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Rings whose non-units are square-nil clean

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Abstract

Purpose – To describe the structure of a special sort of rings whose non-invertible elements are the sum of a nilpotent and a square-idempotent (which commute one another). Specifically, we consider in-depth and characterize in certain aspects the class of so-called *strongly NUS-nil clean rings*, that are those rings whose non-units are *square-nil clean* in the sense that they are a sum of a nilpotent and a square-idempotent that commutes with each other. This class of rings lies properly between the classes of strongly nil clean rings and strongly clean rings.
Design/methodology/approach – We develop an original method of proof based on polynomial expressions.
Findings – It is proved the valuable criterion that a ring R is strongly NUS-nil clean if, and only if, a4 - a2 $\in$ Nil (R) for every a $\notin$ *U(R)*. In particular, a ring R with only trivial idempotents is strongly NUS-nil clean if, and only if, R is a local ring with nil Jacobson radical. Some special matrix constructions and group ring extensions will provide us with new sources of examples of strongly NUS-nil clean rings.
Originality/value – We declare that the obtained by us results are absolutely original and do not duplicate other known results.

Keywords Unit, (Square-)nil clean element (ring), (Strongly) square-nil clean element (ring), Matrix ring, Group ring

Paper type Research article

1. Introduction and basic facts

Throughout (the rest of) the present paper, all rings are supposed to be associative with identity and all modules are unitary. We denote by $U(R)$, Nil(R) and $Id(R)$ the set of invertible elements, the set of nilpotent elements and the set of idempotent elements in R, respectively. Likewise, $J(R)$ denotes the Jacobson radical of R, and $Z(R)$ denotes the centre of R. The ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are, respectively, denoted by $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$.

Mimicking Nicholson [1, 2], a ring is called *clean* if every its element can be written as a sum of a unit element and an idempotent element. Moreover, a ring R is termed *exchange* if, for each $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Every clean ring is known to be exchange, but the converse implication is manifestly *not* valid in general; however, it is true in the abelian case (see, e.g. [1, Proposition 1.8]).

In [2], an element c of a ring R is defined to be *strongly clean* if there is an idempotent $e \in R$, which commutes with c, such that $c - e$ is invertible in R. Analogously, a strongly clean ring is

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the one in which every element is strongly clean. Furthermore, it is shown that a strongly clean element of a ring satisfies a generalized version of the classical Fitting's Lemma. This yields (cf. [3, Section 10]) that each strongly π-regular element of a ring is strongly clean. (We also refer to Ref. [4] for an excellent overview of the relationship between the clean property and other classical ring theory notions.)

Next, imitating Diesl [5], a ring R is said to be *(strongly) nil clean* if, for each $r \in R$, there are $q \in \mathrm{Nil}(R)$ and $e \in Id(R)$ such that $r = q + e$ (in addition, $qe = eq$ for strong nil cleanness). Many fundamental properties of (strongly) nil clean rings were proven in this source as well as their general theory was developed there.

Following Danchev *et al.* [6], a ring R is called *generalized strongly nil clean*, and briefly abbreviated by GSNC, if any non-invertible element of R is strongly nil clean.

Now, we say that an element $a \in R$ is said to be *square-idempotent*, provided $a^2 = a^4$. Motivated by the above discussion, we introduce and study the notion of *NUS-nil clean* rings. So, a ring R is called *(strongly) square-nil clean* if, for each $a \in R$, there are a square-idempotent $e \in R$ and an element $n \in \mathrm{Nil}(R)$ such that $a = e + n$ (in addition, $en = ne$ for strong square-nil cleanness). If any non-unit (i.e. any non-invertible element) in a ring R is (strongly) square-nil clean, we say R is (strongly) NUS-nil clean. The class of strongly NUS-nil clean rings lies strictly between the classes of strongly nil clean rings and strongly clean rings. Specifically, we establish that: a ring R is strongly NUS-nil clean if, and only if, $a^4 - a^2 \in \mathrm{Nil}(R)$ for all $a \notin U(R)$. We, moreover, prove that this property passes to corner rings and is *not* Morita invariant. Also, the direct product of strongly NUS-nil clean rings is too a strongly NUS-nil clean ring. In particular, strongly NUS-nil clean rings are always strongly π-regular. Likewise, the Jacobson radical of a strongly NUS-nil clean ring is a nil-ideal. Finally, a ring R with only trivial idempotents is strongly NUS-nil clean if, and only if, R is a local ring with nil Jacobson radical. Certain matrix constructions and group ring extensions will reach us with some new types of examples of NUS-nil clean rings.

Our strategic achievements are, respectively, Theorems 2.6, 2.28, 2.30 and 2.37 stated and proved in the sequel.

2. Strongly NUS-nil clean rings

In this section, we officially introduce the concept of strongly NUS-nil clean rings and investigate their elementary but useful properties.

Definition 2.1. Let R be a ring. An element $e \in R$ is said to be *square-idempotent*, provided $e^2 = e^4$. Additionally, an element $a \in R$ is said to be *square-nil clean* if $a = e + n$, where e is a square-idempotent and n is a nilpotent. In particular, if $en = ne$, the element is called *strongly square-nil clean*. Particularly, a ring R is called *(strongly) NUS-nil clean* if every non-unit (i.e., every non-invertible) element in R is (strongly) square-nil clean.

We begin our work with the next series of preliminary technicalities.

Lemma 2.2. Every strongly NUS-nil clean element is clean.

Proof. Assume $a = e + n$, where a is a non-unit, e is a square-idempotent and n is a nilpotent element of a ring R that commutes with e. It is now easily seen that $a = 1 - e^2 + e - 1 + e^2 + n$ and $e - 1 + e^2 + n \in U(R)$, as needed. □

Our immediate consequence is the following.

Corollary 2.3. Every strongly NUS-nil clean ring is strongly clean.

In other words, the class of strongly NUS-nil clean rings lies in a proper way between the classes of strongly nil clean rings and strongly clean rings. Indeed, it is worthwhile noting that $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathrm{M}_2(\mathbb{Z}_3)$ and $\mathrm{M}_2(\mathbb{Z}_2)$ are all strongly NUS-nil clean rings that are *not* strongly nil clean, while $\mathbb{Z}_2[[x]]$ and $\mathbb{Z}_3[[x]]$ are strongly clean rings that are *not* strongly NUS-nil clean.

