_{n+1}t^{n+1}$ satisfies all the conditions of formal deformation. This implies that (μ'_t, m'_t) satisfies the conditions

$$[\mu'_t, \mu'_t]_B = 0, \qquad [\mu'_t, m'_t]_B = 0, \qquad [m'_t, m'_t]_B = 0.$$

The above is the same as the following conditions:

(1) $[\mu_0, \mu_{n+1}]_B = -\frac{1}{2}\sum_{i+j=n+1}[\mu_i, \mu_j]_B$

(2) $[\mu_0, m_{n+1}]_B + [m_0, \mu_{n+1}]_B = -\sum_{i+j=n+1}[\mu_i, m_j]_B$

(3) $[m_0, m_{n+1}]_B = -\frac{1}{2}\sum_{i+j=n+1}[m_i, m_j]_B$.

Now, if (μ_t, m_t) defines an order n deformation of a compatible G-Hom-Leibniz algebra L, we define an element $Obs^n_{G,\alpha} \in LC^3_{G,\alpha}(L,L)$ by

$$Obs^n_{G,\alpha} = \left(-\frac{1}{2}\sum_{i+j=n+1}\left[\mu_i, \mu_j\right]_B, -\sum_{i+j=n+1}[\mu_i, m_j]_B, -\frac{1}{2}\sum_{i+j=n+1}[m_i, m_j]_B\right).$$

Here we note that for $i, j \geq 0$, $x, y \in L$ and $g \in G$, we have $\mu_i(gx, gy) = g\mu_i(x, y)$, $m_i(gx, gy) = gm_i(x, y)$ and $\alpha(gx) = g\alpha(x)$. Hence $gObs^n_{G,\alpha}(x,y,z) = Obs^n_{G,\alpha}(gx, gy, gz)$.

We call this the equivariant obstruction cochain and the corresponding equivariant cohomology class as the obstruction class to extend the deformation (μ_t, m_t) of order n, to a deformation of order $n+1$.

We have the following theorem:

Theorem 4.1. A deformation of order n given by (μ_t, m_t) of a compatible G-Hom-Leibniz algebra is extendable to a deformation of order $n+1$ iff the corresponding obstruction class is trivial.

Proof. Suppose (μ_t, m_t) is extendable to a deformation of order $n+1$ given by $\mu'_t = \mu_t + \mu_{n+1}t^{n+1}$ and $m'_t = m_t + m_{n+1}t^{n+1}$. Then, by considering the obstruction cochain defined in the previous paragraph, we have,

$$\begin{aligned} Obs^n_{G,\alpha} &= \left([\mu_0, \mu_{n+1}]_B, [\mu_0, m_{n+1}]_B + [m_0, \mu_{n+1}]_B, [m_0, m_{n+1}]_B\right) \\ &= d\left(\mu_{n+1}, m_{n+1}\right). \end{aligned}$$

Conversely, suppose the obstruction class of $Obs^n_{G,\alpha}$ vanishes. Then there exists a cochain. $(\mu_{n+1}, m_{n+1}) \in LC^n_{G,\alpha}$ such that $Obs^n_{G,\alpha} = d(\mu_{n+1}, m_{n+1})$.

Define $\mu'_t = \mu_t + \mu_{n+1}t^{n+1}$ and $m'_t = m_t + m_{n+1}t^{n+1}$. Then (μ'_t, m'_t) defines a deformation of order $n+1$ extending the deformation (μ_t, m_t).

□

4.2 Equivariant infinitesimal deformation and Nijenhuis operator

We now study in a little more detail the equivariant deformations of order 1 on a compatible G-Hom-Leibniz algebra. This is known as the equivariant infinitesimal deformation. We then explore equivariant Nijenhuis operators, some results on these operators and establish a relation between equivariant Nijenhuis operators and infinitesimal deformations.

