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On the commutativity of quotient near-rings under specific identities

Slimane El Amrani and Abderrahmane Raji
Department of Mathematics, Faculty of Science and Technology, Sultan Moulay Slimane University, Beni Mellal, Morocco

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Abstract

Purpose – The purpose of this paper is to investigate the commutativity of quotient near-rings endowed with a left derivation and a multiplier satisfying specific differential algebraic identities, and to clarify the role of the 3-prime condition in this context.

Design/methodology/approach – The study considers a right near-ring $\mathcal{N}$ together with a 3-prime ideal $\mathcal{P}$ and a nonzero semigroup ideal $\mathcal{I} \nsubseteq \mathcal{P}$, and defines induced structures on the quotient near-ring $\mathcal{N}/\mathcal{P}$. Known results on derivations, left derivations and multipliers on near-rings are combined with several new lemmas to derive sufficient conditions that ensure $\mathcal{N}/\mathcal{P}$ becomes a commutative ring.

Findings – The main results show, in particular, that if a non-$\mathcal{P}$-trivial left derivation and a suitable multiplier on $\mathcal{N}$ satisfy certain differential identities on $\mathcal{I}$, then the quotient near-ring $\mathcal{N}/\mathcal{P}$ is necessarily commutative. The paper also establishes that, under the imposed hypotheses, there is no non-$\mathcal{P}$-trivial left derivation on $\mathcal{N}$, emphasizing the strength of the structural assumptions.

Originality/value – This work extends existing commutativity criteria for prime and semiprime near-rings by focusing on quotient near-rings equipped simultaneously with left derivations and multipliers that obey new differential identities. By restricting the analysis to suitable ideals of $\mathcal{N}$, the paper highlights new mechanisms through which derivations and multipliers enforce commutativity in quotient near-ring structures.

Keywords Prime near-rings, Left derivations, Multipliers, Commutativity

Paper type Research article

1. Introduction

In the present paper, $\mathcal{N}$ will denote a right near-ring with the multiplicative center $\mathcal{Z}(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x.0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributive gives $0.x = 0$). Recall that $\mathcal{N}$ is 3-prime, that is, For all $x, y \in \mathcal{N}, x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. A non empty subset $\mathcal{I}$ of $\mathcal{N}$ is said to be a semigroup left ideal (resp. semigroup right ideal) ideal of $\mathcal{N}$ if $\mathcal{N}\mathcal{I} \subseteq \mathcal{I}$ (resp. $\mathcal{I}\mathcal{N} \subseteq \mathcal{I}$); and if I is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. An additive mapping $H : \mathcal{N} \to \mathcal{N}$ is a multiplier, if $H(xy) = xH(y) = H(x)y$ for all $x, y \in \mathcal{N}$. A derivation on $\mathcal{N}$ is an additive endomorphism δ of $\mathcal{N}$ satisfying the Leibnitz identity $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in \mathcal{N}$; equivalently, $\delta(xy) = x\delta(y) + \delta(x)y$, as noted by Wang (1994). An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(r^2) = 2rd(r)$) holds for all $x, y \in \mathcal{N}$. It is important to note that, in general, the notion of a derivation is not equivalent to that of a left derivation (see, for instance, [[1], Example 3.3]), also a Jordan left derivation need not be a left derivation (see [[2], Example 1.1]).

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The notions of left derivation and Jordan left derivation were introduced by Brešar *et al.* in Ref. [3], where it was established that if a prime ring R of characteristic different from 2 and 3 admits a nonzero Jordan left derivation, then R is necessarily commutative. Later, Ashraf *et al.* [4] proved that the converse holds when the ring is prime and 2-torsion-free.

Further developments were made by Zaidi *et al.* in Ref. [2], where they considered a Jordan ideal J of a 2-torsion-free prime ring R admitting both a nonzero Jordan left derivation d and an automorphism T, satisfying $d(r^2) = 2T(r)d(r)$ for all $r \in J$. Under these assumptions, they showed that either $J \subseteq Z(R)$ or $d(J) = \{0\}$. Given that every ring is, in particular, a near-ring, Lemma 3 of Boua *et al.* [5] shows that the first condition ensures that R is commutative.

A normal subgroup $\mathcal{P}$ of $(\mathcal{N}, +)$ called a left ideal (resp. a right ideal) if $\mathcal{P}\mathcal{N}\subseteq\mathcal{P}$ (resp. $y(x+p) - yx \in \mathcal{P}$ for all $x, y \in \mathcal{N}, p \in \mathcal{P}$), and if $\mathcal{P}$ is both a left ideal and a right ideal, then $\mathcal{P}$ is said to be an ideal of $\mathcal{N}$. As defined by Groenewald [6], an ideal $\mathcal{P}$ is said to be 3-prime if, for all $x, y \in \mathcal{N}$, the inclusion $x\mathcal{N}y\subseteq\mathcal{P}$ implies that $x \in \mathcal{P}$ or $y \in \mathcal{P}$.

We denote by $\mathcal{N}/\mathcal{P}$ the quotient near-ring and $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ its multiplicative center. Clearly, if $\mathcal{P}$ is 3-prime, then $\mathcal{N}/\mathcal{P}$ is a 3-prime near-ring. In Ref. [7], the authors defined a special derivation $\tilde{d}$ on $\mathcal{N}/\mathcal{P}$ by $\tilde{d}(\overline{x}) = \overline{d(x)}$ for all $x \in \mathcal{N}/\mathcal{P}$, where d is a derivation on $\mathcal{N}$. Motivated by this concept, we define a multiplier $\tilde{H}$ and a left derivation $\tilde{d}$ on $\mathcal{N}/\mathcal{P}$ as follows: $\tilde{H}(\overline{x}) = \overline{H(x)}$ and $\tilde{d}(\overline{x}) = \overline{d(x)}$ for all $x \in \mathcal{N}$, respectively.

In the literature, numerous studies have examined the commutativity of 3-prime and semiprime near-rings admitting derivations, generalized derivations, or multiplicative derivations that satisfy appropriate algebraic constraints on certain subsets. Further related results can be found in Refs. [8–18]. This work continues this line of investigation by considering quotient near-rings equipped with a left derivation $\tilde{d}$ and a multiplier $\tilde{H}$ that satisfy specific differential identities ensuring the commutativity of the structure.

2. Preliminary results

We begin by stating several key lemmas that are instrumental in proving our main results.