Lemma 2.4. *Let R_i be a ring for all $i \in I$, where $n \geq 2$ is an integer and $I = \{1, \ldots, n\}$. Then, the finite direct product $\prod_{i=1}^{n} R_i$ is strongly NUS-nil clean if, and only if, each direct component R_i is strongly square-nil clean.*

Proof. The sufficiency being clear, we are focussing on necessity. To that aim, letting $a_i \in R_i$ for some index i, whence

$$(0, \ldots, 0, a_i, 0, \ldots, 0) \notin U\left(\prod_{i=1}^{n} R_i\right).$$

So, $(0, \ldots, 0, a_i, 0, \ldots, 0)$ is strongly square-nil clean, and hence a_i is strongly square-nil clean. That is why, any R_i is strongly square-nil clean, as required. □

Example 2.5. Obvious calculations illustrate that the commutative ring $\mathbb{Z}_5$ is strongly NUS-nil clean, but is *not* strongly square-nil clean.

Recall the pivotal fact that, in view of [7, Lemma 2.6], if $2 \in U(R)$ and $a^3 - a$ is a nilpotent, then there exists a polynomial $\alpha(t) \in \mathbb{Z}[t]$ such that $\alpha(a)^3 = \alpha(a)$ and $a - \alpha(a)$ is a nilpotent.

We now manage to proceed by proving the following helpful necessary and sufficient condition.

Theorem 2.6. *Let R be a ring. Then, R is strongly NUS-nil clean if, and only if, $a^4 - a^2 \in Nil(R)$ for every $a \notin U(R)$.*

Proof. Assume that R is strongly NUS-nil clean. Then, for all $a \notin U(R)$, there are $e^4 = e^2 \in R$ and $n \in \mathrm{Nil}(R)$ such that $a = e + n$ and $en = ne$. Thus, $a^2 = e^2 + n(2e + n)$. So,

$$a^4 = e^2 + n(2e + n)\left(2e^2 + n(2e + n)\right).$$

Therefore, clearly $a^4 - a^2 \in \mathrm{Nil}(R)$, as desired.

Conversely, assume that $a^4 - a^2 \in \mathrm{Nil}(R)$ for every $a \notin U(R)$. It follows that $a^3 - a \in \mathrm{Nil}(R)$ for each $a \notin U(R)$. If, for a moment, $2 \in U(R)$, then the aforementioned [7, Lemma 2.6] applies to get that there exists $\alpha(a) \in \mathbb{Z}[a]$ such that $\alpha(a)^3 = \alpha(a)$ and $a - \alpha(a) \in \mathrm{Nil}(R)$. So, there is $m \in \mathrm{Nil}(R)$ such that $a = \alpha(a) + m$. It is now easy to see that $\alpha(a)^4 = \alpha(a)^2$ and $m\alpha(a) = \alpha(a)m$, so that we are done.

Next, if $2 \notin U(R)$, then, by hypothesis, $12 = 2^4 – 2^2 \in \mathrm{Nil}(R)$. So, $12 = 3 \times 2 \times 2 \in \mathrm{Nil}(R)$ insuring that $6 \in \mathrm{Nil}(R)$. On the other hand, since, for every $a \notin U(R)$, it must be that $a^3 - a \in \mathrm{Nil}(R)$, we get $(a^3 - a)^6 \in \mathrm{Nil}(R)$ and hence

$$a^{18} - 6a^{16} + 15a^{14} - 30a^{12} + 15a^{10} - 6a^8 + a^6 \in \mathrm{Nil}(R).$$

But, as $6 \in \mathrm{Nil}(R)$, we perceive $a^{18} + 3a^{14} + 3a^{10} + a^6 \in \mathrm{Nil}(R)$, and since $a^4 - a^2 \in \mathrm{Nil}(R)$, we obtain that $a^6 - a^4 \in \mathrm{Nil}(R)$ and so $a^6 - a^2 \in \mathrm{Nil}(R)$. Now, from $a^8(a^6 - a^2) \in \mathrm{Nil}(R)$, we may write $a^{14} - a^{10} \in \mathrm{Nil}(R)$ and thus $3a^{14} - 3a^{10} \in \mathrm{Nil}(R)$. Further, as $a^{18} + 3a^{14} + 3a^{10} + a^6 \in \mathrm{Nil}(R)$, we have $a^{18} + 6a^{14} + a^6 \in \mathrm{Nil}(R)$, and since $6 \in \mathrm{Nil}(R)$, we deduce that $a^{18} + a^6 \in \mathrm{Nil}(R)$ giving $a^{13} + a \in \mathrm{Nil}(R)$. However, from $a^3 - a \in \mathrm{Nil}(R)$, we can see that $a^5 - a^3 \in \mathrm{Nil}(R)$ and so $a^5 - a^3 + a^3 - a \in \mathrm{Nil}(R)$. Arguing in a similar way, one can see that $a^7 - a \in \mathrm{Nil}(R)$ and $a^{13} - a \in \mathrm{Nil}(R)$. From this and $a^{13} + a \in \mathrm{Nil}(R)$, we derive $2a \in \mathrm{Nil}(R)$ for all $a \notin U(R)$.

Furthermore, as $a^7 - a \in \mathrm{Nil}(R)$, we deduce $a(a^6 - 1) \in \mathrm{Nil}(R)$. Since $a \notin U(R)$, we can observe that $-a^4 + a \notin U(R)$ whence $-2a^4 + 2a \in \mathrm{Nil}(R)$. Thus, $a(a^6 - 1 - 2a^3 + 2) \in \mathrm{Nil}(R)$ guaranteeing that $a(a^6 - 2a^3 + 1) \in \mathrm{Nil}(R)$. It now follows from $6 \in \mathrm{Nil}(R)$ that $-6a^6 - 6a^2 + 30a^3 - 18a^4 \in \mathrm{Nil}(R)$ which forces that $a(a^6 - 20a^3 + 1 - 6a^5 - 6a + 30a^2) \in \mathrm{Nil}(R)$. Combining this and the relation $15a^5 - 15a^3 \in \mathrm{Nil}(R)$, we arrive at

$$a\left(a^6 - 20a^3 + 1 - 6a^5 - 6a + 30a^2 + 15a^4 - 15a^2\right) \in \mathrm{Nil}(R),$$

which means that

$$a\left(a^6 - 20a^3 + 1 - 6a^5 - 6a + 15a^2 + 15a^4\right) \in \mathrm{Nil}(R)$$

Therefore, $a(a-1)^6 \in \mathrm{Nil}(R)$ so that there is $k \in N$ such that $a^k(a-1)^{6k} = 0$ and thus $a^{6k}(a-1)^{6k} = 0$. So, one infers that $a(a-1) \in \mathrm{Nil}(R)$.