Definition 4.4. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra. Let* $\mu_1, m_1 \in C^2_{G,\alpha}(L, L)$. *Define*

$$\mu_t(x, y) = [x, y] + t\mu_1(x, y), \quad m_t(x, y) = [x, y] + tm_1(x, y), \quad \forall x, y \in L.$$

If for any t, (L, μ_t, m_t, α) is a compatible G-Hom-Leibniz algebra, we say that (L, μ_t, m_t, α) defines an equivariant infinitesimal deformation of $(L, [.,.], \{.,.\}, \alpha)$.
We also say that (μ_1, m_1) generates an equivariant infinitesimal deformation of $(L, [.,.], \{.,.\}, \alpha)$.

For convenience, like earlier, we write $[x, y] = \mu_0(x, y)$ and $\{x, y\} = m_0(x, y)$.
(L, μ_t, m_t, α) is a compatible G-Hom-Leibniz algebra gives us.

(1) $[\mu_0, \mu_0]_B = 0,\ [\mu_0, \mu_1]_B = 0,\ [\mu_1, \mu_1]_B = 0$

(2) $[m_0, m_0]_B = 0,\ [m_0, m_1]_B = 0,\ [m_1, m_1]_B = 0$

(3) $[\mu_0, m_0]_B = 0,\ [\mu_0, m_1]_B + [\mu_1, m_0]_B = 0,\ [\mu_1, m_1]_B = 0.$

Reordering the terms and excluding the trivial equations, we get that (L, μ_t, m_t, α) defines an equivariant infinitesimal deformation of $(L, [.,.], \{.,.\}, \alpha)$ iff

$$[\mu_0, \mu_1]_B = 0, \quad [m_0, m_1]_B = 0, \quad [\mu_0, m_1]_B + [\mu_1, m_0]_B = 0$$

$$[\mu_1, \mu_1]_B = 0, \quad [\mu_1, m_1]_B = 0, \quad [\mu_1, m_1]_B = 0.$$

Note that the first line above implies $d^2(\mu_1, m_1) = 0$ i.e (μ_1, m_1) is a 2-cocycle and the second line implies that (L, μ_1, m_1, α) is a compatible G-Hom-Leibniz algebra.
Hence, we have the following theorem.

Theorem 4.2. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra. If* $(\mu_1, m_1) \in LC^2_{G,\alpha}(L, L)$ *generates an infinitesimal deformation then* (μ_1, m_1) *is a cocycle.*

Definition 4.5. *Two equivariant infinitesimal deformations* (L, μ_t, m_t, α) *and* $(L, \mu'_t, m'_t, \alpha)$ *of compatible G-Hom-Leibniz algebra* $(L, [.,.], \{.,.\}, \alpha)$ *are said to be equivalent if there exists a linear map* $N: L \to L$ *such that*

$$Id + tN : (L, \mu_t, m_t, \alpha) \to \left(L, \mu'_t, m'_t, \alpha\right)$$

is a compatible G-Hom-Leibniz algebra G-homomorphism.

Observe that, $Id + tN$ being a compatible G-Hom-Leibniz algebra G-homomorphism implies that $\forall x, y \in L, g \in G$.

(1) $[x, y] = [x,y]'$

(2) $\mu_1(x, y) - \mu'_1(x, y) = [x, N(y)] + [N(x), y] - N[x, y]$

(3) $N\mu_1(x, y) = \mu'_1(x, N(y)) + \mu'_1(N(x), y) + [N(x), N(y)]$

(4) $\mu'_1(N(x), N(y)) = 0$

(5) $\{x, y\} = \{x,y\}'$

(6) $m_1(x, y) - m'_1(x, y) = \{x, N(y)\} + \{N(x), y\} - N\{x, y\}$

(7) $Nm_1(x, y) = m'_1(x, N(y)) + m'_1(N(x), y) + \{N(x), N(y)\}$

(8) $m'_1(N(x), N(y)) = 0$

(9) $Ng = gN$

(10) $N\alpha = \alpha N$.

Now 2 and 6 give

$$\begin{aligned}(\mu_1 - \mu_1', m_1 - m_1')(x,y) &= ([x,N(y)] + [N(x),y] - N[x,y], \{x,N(y)\} + \{N(x),y\} - N\{x,y\}) \\ &= ([\mu_0, N]_B, [m_0, N]_B) = d^1 N.\end{aligned}$$

Thus, we have the following theorem.