Lemma 1. Let $\mathcal{N}$ be a near-ring, $\mathcal{P}$ a 3-prime ideal of $\mathcal{N}$, and $\mathcal{I}$ a nonzero semigroup ideal of $\mathcal{N}$ such that $\mathcal{I} \nsubseteq \mathcal{P}$. Then,

i. *If $\overline{x} \in \mathcal{N}/\mathcal{P}$ and $\overline{x}\,\overline{\mathcal{I}} = \{\overline{0}\}$ or $\overline{\mathcal{I}}\,\overline{x} = \{\overline{0}\}$, then $\overline{x} = \overline{0}$.*

ii. *If $\overline{x}, \overline{y} \in \mathcal{N}/\mathcal{P}$ and $\overline{x}\,\overline{\mathcal{I}}\,\overline{y} = \{\overline{0}\}$, then $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$.*

Proof. Since $\mathcal{P}$ is 3-prime, the quotient $\mathcal{N}/\mathcal{P}$ inherits the structure of a 3-prime near-ring. Consequently, the arguments for (i) and (ii) proceed analogously to those in [[19], Lemmas 1.3 (i) & 1.4(i)], respectively. □

Lemma 2. Let $\mathcal{N}$ be a near-ring and $\mathcal{P}$ be a 3-prime ideal of $\mathcal{N}$.

i. *[[7], Lemma 2(b)] If $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ contains a nonzero element $\overline{z}$ for which $\overline{z} + \overline{z} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$, then $(\mathcal{N}/\mathcal{P}, +)$ is abelian.*

ii. *[[7], Lemma 2(c)] If $\overline{z} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})\backslash\{\overline{0}\}$ and $\overline{x} \in \mathcal{N}/\mathcal{P}$ such that $\overline{x}\,\overline{z} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ or $\overline{z}\,\overline{x} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$, then $\overline{x} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$.*

iii. *If $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}/\mathcal{P}$ is a commutative ring.*

Remark 1. With $\mathcal{P}$ 3-prime, the quotient $\mathcal{N}/\mathcal{P}$ is a 3-prime near-ring, and the proof of (iii) parallels that of [[20], Lemma 1.5].

Lemma 3. *[[21], Theorem 3.1] Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero left derivation d, then the following properties hold true:*

i. *If there exists a nonzero element a such that $d(a) = 0$ then $a \in \mathcal{Z}(\mathcal{N})$.*

ii. *$(\mathcal{N}, +)$ is abelian, if and only if $\mathcal{N}$ is a commutative ring.*

3. Main results

This section is devoted to investigating the structural behavior of a right near-ring $\mathcal{N}$ under the action of certain algebraic identities, with the analysis restricted to two distinguished subsets of $\mathcal{N}$: a 3-prime ideal $\mathcal{P}$ and a semigroup ideal $\mathcal{I}$, rather than the entire near-ring. The results obtained include, on the one hand, the commutativity of $\mathcal{N}$, and on the other, the nonexistence of a non-$\mathcal{P}$-trivial left derivation d on $\mathcal{N}$, thereby highlighting the significance of the imposed assumptions. The main findings are summarized below.

Theorem 1. *Let $\mathcal{N}$ be a near-ring, $\mathcal{P}$ a 3-prime ideal of $\mathcal{N}$, and $\mathcal{I}$ a nonzero semigroup ideal of $\mathcal{N}$ such that $\mathcal{I} \nsubseteq \mathcal{P}$. If $\mathcal{N}$ admits a non-$\mathcal{P}$-trivial left derivation d such that $\overline{d(\mathcal{I})} \subseteq \mathcal{Z}(\mathcal{N}/\mathcal{P})$, then $\mathcal{N}/\mathcal{P}$ is a commutative ring.*

Proof. Let $i \in \mathcal{I}$. By hypotheses, we have $\overline{d(i^2)} = \overline{(2i)}\ \overline{d(i)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. In virtue of Lemma 2 ($ii$), we find that

$$\overline{d(i)} = \overline{0} \text{ or } \overline{2i} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } i \in \mathcal{I}. \tag{1}$$

Suppose that there exists an element $k \in \mathcal{I}$ such that $\overline{2k} \notin \mathcal{Z}(\mathcal{N}/\mathcal{P})$. From Ref. [4] it follows that $\overline{d(k)} = \tilde{d}(\overline{k}) = \overline{0}$. Since $\tilde{d}$ is additive, we conclude that $\tilde{d}(2\overline{k}) = \overline{d(2k)} = \overline{0}$. By Lemma 3 ($i$), we infer that $\overline{2k} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$; a contradiction. Therefore [4], reduces to $\overline{2i} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i \in \mathcal{I}$. Taking $i = i^2$ in the last relation and using Lemma 2(ii), we obtain $\overline{2i} = \overline{0}$ or $\overline{i} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i \in \mathcal{I}$. Assume that there exists $i_0 \in \mathcal{I}$ such that $\overline{i_0} \notin \mathcal{Z}(\mathcal{N}/\mathcal{P})$, so that, $\overline{i_0} = -\overline{i_0}$. Now, for all $n \in \mathcal{N}$, we have

$$\begin{aligned}
\tilde{d}(\overline{n \circ i_0}) &= \overline{d(ni_0 + i_0 n)} \\
&= \overline{d(ni_0)} + \overline{d(i_0 n)} \\
&= \overline{n}\overline{d(i_0)} + \overline{i_0}\overline{d(n)} + \overline{i_0}\overline{d(n)} + \overline{n}\overline{d(i_0)} \\
&= \overline{d(i_0)}\overline{n} + (\overline{i_0} + \overline{i_0})\overline{d(n)} + \overline{d(i_0)}\overline{n} \\
&= \overline{0}
\end{aligned}$$

and thus by Lemma 3(i), we deduce that $\overline{n \circ i_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $n \in \mathcal{N}$. Putting $n = ni_0$ in the last relation and using Lemma 2(ii), we get $\overline{n \circ i_0} = \overline{0}$ for all $n \in \mathcal{N}$ which means that $\overline{n}\,\overline{i_0} = -\overline{i_0}\,\overline{n}$ for all $n \in \mathcal{N}$. Replacing n by nt, where $t \in \mathcal{N}$, in the previous equation and using it again, we arrive at $[\overline{n}, -\overline{i_0}]\mathcal{N}/\mathcal{P} = \{\overline{0}\}$ for all $n \in \mathcal{N}$. In view of Lemma 1(i), the above result yields $\overline{i_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$, which contradicts our assumption. Therefore, we conclude that $\overline{i} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i \in \mathcal{I}$, which implies that $\overline{\mathcal{I}} \subseteq \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Hence, $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma [20](iii). □

Theorem 2. *Let $\mathcal{N}$ be a near-ring, $\mathcal{P}$ a 3-prime ideal of $\mathcal{N}$, and $\mathcal{I}$ a nonzero semigroup ideal of $\mathcal{N}$ such that $\mathcal{I} \nsubseteq \mathcal{P}$. Suppose that $\mathcal{N}$ admits a left derivation d and a multiplier H satisfying $d(\mathcal{N}) \nsubseteq \mathcal{P}$ and $H(\mathcal{N}) \nsubseteq \mathcal{P}$. If $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ satisfies one of the following conditions:*

i. *$\overline{[d(xy) + H([x, y]), t] \circ r} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y, t \in \mathcal{I}, r \in \mathcal{N}$,*

ii. $\overline{[(d(xy)+H([x,y]))\circ t, r]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ *for all* $x, y, r \in \mathcal{I}, t \in \mathcal{N}$,

iii. $\overline{((d(xy)+H([x,y]))\circ t)\circ r} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ *for all* $x, y \in \mathcal{I}, t, r \in \mathcal{N}$,

then $\mathcal{N}/\mathcal{P}$ is a commutative ring.