Furthermore, owing now to [8, Lemma 3.5], there exists $\beta(a) \in \mathbb{Z}[a]$ such that $\beta(a)^2 = \beta(a)$ and $a - \beta(a) \in \mathrm{Nil}(R)$ (note that the cited lemma works even without the assumption that 2 is not a unit in R). Finally, it follows that $a = \beta(a) + k$, where $k \in \mathrm{Nil}(R)$ and $k\beta(a) = \beta(a)k$, as wanted. □

A ring R is known to be *strongly π-regular*, provided that, for any $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$ (cf. [9]).

We now come to our critical statement.

Proposition 2.7. Every strongly NUS-nil clean ring is strongly π-regular.

Proof. Let R be a strongly NUS-nil clean ring and $a \in R$. If $a \in U(R)$, then a is obviously a strongly π-regular element. If, however, $a \notin U(R)$, then, in virtue of Theorem 2.6, there exists $m \in \mathbb{N}$ such that $(a^4 - a^2)^m = 0$. It, thereby, follows at once that $a^{2m} = a^{2m+1}r$ for some $r \in R$. Consequently, R is a strongly π-regular ring, as expected. □

We next continue with further crucial claims.

Lemma 2.8. Let R be a ring and $0 \neq e = e^2 \in R$. If R is strongly NUS-nil clean, then so is the corner subring eRe.

Proof. Let $a \in eRe$ such that $a \notin U(eRe)$. We write $a = ea = ae = eae$. If $a \in U(R)$, then there exists $b \in R$ such that $ab = ba = 1$, which assures $a(ebe) = (ebe)a = e$, a contradiction. Therefore, $a \notin U(R)$. Hence, according to Theorem 2.6, we have

$$a^4 - a^2 \in \mathrm{Nil}(R) \cap eRe \subseteq \mathrm{Nil}(eRe)$$

as promised. □

Lemma 2.9. Let R be a strongly NUS-nil clean ring. Then, $J(R)$ is a nil-ideal of R.

Proof. Let $j \in J(R)$. Thus, $j \notin U(R)$. Now, using Theorem 2.6, we detect $j^4 - j^2 \in \mathrm{Nil}(R)$. It apparently follows that $j^2(j^2 - 1) \in \mathrm{Nil}(R)$. So, $j^{2t}(j^2 - 1)^t = 0$ for some positive integer t. However, as $j^2 - 1 \in U(R)$, we get $j^{2t} = 0$ and so $j \in \mathrm{Nil}(R)$, as pursued. □

Proposition 2.10. Let R be a ring. The following hold:

(i) *For any nil-ideal $I \subseteq R$, R is strongly NUS-nil clean if, and only if, R/I is strongly NUS-nil clean.*

(ii) *A ring R is strongly NUS-nil clean if, and only if, $J(R)$ is nil and $R/J(R)$ is strongly NUS-nil clean.*

Proof. (i) Suppose R is a strongly NUS-nil clean ring and put $\overline{R} := R/I$. If $\overline{a} \notin U(\overline{R})$, then $a \notin U(R)$, which ensures with the aid of Theorem 2.6 that $a^4 - a^2 \in \mathrm{Nil}(R)$, so that $\overline{a}^4 - \overline{a}^2 \in \mathrm{Nil}(\overline{R})$.

Conversely, suppose $\overline{R}$ is a strongly NUS-nil clean ring. If $a \notin U(R)$, then $\overline{a} \notin U(\overline{R})$, and thus Theorem 2.6 guarantees that $\overline{a}^4 - \overline{a}^2 \in \mathrm{Nil}(\overline{R})$. Therefore, there exists $k \in \mathbb{N}$ such that $(a^4 - a^2)^k \in I \subseteq \mathrm{Nil}(R)$.

(ii) Utilizing Lemma 2.9 and part (i) of the proof, the arguments are complete. □

Corollary 2.11. *Let I be an ideal of a ring R. Then, the following are equivalent:*

(i) *R/I is strongly NUS-nil clean.*

(ii) *R/I^n is strongly NUS-nil clean for all $n \in \mathbb{N}$.*

(iii) *R/I^n is strongly NUS-nil clean for some $n \in \mathbb{N}$.*

Proof. (i) $\Rightarrow$ (ii). For any $n \in \mathbb{N}$, we have $\frac{R/I^n}{I/I^n} \cong R/I$. Since I/I^n is a nil-ideal of R/I^n and R/I is strongly NUS-nil clean, then Proposition 2.10 gives that R/I^n is a strongly NUS-nil clean ring.

(ii) $\Rightarrow$ (iii). This implication is trivial.

(iii) $\Rightarrow$ (i). For any ideal I of R, we have $\frac{R/I^n}{I/I^n} \cong R/I$, so that, via Proposition 2.10, we can conclude that R/I is strongly NUS-nil clean, as stated. □

Proposition 2.12. *Let R be a ring. Then, the following are equivalent:*

(i) *R is strongly square-nil clean.*

(ii) *$T_n(R)$ is strongly NUS-nil clean for all $n \in \mathbb{N}$.*

(iii) *$T_n(R)$ is strongly NUS-nil clean for some $n \geq 2$.*

Proof. (ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Let $a \in R$. Then, it must be that

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{T}_2(R).$$

It is evident that A is non-invertible in $\mathrm{T}_2(R)$. By hypothesis, we can find a square idempotent $E = \begin{pmatrix} e_{11} & e_{12} \\ 0 & e_{22} \end{pmatrix}$ and a nilpotent $Q = \begin{pmatrix} q_{11} & q_{12} \\ 0 & q_{22} \end{pmatrix}$ such that $A = E + Q$ and $EQ = QE$. It now follows by plain inspection that $a = e_{11} + q_{11}$ and $e_{11}q_{11} = q_{11}e_{11}$. Hence, $e_{11}^4 = e_{11}^2$, and q_{11} is a nilpotent in R. Thus, a has strongly NUS-nil clean decomposition. Therefore, point (i) is valid.

(i) $\Rightarrow$ (ii). It suffices with Proposition 2.10 (ii) at hand to prove only that the quotient $\mathrm{T}_n(R)/J(\mathrm{T}_n(R))$ is strongly NUS-nil clean. To this target, observe that

$$\mathrm{T}_n(R)/J(\mathrm{T}_n(R)) \cong \prod_{i=1}^{n} R/J(R).$$

So, we need just to show that $\prod_{i=1}^{n} R/J(R)$ is strongly NUS-nil clean. In fact, Lemma 2.4 is a guarantor that $\prod_{i=1}^{n} R/J(R)$ is strongly NUS-nil clean if, and only if, $R/J(R)$ is strongly square-nil clean. It is now easy to see that, if R is strongly square-nil clean, then $R/J(R)$ is strongly-square nil clean, as asked for. □

Let R be a ring and M a bi-module over R. The *trivial extension* of R and M is stated as

$$\mathrm{T}(R, M) = \{(r, m) : r \in R \text{ and } m \in M\},$$

with addition defined component-wise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The next two consequences arrived quite naturally.