Theorem 4.3. If two equivariant infinitesimal deformations (L, μ_t, m_t, α) *and* $(L, \mu_t', m_t', \alpha)$ *of a compatible G-Hom-Leibniz algebra* (L, μ_0, m_0, α) *are equivalent then,* (μ_1, μ_1') *and* (m_1, m_1') *are in the same cohomology class.*

Definition 4.6. Let $(L, [.,.], \alpha)$ *be a G-Hom-Leibniz algebra. A linear map* $N: L \to L$ *is said to be a G-Rota-Baxter operator on L if*

$$N([x,N(y)] + [N(x),y]) = [N(x),N(y)] \quad \forall x,y \in L \text{ and}$$

$$Ng = gN \text{ and } \alpha N = N\alpha.$$

Definition 4.7. Let $(L, [.,.], \alpha)$ *be a G-Hom-Leibniz algebra. A linear map* $N: L \to L$ *is said to be a G-Nijenhuis operator on L if*

$$N([x,N(y)] + [N(x),y] - N[x,y]) = [N(x),N(y)] \quad \forall x,y \in L$$

$$Ng = gN \text{ and } \alpha N = N\alpha.$$

We define a linear map $[.,.]_N: L \otimes L \to L$ as

$$[x,y]_N = [x,N(y)] + [N(x),y] - N[x,y].$$

It is straightforward to see $g[x,y]_N = [gx,gy]_N$. Further, using the multiplicativity of α and the fact that $N\alpha = \alpha N$, we get

$$\alpha[x,y]_N = [\alpha(x),\alpha(y)]_N.$$

$T_{[.,.]}N: L \otimes L \to L$ denotes the *Nijenhuis torsion* of N defined as

$$T_{[.,.]}N(x,y) = N([x,y]_N) - [N(x),N(y)], \quad \forall x,y \in L.$$

When N is a Nijenhuis operator, we get that $T_{[.,.]}N = 0$.

Example 4.1. The identity map $I: L \to L$ *is a Nijenhuis operator on any G-Hom-Leibniz algebra* $(L, [.,.], \alpha)$.

Proposition 4.1. If $N: L \to L$ *is a G-Nijenhuis operator on G-Hom-Leibniz algebra* $(L, [.,.], \alpha)$*, then* $(L, [.,.]_N, \alpha)$ *is also a G-Hom-Leibniz algebra. Further, N is a Leibniz algebra G-homomorphism from* $(L, [.,.]_N, \alpha)$ *to* $(L, [.,.], \alpha)$ *and* $(L, [.,.], [.,.]_N, \alpha)$ *forms a compatible G-Hom-Leibniz algebra.*

Proof. For every $x, y \in L$ put $[x,y]_N = \pi_N(x, y)$ and $[x, y] = \pi(x, y)$.

Using the Balavoine bracket, we get,

$$[\pi_N, \pi_N]_B(x,y,z) = 2(\pi_N(\pi_N(x,y),\alpha(z)) - \pi_N(\alpha(x),\pi_N(y,z)) + \pi_N(\alpha(y),\pi_N(x,z)))$$

and $g\pi_N = \pi_N g$ Thus $\pi_N = [.,.]_N$ defines a G-Hom-Leibniz algebra structure on L.

Further, $N([x,y]_N) = [N(x), N(y)]$, $gN = Ng\ \forall\, x, y \in L, g \in G$ and $N\alpha = \alpha N$ follows from the definition of G-Nijenhuis operator and $[.,.]_N$.

To show $(L, [.,.], [.,.]_N, \alpha)$ is a compatible G-Hom-Leibniz algebras we first note that $\pi_N = [\pi,N]_B$. For any k_1 and $k_2 \in K$,

$$\begin{aligned} [k_1\pi + k_2\pi_N, k_1\pi + k_2\pi_N]_B &= k_1k_2([\pi,\pi_N]_B + [\pi_N,\pi]_B) \\ &= 2k_1k_2[\pi,\pi_N]_B \\ &= 2k_1k_2[\pi,[\pi,N]_B]_B \\ &= 0. \end{aligned}$$

□

Definition 4.8. *Suppose* $(L, [.,.], \{.,.\}, \alpha)$ *is a compatible G-Hom-Leibniz algebra. A G-linear map* $N: L \to L$ *is said to be a G-Rota-Baxter operator on* $(L, [.,.], \{.,.\}, \alpha)$ *if N is a G-Rota-Baxter operator on the G-Hom-Leibniz algebras* $(L, [.,.], \alpha)$ *and* $(L, \{.,.\}, \alpha)$.