Proof. (i) Assume that

$$\overline{[d(xy)+H([x,y]),t]\circ r} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t \in \mathcal{I}, r \in \mathcal{N}. \tag{2}$$

Substituting $r[d(xy)+H([x,y]),t]$ for r in Ref. [20], and noting that $m \circ nm = (m \circ n)m$, we find that $\left(\overline{[d(xy)+H([x,y]),t]\circ r}\right)\left(\overline{[d(xy)+H([x,y]),t]}\right) \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t \in \mathcal{I}, r \in \mathcal{N}$. In view of Lemma 2($ii$) together with [20], we deduce that

$$\overline{[d(xy)+H([x,y]),t]\circ r} = \overline{0} \text{ or } \overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t \in \mathcal{I}, r \in \mathcal{N}. \tag{3}$$

Let $x,y,t \in \mathcal{I}$, and suppose that $\overline{[d(xy)+H([x,y]),t]\circ r} = \overline{0}$ for all $r \in \mathcal{N}$, which means that $\left(-\overline{[d(xy)+H([x,y]),t]}\right)\overline{r} = \overline{r}\,\overline{[d(xy)+H([x,y]),t]}$ for all $r \in \mathcal{N}$. Putting $r = nr$, where $n \in \mathcal{N}$, in the last expression, we arrive at $\left(-\overline{[d(xy)+H([x,y]),t]}\right)\overline{n}\,\overline{r} = \overline{n}\left(-\overline{[d(xy)+H([x,y]),t]}\right)\overline{r}$ which implies that $\left[-\overline{[d(xy)+H([x,y]),t]}, \overline{n}\right]\mathcal{N}/\mathcal{P} = \{\overline{0}\}$ for all $x,y,t \in \mathcal{I}$. By Lemma 1(i), it follows that $-\overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Consequently [19], reduces to the following

$$\overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \text{ or } -\overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t \in \mathcal{I}. \tag{4}$$

Suppose that there are $x_0, y_0, t_0 \in \mathcal{N}$ such that $\overline{[d(x_0y_0)+H([x_0,y_0]),t_0]} \notin \mathcal{Z}(\mathcal{N}/\mathcal{P})$. From Ref. [8], it necessarily follows that $-\overline{[d(x_0y_0)+H([x_0,y_0]),t_0]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})\backslash\{\overline{0}\}$. Substituting r with $-[d(x_0y_0)+H([x_0,y_0]),t_0]$ in Ref. [20] and applying Lemma 2(ii), we deduce that $2\overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t \in \mathcal{I}$. Once again, choosing $r = [d(xy)+H([x,y]),t]$ in Ref. [20] and in the light of Lemma 2(ii), we conclude that

$$2\overline{[d(xy)+H([x,y]),t]} = \overline{0} \text{ or } \overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t \in \mathcal{I}. \tag{5}$$

Let us now assume that there exist elements $x_1, y_1, t_1 \in \mathcal{N}$ such that $2\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]} = \overline{0}$. Taking $x = x_1$, $y = y_1$, $t = t_1$ and $r = r[d(x_1y_1)+H([x_1,y_1]),t_1]$ in Ref. [20], we obtain

$$\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]\circ r}\,\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } r \in \mathcal{N}.$$

By combining Lemma 2(ii) with equation (2), the preceding result leads to $\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]\circ r} = \overline{0}$ or $\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $r \in \mathcal{N}$. Assume that $\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]\circ r} = \overline{0}$ for all $r \in \mathcal{N}$, it follows that

$$\overline{r}\,\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]} = \left(-\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]}\right)\overline{r} = \overline{[d(x_1y_1)+H([x_1,y_1]),t_1]}\,\overline{r}$$

for all $r \in \mathcal{N}$, implying that $\overline{[d(x_1y_1)+H([x_1,y_1]),t_1]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. From Ref. [9], we conclude that $\overline{[d(xy)+H([x,y]),t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t \in \mathcal{I}$. In particular, for $x = x_0$, $y = y_0$ and

$t = t_0$, we have $\overline{[d(x_0y_0) + H([x_0, y_0]), t_0]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ thereby contradicting the assumption that $\overline{[d(x_0y_0) + H([x_0, y_0]), t_0]} \notin \mathcal{Z}(\mathcal{N}/\mathcal{P})$ and therefore

$$\overline{[d(xy) + H([x, y]), t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y, t \in \mathcal{I}. \tag{6}$$

Which leads to $\overline{[[d(xy) + H([x, y]), t], s]} = \overline{0}$ for all $x, y, t \in \mathcal{I}, s \in \mathcal{N}$. Replacing t by $t(d(xy) + H([x, y]))$ in the latter relation and using [10], we obtain $\overline{[d(xy) + H([x, y]), t]}\ \overline{[d(xy) + H([x, y]), s]} = \overline{0}$ for all $x, y, t, \in \mathcal{I}, s \in \mathcal{N}$. On the other hand, by left-multiplying the above equation by $m \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ and using [10], we obtain $\overline{[d(xy) + H([x, y]), t]}\,\overline{m}\,\overline{[d(xy) + H([x, y]), s]} = \{\overline{0}\}$ for all $x, y, t \in \mathcal{I}, s, m \in \mathcal{N}$ which mean that $\overline{[d(xy) + H([x, y]), t]}\ \mathcal{N}/\mathcal{P}\ \overline{[d(xy) + H([x, y]), s]} = \{\overline{0}\}$ for all $x, y, t \in \mathcal{I}, s \in \mathcal{N}$. By 3-primeness of $\mathcal{N}/\mathcal{P}$, the previous relation forces

$$\overline{[d(xy) + H([x, y]), t]} = \overline{0} \text{ or } \overline{[d(xy) + H([x, y]), s]} = \overline{0} \quad \text{for all } x, y, t \in \mathcal{I}, s \in \mathcal{N}.$$

In both cases, we get

$$\overline{d(xy) + H([x, y])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y \in \mathcal{I}. \tag{7}$$

Taking $y = x$, where $x \in \mathcal{I}$, in Ref. [5], we find that $\overline{d(x^2)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}$. Putting $y = x^3$ in Ref. [5], we obtain $\overline{d(x^4)} = (2\overline{x}^2)\,\overline{d(x^2)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}$. By Lemma 2(*ii*), we obtain $\overline{2x^2} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ or $\overline{d(x^2)} = \overline{0}$ for all $x \in \mathcal{I}$. Invoking Lemma 3(*i*), it follows that $2\overline{x}^2 \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ or $\overline{x}^2 \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}$.

(1) Suppose there exists an element $x_0 \in \mathcal{I}$ such that $\overline{x_0}^2 \neq \overline{0}$. Then, by Lemma 2(*i*), the additive group $(\mathcal{N}/\mathcal{P}, +)$ is abelian. Moreover, using Lemma 3(*ii*), we conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring.