Corollary 2.13. *Let R be a ring and M a bi-module over R. Then, the following statements are equivalent:*

(i) *$T(R, M)$ is a strongly NUS-nil clean ring.*

(ii) *R is a strongly NUS-nil clean ring.*

Proof. Set $A \coloneqq \mathrm{T}(R, M)$ and consider $I \coloneqq \mathrm{T}(0, M)$. It is not too hard to verify that I is a nil-ideal of A such that $\frac{A}{I} \cong R$. So, the result follows directly from Proposition 2.10. □

Let α be an endomorphism of R and n a positive integer. Consider the *skew triangular matrix ring*

$$\mathrm{T}_n(R,\alpha) = \left\{ \left. \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \right| a_i \in R \right\}$$

with addition point-wise and multiplication given by:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ 0 & c_0 & c_1 & \cdots & c_{n-2} \\ 0 & 0 & c_0 & \cdots & c_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix},$$

where

$$c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_0), \quad 1 \le i \le n-1.$$

We, hereafter, denote the elements of $\mathrm{T}_n(R, \alpha)$ by $(a_0, a_1, \ldots, a_{n-1})$. If α is the identity endomorphism, then $\mathrm{T}_n(R, \alpha)$ is a subring of the upper triangular matrix ring $\mathrm{T}_n(R)$.

Corollary 2.14. *Let R be a ring. Then, the following statements are equivalent:*

(i) *$T_n(R, \alpha)$ is a strongly NUS-nil clean ring.*

(ii) *R is a strongly NUS-nil clean ring.*

Proof. Choose

$$I := \left\{ \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \middle| a_{ij} \in R \quad (i < j) \right\}.$$

Then, one easily checks that $I^n = (0)$ and $\frac{\mathrm{T}_n(R,\alpha)}{I} \cong R$. Consequently, Proposition 2.10 employs to get the desired result. □

Furthermore, one mentions that Wang *et al.* introduced in Ref. [10] the matrix ring $\mathrm{S}_{n,m}(R)$ as follows: supposing R is a ring, consider the matrix ring $\mathrm{S}_{n,m}(R) =$

$$\left\{ \begin{pmatrix} a & b_1 & \cdots & b_{n-1} & c_{1n} & \cdots & c_{1n+m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a & b_1 & c_{n-1,n} & \cdots & c_{n-1,n+m-1} \\ 0 & \cdots & 0 & a & d_1 & \cdots & d_{m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a & d_1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a \end{pmatrix} \in \mathrm{T}_{n+m-1}(R) : a, b_i, d_j, c_{i,j} \in R \right\}.$$

Likewise, let $\mathrm{T}_{n,m}(R)$ be

$$\left\{ \begin{pmatrix} \begin{matrix} a & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & a & b_1 & \cdots & b_{n-2} \\ 0 & 0 & a & \cdots & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} a & c_1 & c_2 & \cdots & c_{m-1} \\ 0 & a & c_1 & \cdots & c_{m-2} \\ 0 & 0 & a & \cdots & c_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{matrix} \end{pmatrix} \in \mathrm{T}_{n+m}(R) : a, b_i, c_j \in R \right\},$$

and let we state

$$\mathrm{U}_n(R) = \left\{ \begin{pmatrix} a & b_1 & b_2 & b_3 & b_4 & \cdots & b_{n-1} \\ 0 & a & c_1 & c_2 & c_3 & \cdots & c_{n-2} \\ 0 & 0 & a & b_1 & b_2 & \cdots & b_{n-3} \\ 0 & 0 & 0 & a & c_1 & \cdots & c_{n-4} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix} \in \mathrm{T}_n(R) : a, b_i, c_j \in R \right\}.$$

Thereby, we have the following.

Example 2.15. Let R be a ring. Then, the following statements are equivalent:

(i) $S_{n,m}(R)$ is a NUS-nil clean ring.

(ii) $T_{n,m}(R)$ is a NUS-nil clean ring.

(iii) $U_n(R)$ is a NUS-nil clean ring.

(iv) R is a NUS-nil clean ring.

Let α be an endomorphism of R. We denote by $R[x, \alpha]$ the *skew polynomial ring* whose elements are the polynomials over R; the addition is defined as usual, and the multiplication is defined by the equality $xr = \alpha(r)x$ for any $r \in R$. So, there is a ring isomorphism

$$\varphi : \frac{R[x,\alpha]}{\langle x^n \rangle} \to \mathrm{T}_n(R,\alpha),$$

given by

$$\varphi\left(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n \rangle\right) = (a_0, a_1, \ldots, a_{n-1})$$

with $a_i \in R$, $0 \leq i \leq n-1$. Thus, one deduces that $\mathrm{T}_n(R,\alpha) \cong \frac{R[x,\alpha]}{\langle x^n \rangle}$, where $\langle x^n \rangle$ is the ideal generated by x^n.

Besides, $R[[x, \alpha]]$ denotes the *ring of skew formal power series* over R; that is, all formal power series of x having coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. On the other hand, we know that the isomorphism $\frac{R[x,\alpha]}{\langle x^n \rangle} \cong \frac{R[[x,\alpha]]}{\langle x^n \rangle}$ is fulfilled.

We, thus, extract the following three consequences.

Corollary 2.16. Let R be a ring with an endomorphism α such that $\alpha(1) = 1$. Then, the following statements are equivalent:

(i) $\frac{R[x,\alpha]}{\langle x^n \rangle}$ *is a strongly NUS-nil clean ring.*

(ii) $\frac{R[[x,\alpha]]}{\langle x^n \rangle}$ *is a strongly NUS-nil clean ring.*

(iii) *R is a strongly NUS-nil clean ring.*

Corollary 2.17. Let R be a ring. Then, the following statements are equivalent:

(i) $\frac{R[x]}{\langle x^n \rangle}$ *is a strongly NUS-nil clean ring.*

(ii) $\frac{R[[x]]}{\langle x^n \rangle}$ *is a strongly NUS-nil clean ring.*

(iii) *R is a strongly NUS-nil clean ring.*

Corollary 2.18. Let R be a ring, and let

$S_n(R) := \left\{(a_{ij}) \in T_n(R) \;\middle|\; a_{11} = a_{22} = \cdots = a_{nn}\right\}$.