On similar lines, we have the definition of a G-Nijenhuis operator on a compatible G-Hom-Leibniz algebra.

Definition 4.9. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra. The G-linear map* $N: L \to L$ *is said to be a G-Nijenhuis operator on* $(L, [.,.], \{.,.\}, \alpha)$ *if N is a G-Nijenhuis operator on the G-Hom-Leibniz algebras* $(L, [.,.], \alpha)$ *and* $(L, \{.,.\}, \alpha)$.

The following proposition establishes a simple relation between the Nijenhuis operators and Rota-Baxter operators on the compatible G-Hom-Leibniz algebra given by $(L, [.,.], \{.,.\}, \alpha)$.

Proposition 4.2. *Let* $N: L \to L$ *be a G-linear map satisfying* $N^2 = 0$. *Then N is a G-Nijenhuis operator iff N is a G-Rota-Baxter operator on* $(L, [.,.], \{.,.\}, \alpha)$.

Proof. Suppose N is a G-Nijenhuis operator on $(L, [.,.], \{.,.\}, \alpha)$. Then $gN = Ng$, $\alpha N = N\alpha$ and $\forall x, y \in L$,

$$\begin{aligned} [N(x),N(y)] &= N([x,N(y)] + [N(x),y] - N[x,y]) \\ &= N([x,N(y)] + [N(x),y]) - N^2[x,y] \\ &= N([x,N(y)] + [N(x),y]) \quad \text{and} \\ \{N(x),N(y)\} &= N(\{x,N(y)\} + \{N(x),y\} - N\{x,y\}) \\ &= N(\{x,N(y)\} + \{N(x),y\}) - N^2\{x,y\} \\ &= N(\{x,N(y)\} + \{N(x),y\}). \end{aligned}$$

Thus N is a G-Rota-Baxter operator on $(L, [.,.], \{.,.\}, \alpha)$. The converse is similar. □

We now establish a couple of interesting results on Nijenhuis operators.

Proposition 4.3. *Let* $(L, [.,.], \{.,.\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra. The G-linear map* $N: L \to L$ *is a G-Nijenhuis operator on* $(L, [.,.], \{.,.\}, \alpha)$ *iff for any* k_1, k_2 *in K, N is a G-Nijenhuis operator on the G-Hom-Leibniz algebra* $(L, [[\ \ ]], \alpha)$, *where* $[[x,y]] = k_1[x,y] + k_2\{x,y\},\ \forall x, y \in L$.

Proof. We have

$$\begin{aligned} T_{[[\ ]]}N(x,y) &= N\big([[x,y]]_N\big) - [[N(x),(y)]] \\ &= N(k_1[x,y]+k_2\{x,y\}) - k_1[N(x),N(y)] - k_2\{N(x),N(y)\} \\ &= k_1(N([x,y]_N) - [N(x),N(y)]) + k_2(N(\{x,y\}_N) - \{N(x),N(y)\}) \\ &= k_1T_{[.,.]}N(x,y) + k_2T_{\{.,.\}}N(x,y) \end{aligned}$$

Hence, we have,

$$T_{[[\ ]]}N = 0 \quad \text{iff} \quad T_{[.,.]}N = T_{\{.,.\}}N = 0.$$

Proposition 4.4. Let $(L, [., .], \{., .\}, \alpha)$ *be a compatible G-Hom-Leibniz algebra and* $N: L \to L$ *a G-Nijenhuis operator on* $(L, [., .], \{., .\}, \alpha)$*. Then* $(L, [.,.]_N, \{.,.\}_N, \alpha)$ *is a compatible G-Hom-Leibniz algebra and N is a compatible G-Hom-Leibniz algebra homomorphism from* $(L, [.,.]_N, \{.,.\}_N, \alpha)$ *to* $(L, [., .], \{., .\}, \alpha)$*.*

Proof. Let $N: L \to L$ be a G-Nijenhuis operator on $(L, [., .], \{., .\}, \alpha)$. Then by the previous theorem N is a G-Nijenhuis operator on the G-Hom-Leibniz algebra $(L, [[\ ]], \alpha)$ for any k_1, k_2 in K.

