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On the Roman domination number of the complement of sum annihilating ideal graph of a reduced ring

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Abstract

Purpose – This paper aims to determine the Roman domination number of the complement of sum annihilating ideal graph of a reduced ring with the assumption that its domination number is finite. We have studied in a previous work the domination number of the complement of sum annihilating ideal graph of a commutative ring. As the Roman domination number is of historical importance, we wish to find out the Roman domination number of the mentioned graph in this paper.

Design/methodology/approach – We use techniques and methods from commutative ring theory regarding the minimal prime ideals. We use the graph-theoretic properties of the sum annihilating ideal graph of a commutative ring and its complement.

Findings – We have noted that if the number of minimal prime ideals of a reduced ring R that is not an integral domain is finite, then the domination number of the complement of sum annihilating ideal graph of a reduced ring is finite and in such a case, if the number of minimal prime ideals equals 2, then the Roman domination number of this graph is either 2 or 4 and we are able to characterize R such that the Roman domination number of this graph equals 2 (respectively, 4). If the number of minimal prime ideals of R equals 3, then it is determined that the Roman domination number of this graph is either 4 or 5. If the number of minimal prime ideals of R equals n and if n exceeds 4, then the Roman domination number of this graph is 2n or 2n −1. We are able to characterize R, according to the Roman domination number of the considered graph.

Research limitations/implications – We do not know any necessary and sufficient condition such that the domination number of the complement of sum annihilating ideal graph of a reduced ring is finite. If the number of minimal prime ideals of a reduced ring is finite, then the domination number of the considered graph is finite and in such cases, we have computed the Roman domination number of the considered graph. We have not considered non-reduced rings. The study helps to understand the structure of reduced rings with the desired Roman domination number of the considered graph.

Practical implications – The problem discussed in this work helps that there is an interplay between the graph parameter Roman domination number of a graph and the algebraic structure for which this graph parameter is considered.

Social implications – This paper will help researchers working in Algebra to understand algebraic structures by associating suitable graphs with algebraic structures and study the graph parameter Roman domination number. Researchers working in other fields of Mathematics can also try to undertake such an investigation.

Originality/value – The results mentioned in this paper are original. We first studied some known results proved by Cockayne *et al.* on the Roman domination number of a graph and i also studied similar work on comaximal ideal graphs of rings and determined the Roman domination number of the complement of sum annihilating ideal graph of a reduced ring. The work undertaken in this paper was not considered earlier by others.

Keywords Complement of sum annihilating ideal graph, Minimal prime ideal, Reduced ring, Domination number, Roman domination number

Paper type Research article

1. Introduction

The rings considered in this paper are commutative with identity that admit at least one nonzero zero-divisor. Let R be a ring. Let $Z(R)$ denote the set of all zero-divisors of R, and let us

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denote $Z(R)\setminus\{0\}$ by $Z(R)^*$. Motivated by the work of Beck [1], several researchers have introduced graphs with algebraic structures and studied the interplay between the algebraic properties of the algebraic structures and the graph-theoretic properties of the graphs associated with them. The graphs considered in this article are undirected and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. In this paper, we allow graphs to admit an infinite number of vertices. Recall that the *zero-divisor graph* of R, denoted by $\Gamma(R)$, is an undirected graph with $V(\Gamma(R)) = Z(R)^*$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$ [2]. For an excellent and inspiring survey on the zero-divisor graphs of commutative rings, one can refer to Ref. [3].

With any commutative ring R, Anderson and Badawi [4] have introduced and investigated an undirected graph called the *total graph* of R, denoted by $T(\Gamma(R))$ with $V(T(\Gamma(R))) = R$ and distinct vertices x and y are adjacent in $T(\Gamma(R))$ if and only if $x + y \in Z(R)$. For several interesting theorems which illustrate the interplay between the graph-theoretic properties of $T(\Gamma(R))$ and the ring-theoretic properties of R, see Ref. [4]. For an excellent, interesting, and inspiring book on graphs associated with commutative rings, one can refer to Ref. [5]. Anderson *et al.* [5] have characterized commutative rings according to the properties of graphs constructed over them. They have investigated in detail several graphs such as zero-divisor graphs, generalized zero-divisor graphs, total graphs, generalized total graphs, annihilator graphs, dot product graphs, and more.

Recall that an ideal I of a ring R is said to be an *annihilating ideal* of R if there exists $r \in R\setminus\{0\}$ such that $Ir = (0)$ [6]. We denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\setminus\{(0)\}$ by $\mathbb{A}(R)^*$. Recall that the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is an undirected graph with $V(\mathbb{AG}(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $\mathbb{AG}(R)$ if and only if $IJ = (0)$ [6]. For several interesting and inspiring results on $\mathbb{AG}(R)$, the reader is referred to Refs. [6, 7].