Then, the following statements are equivalent:

(i) $S_n(R)$ *is a strongly NUS-nil clean ring.*

(ii) *R is a strongly NUS-nil clean ring.*

Proof. Assuming $I := \{(a_{ij}) \in \mathrm{S}_n(R): a_{11} = 0\}$, it is rather evident that I is a nil-ideal of $\mathrm{S}_n(R)$ such that $\mathrm{S}_n(R)/I \cong R$ holds, whence Proposition 2.10 (i) works, as inspected. □

The following example demonstrates that the full matrix constructions are rather more complicated than we anticipate in comparison to the triangular matrix rings. Specifically, we exhibit a concrete construction of a strongly NUS-nil clean ring which is *not* strongly square-nil clean.

Example 2.19. Let $R = \mathrm{M}_2(\mathbb{Z}_2)$. It can be shown by a direct computations that

$$R = U(R) \cup Id(R) \cup \mathrm{Nil}(R)$$

Thus, R is a strongly NUS-nil clean ring, but it is *not* strongly square-nil clean. Indeed, by contrary, assume that $\mathrm{M}_2(\mathbb{Z}_2)$ is strongly square-nil clean. Then, we choose $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_2)$, and so there are $E^4 = E^2 \in \mathrm{M}_2(\mathbb{Z}_2)$ and $N \in \mathrm{Nil}(\mathrm{M}_2(\mathbb{Z}_2))$ such that $A = E + N$ and $EN = NE$. It follows now by a plain calculation that $A^4 - A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Nil}(\mathrm{M}_2(\mathbb{Z}_2))$, which is an obvious contradiction as this is exactly the identity matrix. Therefore, $\mathrm{M}_2(\mathbb{Z}_2)$ is really *not* strongly square-nil clean, as claimed.

The next assertion arises logically asking what happens for sizes greater than 2.

Proposition 2.20. For any ring $R \neq 0$ and any integer $n \geq 3$, the ring $\mathrm{M}_n(R)$ is not strongly NUS-nil clean.

Proof. It suffices to establish that $\mathrm{M}_3(R)$ is *not* a strongly NUS-nil clean ring having in mind Lemma 2.8. To this goal, consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin U(\mathrm{M}_3(R)).$$

Then, we have

$$A^4 - A^2 \notin \mathrm{Nil}(\mathrm{M}_3(R)).$$

Consequently, Theorem 2.6 is applicable to get that R cannot be a strongly NUS-nil clean ring, as asserted. $\square$

An immediate consequence is the following one.

Corollary 2.21. Let R be a strongly NUS-nil clean ring. Then, for any $n > 2$, there does not exist $0 \neq e \in Id(R)$ such that $eRe \cong \mathrm{M}_n(S)$ for some non-zero ring S.

Proof. Assume on the contrary that there exists $0 \neq e \in Id(R)$ such that $eRe \cong \mathrm{M}_n(S)$ for some non-zero ring S. Since R is strongly NUS-nil clean, it follows from Lemma 2.8 that eRe has to be strongly NUS-nil clean too, and so $\mathrm{M}_n(S)$ is also strongly NUS-nil clean, implying a contradiction with Proposition 2.20, as expected. $\square$

The following affirmation is somewhat surprising.

Lemma 2.22. Let $\mathrm{M}_2(R)$ be a strongly NUS-nil clean ring. Then, R is a strongly square-nil clean ring.

Proof. Let $a \in R$. Then, one finds that

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \notin U(\mathrm{M}_2(R)).$$

Thus, Lemma 2.6 works to get that $A^4 - A^2 \in \mathrm{Nil}(\mathrm{M}_2(R))$. So, $a^4 - a^2 \in \mathrm{Nil}(R)$ whence $a^3 - a \in \mathrm{Nil}(R)$. Therefore, bearing in mind [7, Theorem 2.12], one writes that $a = e + n$, where

$e^3 = e$, $n \in \mathrm{Nil}(R)$ and $en = ne$. As $e^4 = e^2$, e is a square-idempotent and so R is a strongly square-nil clean ring, as formulated. □

We now intend to examine some structural characterizations.

Lemma 2.23. If R is a local ring with nil $J(R)$, then R is strongly NUS-nil clean.

Proof. Let $a \in R$ and $a \notin U(R)$. Since R is local, $a \in J(R)$ and hence $a \in \mathrm{Nil}(R)$. So, a is a nilpotent element, and thus it is a strongly NUS-nil clean element, as required. □

We now can extract the following criterion.

Corollary 2.24. Let R be a ring with only trivial idempotents. Then, R is strongly NUS-nil clean if, and only if, R is a local ring with $J(R)$ nil.

Proof. Assume that R is a strongly NUS-nil clean ring, so $J(R)$ is nil in accordance with Lemma 2.9. If $a \notin U(R)$, then we have $a = q + u$, where $qu = uq$, $q \in \mathrm{Nil}(R)$ and either $u^2 = 1$ or $u^2 = 0$. Since a is not a unit, it must be that $a = u + q$, where $qu = uq$ and $u^2 = 0$ giving $a \in \mathrm{Nil}(R)$. Thus, exploiting [9, Proposition 19.3], R must be a local ring.

Oppositely, suppose R is a local ring with a nil Jacobson radical $J(R)$. So, for each $a \notin U(R)$, we have $a \in J(R) \subseteq \mathrm{Nil}(R)$, whence a is a nil clean element, as requested. □

Lemma 2.25. Suppose R is a strongly NUS-nil clean ring and $2 \notin U(R)$. Then, either $2 \in \mathrm{Nil}(R)$ or $6 \in \mathrm{Nil}(R)$.

Proof. If $2 \notin \mathrm{Nil}(R)$, then Theorem 2.6 can be applied to derive that $2^4 – 2^2 \in \mathrm{Nil}(R)$. Hence, we get $12 \in \mathrm{Nil}(R)$ and so $6 \in \mathrm{Nil}(R)$. □

Lemma 2.26. Let R be a ring and $2 \in J(R)$. Then, the following conditions are equivalent:

(i) *R is a strongly NUS-nil clean ring.*

(ii) *R is a GSNC ring.*

Proof. (ii) $\Rightarrow$ (i). It is straightforward.