(2) Assume that $\overline{x}^2 = \overline{0}$ for all $x \in \mathcal{I}$. Replacing x by xy, where $y \in \mathcal{I}$, in Ref. [5], we obtain $\overline{H(-y)}\,\overline{x}\,\overline{y} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y \in \mathcal{I}$. Putting $x = tx$ and $y = ty$, where $t \in \mathcal{I}$ in the last relation, we get $\overline{H(-t)}\,\overline{y}\,\overline{t}\,\overline{x}\,\overline{t}\,\overline{y} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y, t \in \mathcal{I}$. In virtue of Lemma 2(*ii*), it follows that

$$\overline{-H(t)}\,\overline{y}\,\overline{t} = \overline{0} \text{ or } \overline{x}\,\overline{t}\,\overline{y} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y, t \in \mathcal{I}. \tag{8}$$

Suppose that $\overline{H(-t)}\,\overline{y}\,\overline{t} = \overline{0}$ for $y, t \in \mathcal{I}$, which can be written as $\overline{H(-t)}\overline{\mathcal{I}}\overline{t} = \{\overline{0}\}$ for all $t \in \mathcal{I}$. Using Lemma 1(*ii*), we find that $H(\mathcal{N}) \subseteq \mathcal{P}$, contradicting the assumption that H is not $\mathcal{P}$-trivial. Then, from Ref. [3], there exist $y_0, t_0 \in \mathcal{I}$ such that $\overline{-H(t_0y_0)}\,\overline{t_0} \neq \overline{0}$ and $\overline{x}\,\overline{t_0}\,\overline{y_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}$. Substituting zx for x, where $z \in \mathcal{N}$, in the last relation, we obtain $\overline{z}\,\overline{x}\,\overline{t_0}\,\overline{y_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}, z \in \mathcal{N}$. In view of Lemma 2(*ii*), the previous result shows that $\overline{x}\,\overline{t_0}\,\overline{y_0} = \overline{0}$ or $\overline{z} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{I}, z \in \mathcal{N}$. If the first condition holds, in this case we can conclude that $\overline{\mathcal{I}}\,\overline{t_0}\,\overline{y_0} = \{\overline{0}\}$ which, in virtue of Lemma 1(*i*), implies that $\overline{t_0}\,\overline{y_0} = \overline{0}$ and hence $\overline{-H(t_0y_0)}\,\overline{t_0} = \overline{0}$; a contradiction. Accordingly, $\overline{z} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $z \in \mathcal{N}$ and thus $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma 2(*iii*).

(*ii*) Suppose that

$$\overline{[(d(xy) + H([x, y]))\circ t, r]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y, r \in \mathcal{I}, t \in \mathcal{N}. \tag{9}$$

Replacing r by $r((d(xy) + H([x, y]))\circ t)$ in Ref. [11] and invoking Lemma 2(*ii*), we find that for all $x, y, r \in \mathcal{I}, t \in \mathcal{N}$, $\overline{[(d(xy) + H([x, y]))\circ t, r]} = \overline{0}$ or $\overline{(d(xy) + H([x, y]))\circ t} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. By expanding the first condition, we reach the same conclusion as in the second one, thereby reinforcing the result

$$\overline{(d(xy)+H([x,y]))\circ t} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y \in \mathcal{I}, t \in \mathcal{N}. \tag{10}$$

(1) In the next step, our objective is to show that $\mathcal{Z}(\mathcal{N}/\mathcal{P}) \neq \{\overline{0}\}$. Indeed, suppose that $\mathcal{Z}(\mathcal{N}/\mathcal{P}) = \{\overline{0}\}$. From Ref. [1], it follows that $\overline{(d(xy)+H([x,y]))\circ t} = \overline{0}$ for all $x,y \in \mathcal{I}, t \in \mathcal{N}$ which leads to $\left(-\left(\overline{d(xy)+H([x,y])}\right)\right)\overline{t} = \overline{t}\,\overline{(d(xy)+H([x,y]))}$ for all $x,y \in \mathcal{I}, t \in \mathcal{N}$. Taking $t = ts$, where $s \in \mathcal{N}$, in the previous equation and applying it again, we get for all $x,y \in \mathcal{I}$ and $t,s \in \mathcal{N}$, $\left(-\left(\overline{d(xy)+H([x,y])}\right)\right)\overline{t}\,\overline{s} = \overline{t}\left(-\left(\overline{d(xy)+H([x,y])}\right)\right)\overline{s}$. So that, $\left[-\left(\overline{d(xy)+H([x,y])}\right), \overline{t}\right]\mathcal{N}/\mathcal{P} = \{\overline{0}\}$ for all $x,y \in \mathcal{I}, t \in \mathcal{N}$. By applying Lemma 1($i$) and under the assumption that $\mathcal{Z}(\mathcal{N}/\mathcal{P}) = \{\overline{0}\}$, we obtain $\overline{d(xy)+H([x,y])} = \overline{0}$ for all $x,y \in \mathcal{I}$. In particular, putting $x = y$ in the last relation, we get $\overline{d(y^2)} = \overline{0}$ for all $y \in \mathcal{I}$ which, in view of Lemma 3(i), implies that $\overline{y}^2 \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Using the same arguments as those used in the first part, we arrive at the conclusion that $\mathcal{N}/\mathcal{P}$ is a commutative ring. Hence, $\mathcal{N}/\mathcal{P} = \mathcal{Z}(\mathcal{N}/\mathcal{P}) = \{\overline{0}\}$, which contradicts the fact that d and H are not P-trivial.

(2) Now, letting $\overline{0} \neq \overline{z_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ and replacing t by z_0 in Ref. [1], we obtain $2\left(\overline{d(xy)+H([x,y])}\right)\overline{z_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y \in \mathcal{I}$. In view of Lemma 2(ii) and the fact that $\overline{z_0} \neq \overline{0}$, we deduce that $2\left(\overline{d(xy)+H([x,y])}\right) \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y \in \mathcal{I}$. Once again, replacing t by $d(xy) + H([x,y])$ in Ref. [1] and invoking Lemma 2(ii), we get

$$2\left(\overline{d(xy)+H([x,y])}\right) = \overline{0} \text{ or } \overline{d(xy)+H([x,y])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y \in \mathcal{I}. \tag{11}$$

Suppose that there exist $x_1, y_1 \in \mathcal{I}$ such that $2\left(\overline{d(x_1y_1)+H([x_1,y_1])}\right) = \overline{0}$, it follows that $\overline{d(x_1y_1)+H([x_1,y_1])} = -\left(\overline{d(x_1y_1)+H([x_1,y_1])}\right)$. Taking $x = x_1, y = y_1$ and $t = t(d(x_1y_1)+H([x_1,y_1]))$ in Ref. [1], we get $\overline{(d(x_1y_1)+H([x_1,y_1])\circ t)}\ \overline{(d(x_1y_1)+H([x_1,y_1]))} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $t \in \mathcal{N}$ which, in view of Lemma 2(ii), implies that $\overline{(d(x_1y_1)+H([x_1,y_1]))\circ t} = \overline{0}$ or $\overline{d(x_1y_1)+H([x_1,y_1]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $t \in \mathcal{N}$. Assume that $\overline{(d(x_1y_1)+H([x_1,y_1]))\circ t} = \overline{0}$ for all $t \in \mathcal{N}$, that is, $\overline{t}\,\overline{(d(x_1y_1)+H([x_1,y_1]))} = (-\overline{(d(x_1y_1)+H([x_1,y_1]))})\overline{t} = \overline{(d(x_1y_1)+H([x_1,y_1]))}\ \overline{t}$ for all $t \in \mathcal{N}$ which leads to $\overline{d(x_1y_1)+H([x_1,y_1])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ and therefore [21] reduces to $\overline{d(xy)+H([x,y])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y \in \mathcal{I}$. The rest of the proof follows the same steps as those used after relation [5] in the proof of (i), we can establish that $\mathcal{N}/\mathcal{P}$ is a commutative ring.