Motivated by the research work of Anderson and Badawi [4] on the total graph of a commutative ring and the research work of Behboodi and Rakeei [6, 7] on the annihilating-ideal graph of a commutative ring, with R, we have introduced and investigated an undirected graph, denoted by $\Omega(R)$ such that $V(\Omega(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $\Omega(R)$ if and only if $I + J \in \mathbb{A}(R)$ [8]. The graph $\Omega(R)$ was also investigated in Ref. [9], and the authors of [9] called $\Omega(R)$, the sum annihilating ideal graph of R. Let $G = (V, E)$ be a simple graph. Recall that the *complement* of G, denoted by G^c, is a graph with vertex set V and distinct vertices u and v are adjacent in G^c if and only if they are not adjacent in G [10, Definition 1.2.13]. We [11] studied the interplay between the graph-theoretic properties of $(\Omega(R))^c$ and the ring-theoretic properties of R.

Let R be a ring (R can be an integral domain). We denote the set of all prime ideals of R by $Spec(R)$, the set of all maximal ideals of R by $Max(R)$, and the set of all minimal prime ideals of R by $Min(R)$. For a subset E of R, we denote the set $\{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supseteq E\}$ by $V(E)$. For any subset A of R, we denote the set $A\setminus\{0\}$ by A^*. For an element $x \in R$, the *annihilator of x in R*, denoted by $Ann_R(x)$ or $Ann(x)$, is defined as $Ann(x) = \{r \in R \mid rx = 0\}$. We denote the set of all proper ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\setminus\{(0)\}$ by $\mathbb{I}(R)^*$. We denote the nilradical of R by $Nil(R)$. We say that R is *reduced* if $Nil(R) = (0)$. If A is a proper subset of a set B, then we denote it by $A \subset B$. For any $n \in \mathbb{N}\setminus\{1\}$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. For definitions and standard results from commutative ring theory, one can refer to Refs. [12–14].

Let $G = (V, E)$ be a graph. Let us recall the following definitions. A set $S \subseteq V$ is called a *dominating set* of G if every vertex $u \in V\setminus S$ has a neighbor $v \in S$ [10, Definition 10.2.1]. A *γ-set* of G is a minimum dominating set of G; that is, a dominating set of G whose cardinality is minimum [10, Definition 10.2.2]. The *domination number* of G is the cardinalty of a minimum dominating set of G and it is denoted by $\gamma(G)$ [10, Definition 10.2.3]. The domination number of graphs associated with commutative rings has been studied by several authors, for example, see Refs. [15–20].

The graph-theoretic properties of Roman domination number of a graph $G = (V, E)$ was investigated by Cockayne *et al.*, see Ref. [21]. For a clear motivation for the concept of Roman domination number of a graph, see [21, p. 12], where Cockayne *et al.* mentioned that the definition of a Roman dominating function was given implicitly in Refs. [22, 23]. Recall that a *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ [21, p. 12]. Let $i \in \{0, 1, 2\}$. Let us denote $\{v \in V | f(v) = i\}$ by V_i. Observe that $V = \bigcup_{i=0}^{2} V_i$ with $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{0, 1, 2\}$. We denote f by $f = (V_0, V_1, V_2)$. The domination number of $\Omega(R)$ (respectively, $(\Omega(R))^c$) was determined in Ref. [19], where we had proved that $\gamma(\Omega(R)) < \infty$, but there are several reduced rings such that $(\Omega(R))^c$ does not admit any finite dominating set, and if $|Min(R)| < \infty$, then $\gamma((\Omega(R))^c) < \infty$. If R is not reduced and $Z(R)$ is an ideal of R, then we were able to characterize R such that $\gamma((\Omega(R))^c) < \infty$. However, if R is not reduced and $Z(R)$ is not an ideal of R, the problem of characterizing R such that $\gamma((\Omega(R))^c) < \infty$ was left open, see [19, Section 3]. As R can have an infinite number of nonzero annihilating ideals and the vertex set of $(\Omega(R))^c$ is $\mathbb{A}(R)^*$, in this paper, we consider graphs $G = (V, E)$ which admit at least one Roman dominating function $f = (V_0, V_1, V_2)$ such that both V_1 and V_2 are finite. Note that a graph $G = (V, E)$ admits a Roman dominating function $f = (V_0, V_1, V_2)$ such that both V_1 and V_2 are finite if and only if $\gamma(G) < \infty$. Hence in this paper, we restrict ourselves to reduced rings R such that $\gamma((\Omega(R))^c) < \infty$ and determine $\gamma_{\mathcal{R}}((\Omega(R))^c)$. We use RDF to denote Roman dominating function. We denote the cardinality of a set A by $|A|$.