(i) $\Rightarrow$ (ii). It is sufficient to prove only that $a^2 - a \in \mathrm{Nil}(R)$ for each $a \in R \setminus U(R)$. To that end, choose $a \in R \setminus U(R)$. Thus, Theorem 2.6 allows us to infer that $a^4 - a^2 \in \mathrm{Nil}(R)$, and so $a^2(1 - a^2) \in \mathrm{Nil}(R)$. But, as $2 \in \mathrm{Nil}(R)$, we get $a^2(1 - a^2 - 2a + 2a^2) \in \mathrm{Nil}(R)$ and, therefore, $a^2(1 - a)^2 \in \mathrm{Nil}(R)$. Thus, $a^2 - a \in \mathrm{Nil}(R)$, as needed. □

Lemma 2.27. Suppose R is a ring such that $2 \notin U(R)$. Then, the following items are equivalent:

(i) *R is a strongly NUS-nil clean ring.*

(ii) *Either R is a GSNC ring or R is a strongly square-nil clean ring.*

Proof. (ii) $\Rightarrow$ (i). This is routine.

(i) $\Rightarrow$ (ii). Applying Lemma 2.25, we have that either $2 \in \mathrm{Nil}(R)$ or $6 \in \mathrm{Nil}(R)$. If, for a moment, $2 \in \mathrm{Nil}(R)$, then R is a GSNC ring using Lemma 2.26. However, if $6 \in \mathrm{Nil}(R)$, we may decompose with the help of the Chinese Remainder Theorem that $R \cong R_1 \oplus R_2$, where $2 \in \mathrm{Nil}(R_1)$ and $3 \in \mathrm{Nil}(R_2)$. Now, Lemma 2.4 tells us that R_1 and R_2 are both strongly square-nil clean rings. Moreover, since $2 \in \mathrm{Nil}(R_1)$, the ring R_1 is even strongly nil clean. It thus follows at once that R is a strongly square-nil clean ring, as promised. □

The following necessary and sufficient condition is key providing us with a satisfactory close transversal between the two new notions of strong square-nil cleanness and strong NUS-nil cleanness as defined above.

Theorem 2.28. Let R be a ring. Then, the following issues are equivalent:

(i) *R is a strongly square-nil clean ring.*

(ii) *R is a strongly NUS-nil clean ring and, for every $u \in U(R)$, $u^2 = 1 + n$, where $n \in \mathrm{Nil}(R)$.*

Proof. (i) $\Rightarrow$ (ii). It is readily to see that R is strongly NUS-nil clean. Let $u \in U(R)$. Thus, $u = e + m$, where $e^2 = e^4$ and $m \in \mathrm{Nil}(R)$ with $em = me$. It follows that $u^2 = e^2 + m(2e + m)$ and $u^4 = e^4 + m(2e + m)(2e^2 + m(2e + m)) = e^2 + m(2e + m)(2e^2 + m(2e + m))$. Hence, $u^4 - u^2 = m(2e + m)(2e^2 + m(2e + m) - 1) \in \mathrm{Nil}(R)$. So, $1 - u^2 \in \mathrm{Nil}(R)$. Consequently, there exists $n \in \mathrm{Nil}(R)$ such that $u^2 = 1 + n$, as desired.

(ii) $\Rightarrow$ (i). Assume that $a \in R$. If $a \notin U(R)$, then $a = f + q$, where $f^2 = f^4$ and $q \in \mathrm{Nil}(R)$ with $qf = fq$ and we are done.

If, however, $a \in U(R)$, then $a^2 = 1 + n$, where $n \in \mathrm{Nil}(R)$. It follows that $a^3 = a^2 a = a + na$ and, similarly, $a^3 = aa^2 = a + an$. So, $an = na$ and hence $a - a^3 = -an \in \mathrm{Nil}(R)$. Now, we will distinguish two possible cases for the element $2 \in R$.

If, firstly, $2 \in U(R)$, then [7, Lemma 2.6] applies to get that there exists $\alpha(a) \in \mathbb{Z}[a]$ such that $\alpha(a)^3 = \alpha(a)$ and $a - \alpha(a) \in \mathrm{Nil}(R)$. So, there is $m \in \mathrm{Nil}(R)$ such that $a = \alpha(a) + m$. It is now easy to see that $\alpha(a)^4 = \alpha(a)^2$ and $m\alpha(a) = \alpha(a)m$, so that we are over.

However, if $2 \notin U(R)$, then we claim that $2 \in \mathrm{Nil}(R)$. Assuming on the contrary $2 \notin \mathrm{Nil}(R)$, then by point (ii) and using Theorem 2.6, we write 2^4–$2^2 \in \mathrm{Nil}(R)$. Consequently, $6 \in \mathrm{Nil}(R)$. Since, for every $a \in U(R)$, we know $a^3 - a \in \mathrm{Nil}(R)$, we perceive $(a^3 - a)^6 \in \mathrm{Nil}(R)$ and hence

$$a^{18} - 6a^{16} + 15a^{14} - 30a^{12} + 15a^{10} - 6a^8 + a^6 \in \mathrm{Nil}(R).$$

But, $6 \in \mathrm{Nil}(R)$ enables us that $a^{18} + 3a^{14} + 3a^{10} + a^6 \in \mathrm{Nil}(R)$, and since $a^4 - a^2 \in \mathrm{Nil}(R)$, we obtain that $a^6 - a^4 \in \mathrm{Nil}(R)$ so that $a^6 - a^2 \in \mathrm{Nil}(R)$. Now, from $a^8(a^6 - a^2) \in \mathrm{Nil}(R)$, we write $a^{14} - a^{10} \in \mathrm{Nil}(R)$ and thus $3a^{14} - 3a^{10} \in \mathrm{Nil}(R)$. But, as $a^{18} + 3a^{14} + 3a^{10} + a^6 \in \mathrm{Nil}(R)$, we have $a^{18} + 6a^{14} + a^6 \in \mathrm{Nil}(R)$, and since $6 \in \mathrm{Nil}(R)$, we deduce that $a^{18} + a^6 \in \mathrm{Nil}(R)$ leading to $a^{13} + a \in \mathrm{Nil}(R)$. Further, as $a^3 - a \in \mathrm{Nil}(R)$, we can see that $a^5 - a^3 \in \mathrm{Nil}(R)$ and so $a^5 - a^3 + a^3 - a \in \mathrm{Nil}(R)$. Utilizing analogous arguments, one can see that $a^{13} - a \in \mathrm{Nil}(R)$. Combining this and $a^{13} + a \in \mathrm{Nil}(R)$, we obtain $2a \in \mathrm{Nil}(R)$ for all $a \in U(R)$. It now follows that $2 \in \mathrm{Nil}(R)$, as claimed.