To prove (iii), we assume that

$$\overline{((d(xy)+H([x,y]))\circ t)\circ r} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y \in \mathcal{I}, t,r \in \mathcal{N}. \tag{12}$$

Suppose that $\mathcal{Z}(\mathcal{N}/\mathcal{P}) = \{\overline{0}\}$. It follows that $\overline{r}(\overline{(d(xy)+H([x,y]))\circ t}) = (-(\overline{(d(xy)+H([x,y]))\circ t}))\overline{r}$ for all $x,y \in \mathcal{I}, t,r \in \mathcal{N}$. Putting $r = rs$, where $s \in \mathcal{N}$, in the previous equation and using it again, we find that $\overline{r}(-(\overline{(d(xy)+H([x,y]))\circ t}))\,\overline{s} = (-(\overline{(d(xy)+H([x,y]))\circ t}))\,\overline{r}\,\overline{s}$ for all $x,y \in \mathcal{I}, r, t, s \in \mathcal{N}$, which implies that $[\overline{r}, -(\overline{(d(xy)+H([x,y]))\circ t})]\mathcal{N}/\mathcal{P} = \{\overline{0}\}$ for all $x,y \in \mathcal{I}, t,s \in \mathcal{N}$. In view of the Lemma 1(i), we obtain $-(\overline{(d(xy)+H([x,y]))\circ t}) \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) = \{\overline{0}\}$ which implies that $\overline{(d(xy)+H([x,y]))\circ t} = 0 \quad$ for all $x,y \in \mathcal{I}, t \in \mathcal{N}$. Since this result coincides with relation

[1], a contradiction follows, and hence $\mathcal{Z}(\mathcal{N}/\mathcal{P}) \neq \{\overline{0}\}$. Let $\overline{0} \neq \overline{z_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Substituting $r = z_0$ into [6] and invoking Lemma 2(*ii*), we obtain $2\overline{(d(xy) + H([x,y]))\circ t)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y \in \mathcal{I}, t \in \mathcal{N}$. Replacing r by $(d(xy) + H([x,y])\circ t$ in Ref. [6], and applying Lemma 2(*ii*), we obtain

$$2(\overline{(d(xy) + H([x,y])\circ t}) = \overline{0} \text{ or } \overline{(d(xy) + H([x,y])\circ t} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y \in \mathcal{I}, t \in \mathcal{N}. \tag{13}$$

Suppose that there exist $x_0, y_0 \in \mathcal{I}, t_0 \in \mathcal{N}$ such that $2(\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0}) = \overline{0}$. Our main in the following is to chow that $\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Putting $x = x_0$, $y = y_0$, $t = t_0$ and $r = r((d(x_0y_0) + H([x_0, y_0]))\circ t_0)$ in Ref. [6], and using Lemma 2(*ii*), we get

$$\overline{((d(x_0y_0) + H([x_0,y_0]))\circ t_0)\circ r} = \overline{0} \text{ or } \overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } r \in \mathcal{N}.$$

Assume that the first condition holds for all $r \in \mathcal{N}$, then

$$\begin{aligned}\overline{r}\,(\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0}) &= (-(\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0}))\,\overline{r} \\ &= (\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0})\,\overline{r}\end{aligned}$$

which implies that $\overline{(d(x_0y_0) + H([x_0,y_0]))\circ t_0} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Thus [12], yields $\overline{(d(xy) + H([x,y]))\circ t} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y \in \mathcal{I}, t \in \mathcal{N}$. Since this condition is similar to Ref. [1], it follows by the same reasoning that $\mathcal{N}/\mathcal{P}$ is a commutative ring. □

Theorem 3. Let $\mathcal{N}$ be a near-ring and $\mathcal{P}$ a 3-prime ideal of $\mathcal{N}$. Assume that $\mathcal{N}$ admits a left derivation d and a multiplier H such that $d(\mathcal{N}) \nsubseteq \mathcal{P}$ and $H(\mathcal{N}) \nsubseteq \mathcal{P}$. If $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ satisfies one of the following conditions:

i. $\overline{[d(x)y + H([x,y]), t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ *for all* $x, y, t \in \mathcal{N}$,
ii. $\overline{[d(x)y + H([x,y]), t]\circ r} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ *for all* $x, y, t, r \in \mathcal{N}$,
iii. $\overline{[(d(x)y + H([x,y]))\circ t, r]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ *for all* $x, y, t, r \in \mathcal{N}$,

then $\mathcal{N}/\mathcal{P}$ is a commutative ring.

Proof. (*i*) By hypothesis, we have

$$\overline{[d(x)y + H([x,y]), t]} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y, t \in \mathcal{N}. \tag{14}$$

Substituting $t(d(x)y + H([x, y]))$ for t in Ref. [13] and applying Lemma 2(*ii*), it follows that either $\overline{[d(x)y + H([x,y]), t]} = \overline{0}$ or $\overline{d(x)y + H([x,y])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y, t \in \mathcal{N}$. In both cases, we arrive at the conclusion

$$\overline{d(x)y + H([x,y])} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x, y \in \mathcal{N}. \tag{15}$$

Substituting yx for y in Ref. [14] and using Lemma 2(*ii*), we conclude that either $\overline{d(x)y + H([x,y])} = \overline{0}$ or $\overline{x} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x, y \in \mathcal{N}$. Suppose that

$$\overline{d(x)y + H([x,y])} = \overline{0} \quad \text{for all } x, y \in \mathcal{N}. \tag{16}$$