Let $G = (V, E)$ be a graph such that $\gamma(G) < \infty$. Let $f = (V_0, V_1, V_2)$ be an RDF on G such that both V_1 and V_2 are finite. Then, the *weight* of f is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G, denoted by $\gamma_{\mathcal{R}}(G)$. We say that an RDF $f = (V_0, V_1, V_2)$ on G is a $\gamma_{\mathcal{R}}$-function if $f(V) = \gamma_{\mathcal{R}}(G)$, see [21, p. 13].

Throughout this paper, we use R to denote a ring. Unless otherwise specified, we assume that R is reduced, $\mathbb{A}(R)^* \neq \emptyset$ and $\gamma((\Omega(R))^c) < \infty$. This paper aims to compute $\gamma_{\mathcal{R}}((\Omega(R))^c)$. This paper consists of three sections including the introduction. In Section 2, we state and prove some preliminary results that are used in proving the main results of this paper. In Section 3, we try to compute $\gamma_{\mathcal{R}}((\Omega(R))^c)$. As $|Min(R)| < \infty$ is a sufficient condition for the domination number of $(\Omega(R))^c$ to be finite, see [19, Section 3], we assume that $|Min(R)| < \infty$, and try to determine $\gamma_{\mathcal{R}}((\Omega(R))^c)$. If $|Min(R)| = 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) \in \{2, 3, 4\}$ (Proposition 3.2); $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ if and only if R is ring-isomorphic to $R_1 \times R_2$, where R_i is an integral domain for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$ (Proposition 3.3); $\gamma_{\mathcal{R}}((\Omega(R))^c) \neq 3$ (Proposition 3.4); $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ if and only if $Min(R) \cap Max(R) = \emptyset$ (Remark 3.5). Let $|Min(R)| = n$ for some $n \in \mathbb{N}$ with $n \geq 3$. If $Min(R) \cap Max(R) = \emptyset$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$. If $n = 3$, and if $|Min(R) \cap Max(R)| \geq 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ and R is ring-isomorphic to $D \times F_1 \times F_2$, where D is an integral domain (D can be a field) and F_i is a field for each $i \in \{1, 2\}$. If $n = 3$, and if $|Min(R) \cap Max(R)| = 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$, and R is ring-isomorphic to $T \times F$, where T is a reduced ring with $|Min(T)| = 2$, $Min(T) \cap Max(T) = \emptyset$, and F is a field. If $n \geq 4$, and $|Min(R) \cap Max(R)| \geq 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n - 1$, and R is ring-isomorphic to $T \times F$, where T is a reduced ring with $|Min(T)| = n - 1$, $Min(T) \cap Max(T)$ can be nonempty, and F is a field (Theorem 3.22). Examples 3.7 and 3.23 illustrate the results proved in this section.

2. Some preliminary results

In this paper, we consider graphs $G = (V, E)$ with no restriction on $|V|$ and such that they admit at least one Roman dominating function $f = (V_0, V_1, V_2)$ with both V_1 and V_2 are finite. The graph G satisfies the property that it admits at least one Roman dominating function

$f=(V_0,V_1,V_2)$ with both V_1 and V_2 are finite if and only if $\gamma(G)<\infty$. In this section, we state and prove some preliminary results that are used for proving the main results of this paper.

Remark 2.1. Assume that $G=(V,E)$ is a graph such that $\gamma(G)<\infty$. Let $f=(V_0,V_1,V_2)$ be a $\gamma_{\mathcal{R}}$-function on G. Then, $V_1\cup V_2$ is a dominating set of G. So, $\gamma(G)\le|V_1|+|V_2|\le|V_1|+2|V_2|=f(V)=\gamma_{\mathcal{R}}(G)$ [21, Proposition 1].

Remark 2.2. Let $G=(V,E)$ be a graph such that $\gamma(G)<\infty$. Let $S\subseteq V$ be such that $\gamma(G)=|S|$. Then, $f=(V\setminus S,\varnothing,S)$ is an RDF on G. As $f(V)=2|S|=2\gamma(G)$, it follows that $\gamma_{\mathcal{R}}(G)\le 2|S|=2\gamma(G)$ [21, Proposition 1].