Next, we assert that $1 - a \notin U(R)$. Acting on contrary, we assume that $1 - a \in U(R)$ and then, by assumption, there is $m \in \mathrm{Nil}(R)$ such that $(1 - a)^2 = 1 + m$ and $m(1 - a) = (1 - a)m$. It, thus, follows that $1 - 2a + a^2 = 1 + m$. Since $a^2 = 1 + n$, we have that $1 - 2a + 1 + n = 1 + m$ and so $1 - 2a = m - n$. As $ma = am$ and $n = a^2 - 1$, we find $mn = nm$, thus showing that $m - n \in \mathrm{Nil}(R)$. So, $1 - 2a \in \mathrm{Nil}(R)$, and since $2 \in \mathrm{Nil}(R)$, we detect $1 \in \mathrm{Nil}(R)$, being an absurd. Therefore, $1 - a \notin U(R)$, as asserted, and, via (ii), we inspect that

$$(1 - a)\left[(1 - a)^3 - (1 - a)\right] \in \mathrm{Nil}(R).$$

It, thereby, follows that

$$(1 - a)\left(1 - 3a + 3a^2 - a^3 - 1 + a\right) = (1 - a)\left(-2a + 3a^2 - a^3\right) \in \mathrm{Nil}(R).$$

Furthermore, as $a^3 - a \in \mathrm{Nil}(R)$, we can write $(1 - a)(a^3 - a) \in \mathrm{Nil}(R)$. Consequently, $(1 - a)(-3a + 3a^2) \in \mathrm{Nil}(R)$. But, since $2 \in \mathrm{Nil}(R)$, we arrive at $(1 - a)(2a - 2a^2) \in \mathrm{Nil}(R)$. It, therefore, follows that

$$(1 - a)\left(-a + a^2\right) = (1 - a)\left(-3a + 3a^2\right) + (1 - a)\left(2a - 2a^2\right) \in \mathrm{Nil}(R).$$

Thus, $a(1 - a) = (1 - a)a \in \mathrm{Nil}(R)$ for all $a \in U(R)$ meaning that $1 - a \in \mathrm{Nil}(R)$ with $a \in 1 + \mathrm{Nil}(R)$, because obviously a commutes with the existing element from $\mathrm{Nil}(R)$.

Finally, in either case, the element $a \in U(R)$ has a strongly square-nil clean decomposition, as wanted. □

The next comments are well-positioned.

Remark 2.29. All rings R having the property considered in the last lemma, namely that $u^2 \in 1 + \mathrm{Nil}(R)\ \forall\ u \in U(R)$, are firstly introduced in Ref. [11] under name *2-UU rings*. Moreover, the given above equality $u^2 = 1 + n$ does *not* automatically allow us to conclude that the element $u = u^{-1} + u^{-1}n$ possesses a strong square-nil clean decomposition, because the identity $(u^{-1})^2 = (u^{-1})^4$ being equivalent to $u^2 = 1$ leads to $n = 0$.

We are now prepared to prove our principal result that sounds somewhat curious and is a partial converse of Lemma 2.22.

Theorem 2.30. Let R be simultaneously a Noetherian local ring and a strongly square-nil clean ring. Then, $\mathrm{M}_2(R)$ is a strongly NUS-nil clean ring.

Proof. Firstly, we show that $6 \in \mathrm{Nil}(R)$. If, for a moment, $2 \notin U(R)$, then Lemma 2.25 employs to get either $2 \in \mathrm{Nil}(R)$ or $6 \in \mathrm{Nil}(R)$. So, in this case, we have $6 \in \mathrm{Nil}(R)$, because $2 \in \mathrm{Nil}(R)$ would imply that $6 \in \mathrm{Nil}(R)$.

If $2 \in U(R)$, then, by hypothesis, there exist $e^2 = e^4 \in R$ and $n \in \mathrm{Nil}(R)$ such that $2 = e + n$ and $en = ne$. So,

$$2^3 - 2 = e^3 - e + n\left(3e^2 + 3en + n^2 - 1\right) \in \mathrm{Nil}(R).$$

Therefore, $6 \in \mathrm{Nil}(R)$ in either case.

Thus, one can decompose with the aid of the Chinese Remainder Theorem that $R \simeq R_1 \times R_2$, where $2 \in \mathrm{Nil}(R_1)$ and $3 \in \mathrm{Nil}(R_2)$. But, since R is local, we have either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$. So, $\mathrm{M}_2(R/J(R))$ is a strongly NUS-nil clean ring. However, we know that

$$\mathrm{M}_2(R/J(R)) \cong \mathrm{M}_2(R)/\mathrm{M}_2(J(R))$$

as well as that $J(R)$ is a nilpotent ideal of R, because R is Noetherian. Consequently, one knows that $\mathrm{M}_2(J(R))$ is a nilpotent ideal of $\mathrm{M}_2(R)$. Finally, Proposition 2.10 leads to the fact that $\mathrm{M}_2(R)$ is a strongly NUS-nil clean ring, as asked. □

The next assertion also sounds somewhat surprisingly.

Lemma 2.31. Let R be a strongly NUS-nil clean ring with $2 \in U(R)$ and, for any $u \in U(R)$, we have $u^2 = 1$. Then, R is a commutative ring.

Proof. For all $u, v \in U(R)$, we have $u^2 = v^2 = (uv)^2 = 1$. Therefore, $uv = (uv)^{-1} = v^{-1}u^{-1} = vu$. Hence, the invertible elements commute with each other.

Now, we will illustrate that R is abelian. In fact, for each idempotent e in R and $a \in R$, we know $2e - 1 \in U(R)$ and $(1 + ea(1 - e)) \in U(R)$. Since, by what we have shown above, the invertible elements commute with each other, we have $2ea(1 - e) = 0$. Since $2 \in U(R)$, we get $ea(1 - e) = 0$, which means $ea = eae$.

On the other hand, since $2e - 1 \in U(R)$ and $(1 + (1 - e)ae) \in U(R)$, we obtain $2(1 - e)ae = 0$. So, $ae = eae$. Therefore, R is abelian, as claimed.

On the other side, since $1 + \mathrm{Nil}(R) \subseteq U(R)$ and all invertible elements commute with each other, the nilpotent elements also commute with each other.

On the other hand, since $2 \in U(R)$ and, for every $u \in U(R)$, $u^2 = 1$, we receive $2^2 = 1$. It now follows that $3 \in \mathrm{Nil}(R)$. Next, we demonstrate that $a^3 - a \in \mathrm{Nil}(R)$ for any $a \in R$. To that purpose, choose $a \in R$. If $a \in U(R)$, then $a^2 = 1$ and so $a^3 - a = 0$.