One hand, taking $y = x$ in Ref. [15], we get $\overline{d(x)x} = \overline{0}$ for all $x \in \mathcal{N}$. On the other hand, putting $y = yd(x)$ in Ref. [15], where $y \in \mathcal{N}$, and using the previous result, we arrive at $(\overline{d(x) + H(x)})\,\mathcal{N}/\mathcal{P}\,\overline{d(x)} = \{\overline{0}\}$ for all $x \in \mathcal{N}$. By 3-primeness of $\mathcal{N}/\mathcal{P}$, it follows that either $\overline{d(x)+H(x)}=\overline{0}$ or $\overline{d(x)}=\overline{0}$ for all $x\in\mathcal{N}$. Given that $\overline{d}\neq -\overline{H}$ then there exists an element $x_0\in\mathcal{N}\backslash\mathcal{P}$ such that $\overline{d(x_0)}=\overline{0}$ and thus, in view Lemma 3(i), $\overline{x_0}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Replacing y by x_0 in Ref. [15], we find that $\overline{d(x)}\,\mathcal{N}/\mathcal{P}\,\overline{x_0}=\{\overline{0}\}$ for all $x\in\mathcal{N}$. The 3-primeness of $\mathcal{N}/\mathcal{P}$ together with $\overline{x_0}\neq\overline{0}$ imply that $\overline{d(x)}=\overline{0}$ for all $x\in\mathcal{N}$, which yields a contradiction. Hence, there exist $x_1,y_1\in\mathcal{N}$ such that $\overline{d(x_1)y_1+H([x_1,y_1])}\neq\overline{0}$ and $\overline{x_1}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\backslash\{\overline{0}\}$. Setting $y=x_1$ in Ref. [14] and using Lemma 2(ii), we obtain $\overline{d(x)}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x\in\mathcal{N}$, i.e., $\overline{d(\mathcal{N})}\subseteq\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Consequently, by Theorem 1, $\mathcal{N}/\mathcal{P}$ is a commutative ring.

(ii) Assume that

$$\overline{[d(x)y+H([x,y]),t]\circ r}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\quad\text{for all } x,y,t,r\in\mathcal{N}. \tag{17}$$

This implies that $\left[\overline{[d(x)y+H([x,y]),t]\circ r},\overline{m}\right]=\overline{0}$ for all $x,y,t,r,m\in\mathcal{N}$. Replacing r by $r[d(x)y+H([x,y]),t]$ in the above relation and applying [16], we obtain, for all $x,y,t,r,m\in\mathcal{N}$,

$$\left(\overline{[d(x)y+H([x,y]),t]\circ r}\right)\left[\overline{[d(x)y+H([x,y]),t]},\overline{m}\right]=\overline{0}.$$

Left-multiplying both sides of the preceding equation by $\overline{k}$, where $k\in\mathcal{N}$, and applying [16], we get $\left(\overline{[d(x)y+H([x,y]),t]\circ r}\right)\overline{k}\left[\overline{[d(x)y+H([x,y]),t]},\overline{m}\right]=\{\overline{0}\}$ for all $x,y,t,r,m,k\in\mathcal{N}$ which means that $\left(\overline{[d(x)y+H([x,y]),t]\circ r}\right)\mathcal{N}/\mathcal{P}\left[\overline{[d(x)y+H([x,y]),t]},\overline{m}\right]=\{\overline{0}\}$ for all $x,y,t,r,m\in\mathcal{N}$. By 3-primeness of $\mathcal{N}/\mathcal{P}$, we deduce that

$$\overline{[d(x)y+H([x,y]),t]\circ r}=\overline{0}\text{ or }\left[\overline{[d(x)y+H([x,y]),t]},\overline{m}\right]=\overline{0}\quad\text{for all } x,y,t,r,m\in\mathcal{N}. \tag{18}$$

Let $x,y,t\in\mathcal{N}$. If the second condition of [17] holds for all $m\in\mathcal{N}$, then $\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Now, suppose that the first condition holds for all $r\in\mathcal{N}$, i.e, $\overline{[d(x)y+H([x,y]),t]\circ r}=\overline{0}$ for all $r\in\mathcal{N}$. It follows that $\left(-\overline{[d(x)y+H([x,y]),t]}\right)\overline{r}=\overline{r}\,\overline{[d(x)y+H([x,y]),t]}$ for all $r\in\mathcal{N}$. Putting $r=nr$, where $n\in\mathcal{N}$, in the preceding relation and applying it again, we thereby obtaining $-\overline{[d(x)y+H([x,y]),t]})\,\overline{n}\,\overline{r}=\overline{n}\left(-\overline{[d(x)y+H([x,y]),t]}\right)\overline{r}$ for all $n\in\mathcal{N}$ which leads to $\left[-\overline{[d(x)y+H([x,y]),t]},\overline{n}\right]\mathcal{N}/\mathcal{P}=\{\overline{0}\}$ for all $n\in\mathcal{N}$. Using Lemma 1(i), we conclude that $-\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Consequently [17], reduces to

$$\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\text{ or }-\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\text{ for all } x,y,t\in\mathcal{N}. \tag{19}$$

Assume that there exist $x_0,y_0,t_0\in\mathcal{N}$ such that $\overline{[d(x_0)y_0+H([x_0,y_0]),t_0]}\notin\mathcal{Z}(\mathcal{N}/\mathcal{P})$. In virtue of [7], necessarily $-\overline{[d(x_0)y_0+H([x_0,y_0]),t_0]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\{\overline{0}\}$. Replacing r by $-[d(x_0)y_0+H([x_0,y_0]),t_0]$ in Ref. [16] and applying Lemma 2(ii), we obtain $2\left(\overline{[d(x)y+H([x,y]),t]}\right)\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t\in\mathcal{N}$. So, in particular for $r=[d(x)y+H([x,y]),t]$ in Ref. [16] and in view of Lemma 2(ii), we find that

$$2\overline{[d(x)y+H([x,y]),t]}=\overline{0} \text{ or } \overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t\in\mathcal{N}. \tag{20}$$

Assume now there exists $x_1, y_1, t_1 \in \mathcal{N}$ such that $2\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]}=\overline{0}$ Taking $x=x_1, y=y_1, t=t_1$ and $r=r[d(x_1)y_1+H([x_1,y_1]),t_1]$ in Ref. [16] and invoking Lemma 2(*ii*), we obtain

$$\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]\circ r}=\overline{0} \text{ or } \overline{[d(x_1)y_1+H([x_1,y_1]),t_1]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } r\in\mathcal{N}$$

Suppose that $\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]\circ r}=\overline{0}$ for all $r\in\mathcal{N}$. Developing this expression and using the fact that $\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]}=-\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]}$, we get $\overline{[d(x_1)y_1+H([x_1,y_1]),t_1]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ and therefore [2] gives $\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t\in\mathcal{N}$. In particular, for $x=x_0, y=y_0$ and $t=t_0$ we obtain $\overline{[d(x_0)y_0+H([x_0,y_0]),t_0]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ which contradicts our assumption that $\overline{[d(x_0)y_0+H([x_0,y_0]),t_0]}\notin\mathcal{Z}(\mathcal{N}/\mathcal{P})$ and therefore $\overline{[d(x)y+H([x,y]),t]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t\in\mathcal{N}$, which is identical to Ref. [13]. We thus conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring.