Remark 2.3. If G has at least one edge, then $\gamma_{\mathcal{R}}(G)\ge\gamma(G)+1$ by [21, Proposition 2]. For completeness and easy reference, we include a proof. If $\gamma_{\mathcal{R}}(G)=\gamma(G)$, then there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that the weight of f equals $\gamma(G)$. Thus, $\gamma(G)=\gamma_{\mathcal{R}}(G)=|V_1|+2|V_2|$. By Remark 2.1, $\gamma(G)\le|V_1|+|V_2|\le|V_1|+2|V_2|$. Therefore, $|V_1|+|V_2|=|V_1|+2|V_2|$. Hence, $V_2=\varnothing$, so $V_0=\varnothing$. Therefore, $V(G)=V_1$ is a minimum dominating set of G. So, G has no edges, a contradiction to the assumption that G has at least one edge. Therefore, $\gamma_{\mathcal{R}}(G)\ge\gamma(G)+1$.

Remark 2.4. Let $G=(V,E)$ be a complete bipartite graph with vertex partition $V=A_1\cup A_2$. Then, $\gamma_{\mathcal{R}}(G)=4$ if $|A_i|\ge 3$ for each $i\in\{1,2\}$, $\gamma_{\mathcal{R}}(G)=3$ if $|A_i|\ge 2$ for each $i\in\{1,2\}$ and $|A_j|=2$ for some $j\in\{1,2\}$, and $\gamma_{\mathcal{R}}(G)=2$ if $|A_i|=1$ for some $i\in\{1,2\}$ [21, Proposition 8].

Lemma 2.5. Let $G=(V,E)$ be a graph such that $|V|\ge 3$. If $\gamma(G)=2$, then $3\le\gamma_{\mathcal{R}}(G)\le 4$.

Proof. Assume that $\gamma(G)=2$. We obtain from Remark 2.1 that $2\le\gamma_{\mathcal{R}}(G)$. By Remark 2.2, $\gamma_{\mathcal{R}}(G)\le 4$. Let $D=\{v_1,v_2\}$ be a dominating set of G. As $|V|\ge 3$ by assumption, there exists $v\in V\setminus D$. Hence, v and v_i are adjacent in G for some $i\in\{1,2\}$. Thus, G admits at least one edge. So, $\gamma_{\mathcal{R}}(G)\ge\gamma(G)+1=3$ by Remark 2.3. Therefore, $3\le\gamma_{\mathcal{R}}(G)\le 4$. □

Lemma 2.6. Let $G=(V,E)$ be a graph such that $|V|\ge 3$. If $\gamma(G)=2$, then $\gamma_{\mathcal{R}}(G)=3$ if and only if there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that either $V_2=V_0=\varnothing$, and $|V_1|=3$ or $|V_1|=|V_2|=1$, and $V_0=V\setminus(V_1\cup V_2)$.

Proof. Assume that $|V|\ge 3$ and $\gamma(G)=2$. By Lemma 2.5, $3\le\gamma_{\mathcal{R}}(G)\le 4$. Note that $\gamma_{\mathcal{R}}(G)=3$ if and only if there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that the weight of f equals 3, if and only if $|V_1|+2|V_2|=3$, if and only if either $V_2=\varnothing=V_0$, and $|V_1|=3$ or $|V_1|=|V_2|=1$, and $V_0=V\setminus(V_1\cup V_2)$. □

For a graph G, we denote the degree of a vertex v in G by $\deg(v)$.

Corollary 2.7. Let $G=(V,E)$ be a graph such that $|V|\ge 4$. If $\gamma(G)=2$, then the following statements are equivalent:

(1) $\gamma_{\mathcal{R}}(G)=3$.

(2) There exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that $|V_1|=|V_2|=1$, and $V_0=V\setminus(V_1\cup V_2)$.

(3) There exists $x\in V$ such that $\deg(x)=1$ in G^c.

Proof. By hypothesis, $|V|\ge 4$ and $\gamma(G)=2$.

$(1)\Leftrightarrow(2)$ If there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that $V_2=V_0=\varnothing$, and $|V_1|=3$, then $|V|=3$, since $V=\bigcup_{i=0}^{2}V_i$, contradicting the assumption that $|V|\ge 4$. Therefore, we obtain from Lemma 2.6 that $\gamma_{\mathcal{R}}(G)=3$ if and only if there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that $|V_1|=|V_2|=1$, and $V_0=V\setminus(V_1\cup V_2)$.

$(2)\Rightarrow(3)$ Assume that there exists a $\gamma_{\mathcal{R}}$-function $f=(V_0,V_1,V_2)$ on G such that $|V_1|=|V_2|=1$, and $V_0=V\setminus(V_1\cup V_2)$. Let $V_2=\{x\}$, and let $V_1=\{y\}$. As y and x are not adjacent in G by [21, Proposition 3(b)], it follows that y and x are adjacent in G^c. Since f is a $\gamma_{\mathcal{R}}$-function on G, each element of V_0 is adjacent to x in G. Therefore, $\deg(x)=1$ in G^c.