If, however, $a \notin U(R)$, then Theorem 2.6 informs us that $a^4 - a^2 \in \mathrm{Nil}(R)$. So, $a^3 - a \in \mathrm{Nil}(R)$.

Now, given $x, y \in R$. Consulting with [7, Proposition 2.8], one writes that $x = e_1 + e_2 + m$ and $y = f_1 + f_2 + n$, where $e_1, e_2, f_1, f_2 \in Id(R)$ and $m, n \in \mathrm{Nil}(R)$. As all idempotents are known by the established above to be central in R, and nilpotent elements are commuting with each other, we finally conclude that $xy = yx$. Thus, R is a commutative ring, indeed. □

Let A, B be two rings and let M, N be (A, B)-bi-module and (B, A)-bi-module, respectively. Moreover, we consider the bilinear maps $\phi: M \otimes_B N \to A$ and $\psi: N \otimes_A M \to B$ that apply to the following properties:

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now, the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring with obvious matrix operations that is called a *Morita context ring*. Designate the two-sided ideals Imϕ and Imψ to MN and NM, respectively, that are called the *trace ideals* of the Morita context.

We now have at our disposal all the instruments necessary to prove the following criterion.

Proposition 2.32. Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring such that MN and NM are nilpotent ideals of A and B, respectively. Then, R is a strongly NUS-nil clean ring if, and only if, both A and B are strongly square-nil clean rings.

Proof. Since, MN and NM are nilpotent ideals of A and B, respectively, one can says that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Therefore, adapting [12], we argue that

$$J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$$

and hence the isomorphism

$$\frac{R}{J(R)} \cong \frac{A}{J(A)} \times \frac{B}{J(B)}$$

is fulfilled. Notice that R is a strongly NUS-nil clean ring if, and only if, so is the factor-ring $R/J(R)$.

Furthermore, thanking to Lemma 2.4, the quotient $R/J(R)$ is strongly NUS-nil clean ring if, and only if, $\frac{A}{J(A)}$ and $\frac{B}{J(B)}$ are both strongly square-nil clean rings.

Now, if R is strongly NUS-nil clean, then by what we have established so far $J(R)$ is a nil-ideal and so $J(A)$ and $J(B)$ are nil as well. Observe also that Proposition 2.10 can be applied to get that both A and B are strongly square-nil clean rings, as stated.

Conversely, if A and B are strongly square-nil clean rings, then both $J(A)$ and $J(B)$ are nil, and besides $\frac{A}{J(A)}$ and $\frac{B}{J(B)}$ are strongly square-nil clean rings yielding that $R/J(R)$ is a strongly NUS-nil clean ring. But, as $J(A)$ and $J(B)$ are nil, we elementarily deduce that $J(R)$ is a nil-ideal of R. It now follows that R is a strongly NUS-nil clean ring, as formulated. □

Next, let R, S be two rings, and let M be an (R, S)-bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-module M, we consider the *formal triangular matrix ring*

$$\mathrm{T}(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it obviously forms a ring with the usual matrix operations. Regarding Proposition 2.32, if we set $N = \{0\}$, then we will obtain the following immediate consequence.

Corollary 2.33. Let R, S be rings and let M be an (R, S)-bi-module. Then, $\mathrm{T}(R, S, M)$ is strongly NUS-nil clean ring if, and only if, both R, S are strongly square-nil clean rings.

We now deal with group ring extensions of the NUS-nil clean property as follows: let R be a ring, G a group, and we traditionally denote by RG the group ring of G over R.

In this section, we intend to establish a suitable criterion for a group ring to be strongly NUS-nil clean under some sensible circumstances on the former group and ring objects. Concretely, we are able to achieve this, provided the whole group is locally finite. Recall that a group G is a p-group if every element of G has order which is a power of the prime number p. Also, we recollect that a group is locally finite if each its finitely generated subgroup is finite.

Suppose now that G is an arbitrary group and R is an arbitrary ring. Standardly, in RG there is a homomorphism $\varepsilon: RG \to R$, defined by

$$\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g,$$

which is called the *augmentation map* of RG and its kernel, denoted by $\Delta(RG)$, is called the *augmentation ideal* of RG.

We now proceed by showing the following assertions.

Lemma 2.34. *If RG is a strongly NUS-nil clean ring, then so is R.*

Proof. We know that $RG/\Delta(RG) \cong R$. Thus, it follows at once that R must be a strongly NUS-nil clean ring as being an epimorphic image of RG. □

Proposition 2.35. *Let R be a strongly NUS-nil clean ring with $p \in \mathrm{Nil}(R)$, and let G be a locally finite p-group, where p is a prime. Then, the group ring RG is a strongly NUS-nil clean ring.*

Proof. Knowing [13, Proposition 16], we see that $\Delta(RG)$ is a nil-ideal. Thus, since $RG/\Delta(RG) \cong R$, referring to Proposition 2.10 we infer that RG is a strongly NUS-nil clean ring. □

We also record the following interesting fact.

Lemma 2.36. *[14, Lemma 2]. Let p be a prime number with $p \in J(R)$. If G is a locally finite p-group, then $\Delta(RG) \subseteq J(RG)$.*

Having in hand the preceding three statements, we now have at our disposal all the machinery needed to establish the following major assertion.

Theorem 2.37. *Let R be a ring and let G be a locally finite p-group, where p is a prime number with $p \in J(R)$. Then, RG is a strongly NUS-nil clean ring if, and only if, R is a strongly NUS-nil clean ring and $\Delta(RG)$ is a nil-ideal of RG.*

We finish off our research work with the following claim.

Lemma 2.38. *Let R be a strongly NUS-nil clean ring, and let G be a group such that $\Delta(RG) \subseteq J(RG)$. Then, $RG/J(RG)$ is a strongly NUS-nil clean ring.*

Proof. Seeing that $RG = \Delta(RG) + R$, because $\Delta(RG) \subseteq J(RG)$, we write $RG = J(RG) + R$. Therefore, the isomorphism

$$R/(J(RG) \cap R) \cong (J(RG) + R)/J(RG) = RG/J(RG)$$

is true, and since R is strongly NUS-nil clean, one finds that $R/(J(RG) \cap R)$ is strongly NUS-nil clean too. Consequently, we conclude that $RG/J(RG)$ is a strongly NUS-nil clean ring, as requested.

Data availability

No data was used for the research described in the article.

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