(iii) By hypothesis given, we have $\overline{[(d(x)y+H([x,y]))\circ t,r]}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y,t,r\in\mathcal{N}$. Taking $r=r((d(x)y+H([x,y]))\circ t)$ and invoking Lemma 2(*ii*), we infer that

$$\overline{(d(x)y+H([x,y]))\circ t}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x,y,t\in\mathcal{N}. \tag{21}$$

In the next step, our goal is to show that $\mathcal{Z}(\mathcal{N}/\mathcal{P})$ is nonzero. Suppose, to the contrary, that $\mathcal{Z}(\mathcal{N}/\mathcal{P})=\{\overline{0}\}$. Then, by Ref. [18], we have $\overline{t}\,\overline{(d(x)y+H([x,y]))}=\left(-\overline{(d(x)y+H([x,y]))}\right)\overline{t}$ for all $x,y,t\in\mathcal{N}$. For $t=tm$, where $m\in\mathcal{N}$, we find that $\overline{t}\left(-\overline{(d(x)y+H([x,y]))}\right)\overline{m}=\left(-\overline{(d(x)y+H([x,y]))}\right)\overline{tm}$ for all $x,y,m,t\in\mathcal{N}$. Accordingly, $\left[\overline{t},-\overline{(d(x)y+H([x,y]))}\right]\mathcal{N}/\mathcal{P}=\{\overline{0}\}$ for all $x,y,t\in\mathcal{N}$. Invoking Lemma 1(*i*) and using our assumption that $\mathcal{Z}(\mathcal{N}/\mathcal{P})=\{\overline{0}\}$, we obtain $\overline{(d(x)y+H([x,y]))}=\overline{0}$ for all $x,y\in\mathcal{N}$. As this result is the same as [15], we conclude that $\mathcal{Z}(\mathcal{N}/\mathcal{P})\neq\{\overline{0}\}$. Letting $0\neq\overline{z}_0\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ and replacing t by z_0 in Ref. [18] and using Lemma 2(*ii*), we get $2\overline{(d(x)y+H([x,y]))}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y\in\mathcal{N}$. Putting $t=d(x)y+H([x,y])$ in Ref. [18] and invoking Lemma 2(*ii*), we obtain

$$2\overline{(d(x)y+H([x,y]))}=\overline{0} \text{ or } \overline{d(x)y+H([x,y])}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}) \text{ for all } x,y\in\mathcal{N}. \tag{22}$$

Suppose that there exist $x_0, y_0\in\mathcal{N}$ such that $2\overline{(d(x_0)y_0+H([x_0,y_0]))}=\overline{0}$. More specifically taking $x=x_0, y=y_0$ and $t=t(d(x_0)y_0+H([x_0,y_0]))$ in Ref. [18] and applying Lemma 2(*ii*), we find that

$$\overline{(d(x_0)y_0+H([x_0,y_0]))\circ t}=\overline{0} \quad \text{for all } t\in\mathcal{N} \text{ or } \overline{d(x_0)y_0+H([x_0,y_0])}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}).$$

Let us note that verifying the first condition also brings us to the same result as the second condition. Consequently, (22) reduces to $\overline{d(x)y+H([x,y])}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $x,y\in\mathcal{N}$ which is identical to Ref. [14], and therefore $\mathcal{N}/\mathcal{P}$ is a commutative ring. □

Theorem 4. Let $\mathcal{N}$ be a near-ring, $\mathcal{P}$ a 3-prime ideal of $\mathcal{N}$, and $\mathcal{I}$ a nonzero semigroup ideal of $\mathcal{N}$ such that $\mathcal{I}\nsubseteq\mathcal{P}$. Then, $\mathcal{N}$ admits no non-$\mathcal{P}$-trivial left derivations d_1 and d_2 satisfying one of the following assertions:

i. $\overline{d_1([x,i])}=\overline{d_2(x)}\,\overline{i}$ *for all* $i\in\mathcal{I}, x\in\mathcal{N}$;

ii. $\overline{d_1([x,i])}=\overline{d_2(ix)}$ *for all* $i\in\mathcal{I}, x\in\mathcal{N}$.

Proof. (i) Suppose $\mathcal{N}$ admits two left derivations d_1 and d_2 which are not $\mathcal{P}$-trivial such that

$$\overline{d_1([x,i])} = \overline{d_2(x)}\,\overline{i} \quad \text{for all } i\in\mathcal{I}, x\in\mathcal{N}. \tag{23}$$

Obviously, we can see that $\overline{d_2(i)i} = \overline{0}$ for all $i\in\mathcal{I}$. Now, replacing x by xi in (23) and applying the definition of d_1 and d_2, we get $\overline{[x,i]}\,\overline{d_1(i)} + \overline{i}\,\overline{d_1([x,i])} = \overline{x}\,\overline{d_2(i)}\,\overline{i} + \overline{i}\,\overline{d_2(x)}\,\overline{i}$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. After simplifying, we get $\overline{x}\,\overline{i}\,\overline{d_1(i)} = \overline{i}\,\overline{x}\,\overline{d_1(i)}$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. Putting xy instead of x, where $y\in\mathcal{N}$, in the above equation and applying it again, we may write $\overline{[x,i]}\,\mathcal{N}/\mathcal{P}\,\overline{d_1(i)} = \{\overline{0}\}$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. By 3-primeness of $\mathcal{N}/\mathcal{P}$, we arrive at $\overline{[x,i]} = \overline{0}$ or $\overline{d_1(i)} = \overline{0}$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. If there exists $i_0\in\mathcal{I}$ such that $\overline{d_1(i_0)} = \overline{0}$, then $\overline{i_0}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ by Lemma 3(i). Then both cases force that $\overline{\mathcal{I}}\subseteq\mathcal{Z}(\mathcal{N}/\mathcal{P})$. So, $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma 2(iii). In this case, our hypothesis becomes $\overline{d_2(x)}\,\overline{\mathcal{I}} = \{\overline{0}\}$ for all $x\in\mathcal{N}$. By Lemma 1 (i), it follows that $d_2(\mathcal{N})\subseteq\mathcal{P}$ but this contradicts our initial assumption that $d_2(\mathcal{N})\not\subseteq\mathcal{P}$.

(ii) Suppose there exist two non-$\mathcal{P}$-trivial left derivations d_1 and d_2 such that

$$\overline{d_1([x,i])} = \overline{d_2(ix)} \quad \text{for all } i\in\mathcal{I}, x\in\mathcal{N}. \tag{24}$$