$(3) \Rightarrow (1)$ Assume that there exists $x \in V$ such that $\deg(x) = 1$ in G^c. Hence, there exists $y \in V$ such that y and x are adjacent in G^c, but z and x are adjacent in G for all $z \in V \setminus \{x, y\}$. Let $V_1 = \{y\}$, $V_2 = \{x\}$, and let $V_0 = V \setminus (V_1 \cup V_2)$. Then, $f = (V_0, V_1, V_2)$ is an RDF on G and the weight of f equals $|V_1| + 2|V_2| = 3$. Therefore, $\gamma_{\mathcal{R}}(G) \leq 3$. As $\gamma_{\mathcal{R}}(G) \geq 3$ by Lemma 2.5, we obtain that $\gamma_{\mathcal{R}}(G) = 3$. ☐

Lemma 2.8. Let R be a reduced ring. If $I \in \mathbb{A}(R)^*$, then $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in Min(R)$.

Proof. By hypothesis, R is a reduced ring. We obtain from [12, Proposition 1.8] and [14, Theorem 10] that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. Let $I \in \mathbb{A}(R)^*$. Then, $Ir = (0)$ for some $r \in R \setminus \{0\}$. As $r \neq 0$, we obtain that $r \notin \mathfrak{p}$ for some $\mathfrak{p} \in Min(R)$. From $Ir = (0) \subset \mathfrak{p}$, it follows that $I \subseteq \mathfrak{p}$. ☐

Lemma 2.9. Let R be reduced with $|Min(R)| = n$ for some $n \in \mathbb{N} \setminus \{1\}$. Then, $\mathfrak{p} \in \mathbb{A}(R)^*$ for each $\mathfrak{p} \in Min(R)$, and for any $\mathfrak{p} \in Min(R)$, there exists $x \in R^*$ (x depends on $\mathfrak{p}$) such that $\mathfrak{p} = Ann(x)$.

Proof. By hypothesis, R is a reduced ring with $|Min(R)| = n$ for some $n \in \mathbb{N} \setminus \{1\}$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Then, $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$ and $Z(R) = \bigcup_{i=1}^n \mathfrak{p}_i$.

Let $i \in \{1, 2, \ldots, n\}$. Let us denote $\{1, 2, \ldots, n\} \setminus \{i\}$ by A_i. Since distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\mathfrak{p}_i \not\supseteq \bigcap_{j \in A_i} \mathfrak{p}_j$. Hence, we can find $x_i \in (\bigcap_{j \in A_i} \mathfrak{p}_j) \setminus \mathfrak{p}_i$. Observe that $x_i \neq 0$ and $\mathfrak{p}_i x_i = (0)$. Thus, $\mathfrak{p}_i \in \mathbb{A}(R)$. As $\mathfrak{p}_i \neq (0)$, we get that $\mathfrak{p}_i \in \mathbb{A}(R)^*$. From $\mathfrak{p}_i x_i = (0)$, it follows that $\mathfrak{p}_i \subseteq Ann(x_i)$. If $r \in Ann(x_i)$, then $rx_i = 0$. From $x_i \notin \mathfrak{p}_i$, we obtain that that $r \in \mathfrak{p}_i$. Therefore, $Ann(x_i) \subseteq \mathfrak{p}_i$. So, $\mathfrak{p}_i = Ann(x_i)$. ☐

Lemma 2.10. Let R, n be as in the statement of Lemma 2.9. If W is a nonempty proper subset of $Min(R)$ such that $|\{I \in \mathbb{A}(R)^* \mid V(I) \cap Min(R) = W\}| < \infty$, then $\mathfrak{q} \in Max(R)$ for each $\mathfrak{q} \in Min(R) \setminus W$.