Using proposition 4.1 we get that $(L, [[\ ]]_N, \alpha)$ is a G-Hom-Leibniz algebra and N is a G-Leibniz algebra G-homomorphism from $(L, [[\ ]]_N, \alpha)$ to $(L, [[\ ]], \alpha)$.

Hence we get that $(L, [.,.]_N, \{.,.\}_N, \alpha)$ is a compatible G-Hom-Leibniz algebra. Further, we also get that N is a compatible G-Hom-Leibniz algebra homomorphism from $(L, [.,.]_N, \{.,.\}_N, \alpha)$ to $(L, [., .], \{., .\}, \alpha)$.

Definition 4.10. An equivariant infinitesimal deformation (L, μ_t, m_t, α) *of compatible G-Hom-Leibniz algebra* (L, μ_0, m_0) *generated by* (μ_1, m_1) *is trivial if there exists a G-linear* $N: L \to L$ *such that* $Id + tN: (L, \mu_t, m_t, \alpha) \to (L, \mu_0, m_0, \alpha)$ *is a compatible Hom-Leibniz algebra G-homomorphism.*

Now, $Id + tN$ is a compatible Hom-Leibniz algebra homomorphism iff $\forall x, y \in L, g \in G$.

(1) $\mu_1(x, y) = [x, N(y)] + [N(x), y] - N[x, y]$

(2) $m_1(x, y) = \{x, N(y)\} + \{N(x), y\} - N\{x, y\}$

(3) $N\mu_1(x, y) = [N(x), N(y)]$

(4) $Nm_1(x, y) = \{N(x), N(y)\}$

(5) $N\alpha = \alpha N$

(6) $Ng = gN$.

Note that 1,3,5 and 6 give that N is a G-Nijenhuis operator on (L, μ_0, α). Also, note that 2,4,5 and 6 give that N is a G-Nijenhuis operator on (L, m_0, α).

Thus, we have the following theorem.

Theorem 4.4. A trivial equivariant infinitesimal deformation of a compatible G-Hom-Leibniz algebra gives rise to a G-Nijenhuis operator.

Theorem 4.5. A G-Nijenhuis operator on a compatible G-Hom-Leibniz algebra $(L, [., .], \{., .\}, \alpha)$ *gives rise to a trivial deformation.*

Proof. Let N be a G-Nijenhuis operator on a compatible G-Hom-Leibniz algebra $(L, [., .], \{., .\}, \alpha)$. Take

$$\mu_1(x,y) = [x, N(y)] + [N(x), y] - N[x,y]$$

$$m_1(x,y) = \{x, N(y)\} + \{N(x), y\} - N\{x,y\}$$

for any $x, y \in L$. Then

$$\begin{aligned} d^1N(x,y) &= \big([\mu_0, N]_B, [m_0, N]_B\big)(x,y) \\ &= \big([x, N(y)] + [N(x), y] - N[x,y], \{x, N(y)\} + \{N(x), y\} - N\{x,y\}\big) \\ &= (\mu_1(x,y), m_1(x,y)). \end{aligned}$$

i.e., (μ_1, m_1) is a 2-cocycle. Further, since N is a G-Nijenhuis operator on $(L, [.,.], \{.,.\}, \alpha)$ and $\mu_1 = [.,.]_N$ and $m_1 = \{.,.\}_N$, by proposition (4.4) we get that $(L, [.,.]_N, \{.,.\}_N, \alpha)$ is a compatible G-Hom-Leibniz algebra. These two statements imply that (μ_1, m_1) give rise to an infinitesimal deformation of L. Showing that the deformation is trivial is straightforward.

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Further reading

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Corresponding author
Rinkila Bhutia can be contacted at: rbhutia@cus.ac.in