Replacing x by i in (24), we get $\overline{d_2(i^2)} = \overline{0}$ for all $i\in\mathcal{I}$ which, in view of Lemma 3(i), implies that $\overline{i}^2\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Also, taking $i = i^2$ in (24) and using Lemma 3(i), we get $\overline{i}^2\,\overline{x}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. By Lemma 2($ii$), we infer that

$$\overline{i}^2 = \overline{0} \quad \text{for all } i\in\mathcal{I} \text{ or } \overline{x}\in\mathcal{Z}(\mathcal{N}/\mathcal{P}) \quad \text{for all } x\in\mathcal{N} \tag{25}$$

but each condition leads to a contradiction. In fact, suppose that $\overline{i}^2 = \overline{0}$ for all $i\in\mathcal{I}$. Replacing x by ix in (24) and invoking Lemma 3(i), we find that $\overline{i}\,\overline{x}\,\overline{i}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. On the other hand, putting iti instead of i in (24), where $t\in\mathcal{N}$, and using Lemma 3(i), we obtain $\overline{i}\,\overline{t}\,\overline{i}\,\overline{x}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i\in\mathcal{I}, x, t\in\mathcal{N}$. In the light of Lemma 2(ii), the previous relation shows that $\overline{i}\,\overline{t}\,\overline{i} = \overline{0}$ or $\overline{x}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $i\in\mathcal{I}, x, t\in\mathcal{N}$. Since $\mathcal{N}/\mathcal{P}$ is 3-prime, the first condition gives $\mathcal{I}\subseteq\mathcal{P}$ which contradicts our assumption that $\mathcal{I}\not\subseteq\mathcal{P}$. To complete the proof, it remains to discuss the case where $\overline{x}\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$. If this holds, then $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma 2(iii), and hence (24) becomes $\overline{d_2(ix)} = \overline{0}$ for all $i\in\mathcal{I}, x\in\mathcal{N}$. Putting $i = ti$ in the previous result, we obtain $\overline{i}\overline{x}\overline{d_2(t)} = \overline{0}$ for all $i\in\mathcal{I}, x, t\in\mathcal{N}$. By the 3-primeness of $\mathcal{N}/\mathcal{P}$, it follows that either $\mathcal{I}\subseteq\mathcal{P}$ or $d_2(\mathcal{N})\subseteq\mathcal{P}$, which leads to a contradiction. □

The following example clearly shows that the assumption of 3-primeness for $\mathcal{P}$, as required in the hypotheses of all the preceding theorems, is essential and cannot be omitted.

Example 1. Let $\mathcal{S}$ be a zero-symmetric noncommutative right near-ring. Define $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{I}$ *by:*

$$\mathcal{N} = \left\{\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle|\, 0, x, y, z\in\mathcal{S}\right\}, \quad \mathcal{P} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle|\, 0, x\in\mathcal{S}\right\},$$

$$\mathcal{I} = \left\{\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\, 0, y\in\mathcal{S}\right\}.$$

It is straightforward to verify that $\mathcal{N}$ forms a right near-ring. Moreover, $\mathcal{P}$ constitutes an ideal of $\mathcal{N}$; however, it does not satisfy the 3-primeness condition and hence fails to be a 3-prime ideal. In contrast, $\mathcal{I}$ is a nonzero semigroup ideal of $\mathcal{N}$. We now define the mappings d_1, d_2, and H on $\mathcal{N}$ as follows:

$$d_1\begin{pmatrix}0&0&x\\0&0&y\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&x+y\\0&0&0\end{pmatrix},\quad d_2\begin{pmatrix}0&0&x\\0&0&y\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&x+2y\\0&0&0\\0&0&0\end{pmatrix}$$

$$and\quad H\begin{pmatrix}0&0&x\\0&0&y\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&x\\0&0&0\\0&0&0\end{pmatrix}.$$

It can be readily verified that d_1 *and* d_2 are nonzero left derivations of $\mathcal{N}$, and H is a nonzero multiplier of $\mathcal{N}$ satisfying all the identities required in our theorems. However, $\mathcal{N}/\mathcal{P}$ is a noncommutative ring.

References

1. Enguady A, Boua A. On Jordan ideals with left derivations in 3-prime near-rings. Extracta Math. 2023; 38(1): 51-66. doi: 10.17398/2605-5686.38.1.51.
2. Zaidi SMA, Ashraf M, Shakir A. On Jordan ideals and left (θ; θ)-derivations in prime rings. Int J Math Math Sci. 2004; 37: 1957-64.
3. Brešar M, Vukman J. On left derivations and related mappings. Proc Amer Math Soc. 1990; 110 (1): 7-16. doi: 10.2307/2048234.
4. Ashraf M, Rehman N. On lie ideals and Jordan left derivation of prime rings. Arch Math. 2000; 36: 201-6.
5. Boua A, Oukhtite L, Raji A. Jordan ideals and derivations in prime near-rings. Comment Math Univ Carolin. 2014; 55(2): 131-9.
6. Groenewald NJ. Different prime ideals in near-rings. Comm Algebra. 1991; 19(10): 2667-75. doi: 10.1080/00927879108824287.
7. Samir M, Boua A. (σ, τ)-P-derivations on left near-rings. Note Mat. 2022; 42(2): 93-107.
8. Boua A, Raji A. Several algebraic inequalities on a 3-prime near-ring. JP J Algebra Number Theor Appl. 2017; 39(1): 105-13. doi: 10.17654/nt039010105.
9. Boua A, Oukhtite L, Raji A. On generalized semiderivations in 3-prime near-rings. Asian-Eur J Math. 2016; 9(21): 1650036. (11 pages). doi: 10.1142/s1793557116500364.
10. Boua A, Raji A, Asma A, Farhat A. On generalized semiderivations of prime near-rings. Int J Math Math Sci. 2015; 2015(1): 867923.
11. El Amrani S, Raji A. On Jordan ideals and semigroup ideals in 3-prime near-rings with multiplicative derivations. An Stiint Univ Al I Cuza Iasi Mat (N.S.), Tomul LXXI. 2025; f.2(2): 145-55. doi: 10.47743/anstim.2025.00011.
12. Oukhtite L, Raji A. On two-sided α-n-derivation in 3-prime near-rings. Acta Math Hungar. 2019; 157(2): 465-77. doi: 10.1007/s10474-018-0899-3.
13. Raji A. On multiplicative derivations in 3-prime near-rings. Beitr Algebra Geom. 2024; 65(2): 343-57. doi: 10.1007/s13366-023-00692-0.
14. Raji A, Oukessou M, Belharrate A. Semigroup ideals with multiplicative semiderivations and commutativity of 3-prime near-rings. Note Mat. 2022; 42(2): 43-52.
15. Raji A, Oukessou M, Belharrate A. Jordan ideals with multiplicative derivations in 3-prime near-rings. Adv Math Models Appl. 2022; 7(2): 214-22.
16. Raji A. Some commutativity criteria for 3-prime near-rings. Algebra Discrete Math. 2021; 32(2): 280-92. doi: 10.12958/adm1439.
17. Raji A. Results on 3-prime near-rings with generalized derivations. Beitr Algebra Geom. 2016; 57 (4): 823-9. doi: 10.1007/s13366-015-0267-1.

18. Zerbane A, El Amrani S, Raji A, Boua A. Study of the structure of near-rings with additive maps. Math. 2024; 66(2): 289-97. doi: 10.24193/mathcluj.2024.2.11.
19. Bell HE, Mason G. On derivations in near-rings. In North-Holland Mathematics Studies. 1987; 137: 31-5. doi: 10.1016/s0304-0208(08)72283-7.
20. Bell HE. On derivations in near-rings II. Netherlands: Kluwer Academic; 1997. p. 191-7.
21. Enguady A, Boua A. On Lie ideals with left derivations in 3-prime near-rings. An. Stiint Univ Al I Cuza Iasi Mat (N.S.). 2022; 68(1): 123-32. doi: 10.47743/anstim.2022.00009.

Corresponding author
Abderrahmane Raji can be contacted at: rajiabd2@gmail.com