Proof. By hypothesis, R is a reduced ring with $|Min(R)| = n$ for some $n \in \mathbb{N} \setminus \{1\}$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Let W be a nonempty proper subset of $Min(R)$ such that $|\{I \in \mathbb{A}(R)^* \mid V(I) \cap Min(R) = W\}| < \infty$. Let $|W| = t$. Then, $1 \leq t < n$. Without loss of generality, we can assume that $W = \{\mathfrak{p}_k \mid k \in \{1, \ldots, t\}\}$. Note that $Min(R) \setminus W = \{\mathfrak{p}_{t+j} \mid j \in \{1, \ldots, n-t\}\}$. Since distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\bigcap_{k=1}^t \mathfrak{p}_k \not\subseteq \bigcup_{j=1}^{n-t} \mathfrak{p}_{t+j}$ by [12, Proposition 1.11(i)]. Let $x \in (\bigcap_{k=1}^t \mathfrak{p}_k) \setminus (\bigcup_{j=1}^{n-t} \mathfrak{p}_{t+j})$. Let $j \in \{1, \ldots, n-t\}$. We claim that $\mathfrak{p}_{t+j} \in Max(R)$. Since $\mathfrak{p}_{t+j} \neq R$, there exists $\mathfrak{m}_{t+j} \in Max(R)$ such that $\mathfrak{p}_{t+j} \subseteq \mathfrak{m}_{t+j}$ by [12, Corollary 1.4]. Suppose that $\mathfrak{p}_{t+j} \neq \mathfrak{m}_{t+j}$. Then, $\mathfrak{p}_{t+j} \subset \mathfrak{m}_{t+j}$. Observe that $\mathfrak{m}_{t+j} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$ by [12, Proposition 1.11(i)]. Let $y \in \mathfrak{m}_{t+j} \setminus (\bigcup_{i=1}^n \mathfrak{p}_i)$. From $Z(R) = \bigcup_{i=1}^n \mathfrak{p}_i$, we get that $y \notin Z(R)$. So, $y^m \notin Z(R)$ for all $m \in \mathbb{N}$. By the choice of x and y, it follows that $Rxy^m \in \mathbb{A}(R)^*$ and $V(Rxy^m) \cap Min(R) = W$ for all $m \in \mathbb{N}$. As $|\{I \in \mathbb{A}(R)^* \mid V(I) \cap Min(R) = W\}| < \infty$ by assumption, we can find $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ such that $Rxy^{m_1} = Rxy^{m_2}$. So, $xy^{m_1} = rxy^{m_2}$ for some $r \in R$. Hence, $xy^{m_1}(1 - ry^{m_2 - m_1}) = 0 \in \mathfrak{p}_{t+j}$. As $xy^{m_1} \notin \mathfrak{p}_{t+j}$, we obtain that $1 - ry^{m_2 - m_1} \in \mathfrak{p}_{t+j} \subset \mathfrak{m}_{t+j}$. Thus, $ry^{m_2 - m_1}, 1 - ry^{m_2 - m_1} \in \mathfrak{m}_{t+j}$, so $1 = 1 - ry^{m_2 - m_1} + ry^{m_2 - m_1} \in \mathfrak{m}_{t+j}$, a contradiction, since $\mathfrak{m}_{t+j} \neq R$. Therefore, $\mathfrak{p}_{t+j} = \mathfrak{m}_{t+j} \in Max(R)$ for each $j \in \{1, \ldots, n-t\}$. ☐

Corollary 2.11. If R is reduced, then $|\mathbb{A}(R)^*| = 2$ if and only if R is ring-isomorphic to $F_1 \times F_2$, where F_i is a field for each $i \in \{1, 2\}$.

Proof. By hypothesis, R is reduced. As $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$ and R is not an integral domain, we get that $|Min(R)| \geq 2$.

Assume that $|\mathbb{A}(R)^*| = 2$. Then, we obtain from [6, Theorem 1.1] that R is Artinian. By [12, Proposition 8.1], we get that $Spec(R) = Max(R) = Min(R)$ and $|Max(R)| < \infty$ by [12, Proposition 8.3]. As each member of $Max(R)$ belongs to $\mathbb{A}(R)^*$ by Lemma 2.9 and $|\mathbb{A}(R)^*| = 2$ by assumption, we obtain that $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i |\ i \in \{1,2\}\}$. From $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ and $\bigcap_{i=1}^{2}\mathfrak{m}_i = (0)$, it follows that the mapping $\phi : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ given by $\phi(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2)$ is an isomorphism of rings by [12, Proposition 1.10(ii) and (iii)]. Let $i \in \{1,2\}$. Let us denote $\frac{R}{\mathfrak{m}_i}$ by F_i. Then, F_1 and F_2 are fields and R is ring-isomorphic to $F_1 \times F_2$.

Conversely, assume that R is ring-isomorphic to $F_1 \times F_2$, where F_i is a field for each $i \in \{1,2\}$. Let us denote $F_1 \times F_2$ by T. Since (0) and F are the only ideals of any field F, it follows that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \{(0) \times F_2, F_1 \times (0)\}$. Therefore, $|\mathbb{A}(T)^*| = 2$, so $|\mathbb{A}(R)^*| = 2$. □

Lemma 2.12. If R is a reduced ring with $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 2)$, and if $|Min(R) \cap Max(R)| \geq 1$, then R is ring-isomorphic to $T \times F$, where T is a reduced ring with $|Min(T)| = n - 1$ and F is a field.

Proof. By hypothesis, R is a reduced ring with $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 2)$. Assume that $|Min(R) \cap Max(R)| \geq 1$. Let $Min(R) = \{\mathfrak{p}_i |\ i \in \{1,2,\ldots,n\}\}$. We can assume without loss of generality that $\mathfrak{p}_n \in Max(R)$. Note that $\bigcap_{i=1}^{n}\mathfrak{p}_i = (0)$. Since distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\bigcap_{k=1}^{n-1}\mathfrak{p}_k \not\subseteq \mathfrak{p}_n$. So, $(\bigcap_{k=1}^{n-1}\mathfrak{p}_k) + \mathfrak{p}_n = R$. Since $(\bigcap_{k=1}^{n-1}\mathfrak{p}_k) \cap \mathfrak{p}_n = (0)$, we obtain that the mapping $\phi : R \to \frac{R}{\bigcap_{k=1}^{n-1}\mathfrak{p}_k} \times \frac{R}{\mathfrak{p}_n}$ given by $\phi(r) = (r + \bigcap_{k=1}^{n-1}\mathfrak{p}_k, r + \mathfrak{p}_n)$ is an isomorphism of rings by [12, Proposition 1.10(ii) and (iii)]. Let us denote $\bigcap_{k=1}^{n-1}\mathfrak{p}_k$ by I, $\frac{R}{I}$ by T, and $\frac{R}{\mathfrak{p}_n}$ by F. Thus, R is ring-isomorphic to $T \times F$. Since I is a radical ideal of R, we get that T is a reduced ring. As $Min(T) = \left\{\frac{\mathfrak{p}_k}{I} |\ k \in \{1,\ldots,n-1\}\right\}$, it follows that $|Min(T)| = n - 1$. Observe that F is a field, since $\mathfrak{p}_n \in Max(R)$. □

3. Some results on $\gamma_{\mathcal{R}}((\Omega(R))^c)$, where R is reduced

As mentioned in Section 1, we use R to denote a ring that is not an integral domain. Unless otherwise specified, we assume that R is reduced and try to determine $\gamma_{\mathcal{R}}((\Omega(R))^c)$ with the assumption that $\gamma((\Omega(R))^c) < \infty$.

Observe that we obtain from [12, Proposition 1.8] and [14, Theorem 10] that $\bigcap_{\mathfrak{p} \in Min(R)}\mathfrak{p} = (0)$. Hence, $|Min(R)| \geq 2$, and $Z(R) = \bigcup_{\mathfrak{p} \in Min(R)}\mathfrak{p}$.

Remark 3.1. We do not know any necessary and sufficient condition such that $\gamma((\Omega(R))^c) < \infty$. If $|Min(R)| < \infty$, then it is known that $\gamma((\Omega(R))^c) < \infty$, see [19, Section 3]. Let $|Min(R)| = n$ for some $n \in \mathbb{N} \setminus \{1\}$. Let $Min(R) = \{\mathfrak{p}_i |\ i \in \{1,2,\ldots,n\}\}$. Then, $\bigcap_{i=1}^{n}\mathfrak{p}_i = (0)$. Let $i \in \{1,2,\ldots,n\}$. By Lemma 2.9, $\mathfrak{p}_i \in \mathbb{A}(R)^*$ for each $i \in \{1,2,\ldots,n\}$. If $n = 2$, then $(\Omega(R))^c$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where $V_i = \{I \in \mathbb{A}(R)^* |\ I \subseteq \mathfrak{p}_i\}$ for each $i \in \{1,2\}$ by the proof of [11, Proposition 2.10 $((ii) \Rightarrow (i))$]. Hence, $\gamma((\Omega(R))^c) \in \{1,2\}$. If $n \geq 3$, then we obtain from [19, Lemma 3.11 and Theorem 3.13] that $\gamma((\Omega(R))^c) = |Min(R)| = n$.

Proposition 3.2. *If* $|Min(R)| = 2$, *then* $\gamma_{\mathcal{R}}((\Omega(R))^c) \in \{2,3,4\}$.

Proof. By hypothesis, R is reduced with $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i |\ i \in \{1,2\}\}$. We know that $(\Omega(R))^c$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where $V_i = \{I \in \mathbb{A}(R)^* |\ I \subseteq \mathfrak{p}_i\}$ for each $i \in \{1,2\}$, see the proof of [11, Proposition 2.10 $((ii) \Rightarrow (i))$]. So, $\gamma((\Omega(R))^c) \in \{1,2\}$. As $(\Omega(R))^c$ has at least one edge, $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq \gamma((\Omega(R))^c) + 1$ by Remark 2.3. By Remark 2.2, $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 2\gamma((\Omega(R))^c)$. Thus, if $\gamma((\Omega(R))^c) = 1$, then

$\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$. If $\gamma((\Omega(R))^c) = 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) \in \{3,4\}$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) \in \{2,3,4\}$. □

Proposition 3.3. If $|Min(R)| = 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ if and only if R is ring-isomorphic to $R_1 \times R_2$, where R_i is an integral domain for each $i \in \{1,2\}$ with R_i is a field for at least one $i \in \{1,2\}$.

Proof. By hypothesis, $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i | \ i \in \{1,2\}\}$. We obtain from the proof of Proposition 3.2 that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ if and only if $\gamma((\Omega(R))^c) = 1$, if and only if $(\Omega(R))^c$ is a star graph. We know from Corollary 2.11 that $|\mathbb{A}(R)^*| = 2$ if and only if R is ring-isomorphic to $F_1 \times F_2$, where F_i is a field for each $i \in \{1,2\}$. If $|\mathbb{A}(R)^*| \geq 3$, then $(\Omega(R))^c$ is a star graph if and only if R is ring-isomorphic to $D \times F$, where F is a field and D is an integral domain that is not a field by [11, Proposition 2.12 $((i) \Leftrightarrow (ii))$]. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ if and only if R is ring-isomorphic to $R_1 \times R_2$, where R_i is an integral domain for each $i \in \{1,2\}$ with R_i is a field for at least one $i \in \{1,2\}$. □

Proposition 3.4. If $|Min(R)| = 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) \neq 3$.

Proof. By hypothesis, $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i | \ i \in \{1,2\}\}$. Note that $(\Omega(R))^c$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where $V_i = \{I \in \mathbb{A}(R)^* | \ I \subseteq \mathfrak{p}_i\}$ for each $i \in \{1,2\}$. Suppose that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 3$. Then, $|V_i| \geq 2$ for each $i \in \{1,2\}$, and one between V_1 and V_2 contains exactly two elements by Remark 2.4. Without loss of generality, we can assume that $|V_1| = 2$ and $|V_2| \geq 2$. As $V_1 = \{I \in \mathbb{A}(R)^* | \ V(I) \cap Min(R) = \{\mathfrak{p}_1\}\}$ and $|V_1| < \infty$, we obtain from Lemma 2.10 that $\mathfrak{p}_2 \in Max(R)$. Therefore, $|Min(R) \cap Max(R)| \geq 1$. Observe that a reduced ring with only one minimal prime ideal is an integral domain. So, we obtain from Lemma 2.12 that R is ring-isomorphic to $D \times F$, where D is an integral domain and F is a field. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ by Proposition 3.3, a contradiction to the assumption that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 3$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) \neq 3$. □

Remark 3.5. Let $|Min(R)| = 2$, and let $Min(R) = \{\mathfrak{p}_i | \ i \in \{1,2\}\}$. Then, we obtain from Propositions 3.2-3.4 that $\gamma_{\mathcal{R}}((\Omega(R))^c) \in \{2,3,4\}$ and $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ if and only if $\mathfrak{p}_i \in Max(R)$ at least one $i \in \{1,2\}$. By Proposition 3.4, $\gamma_{\mathcal{R}}((\Omega(R))^c)$ cannot be equal to 3, so $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ if and only if $\mathfrak{p}_i \notin Max(R)$ for each $i \in \{1,2\}$.

We use the following lemma in the proof of some examples of this paper.

Lemma 3.6. Let $T = K[X_1, X_2, \ldots, X_n]$ $(n \geq 2)$ be the polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over a field K. Let I be the ideal of T given by $I = T(\prod_{i=1}^{n} X_i)$, and let $R = \frac{T}{I}$. Then, R is a reduced ring, $|Min(R)| = n$, and R cannot be expressed as the direct product of two non-trivial rings.

Proof. As $X_1, X_2, \ldots, X_n$ are pairwise non-associate prime elements of T, $TX_i \in Spec(T)$ for each $i \in \{1,2,\ldots,n\}$, and $TX_i \neq TX_j$ for all distinct $i,j \in \{1,2,\ldots,n\}$. Therefore, $I = T(\prod_{i=1}^{n} TX_i) = \bigcap_{i=1}^{n} TX_i$ is a radical ideal of T. So, $R = \frac{T}{I}$ is a reduced ring. For each $i \in \{1,2,\ldots,n\}$, let us denote $X_i + I$ by x_i. Observe that $Rx_i \in Min(R)$ for each $i \in \{1,2,\ldots,n\}$ and $Rx_i \neq Rx_j$ for all distinct $i,j \in \{1,2,\ldots,n\}$. Therefore, $Min(R) = \{Rx_i | \ i \in \{1,2,\ldots,n\}\}$. So, $|Min(R)| = n$. We next verify that R has no non-trivial idempotent element. Let $t \in T$ be such that $t + I$ is an idempotent element of T with $t + I \neq 0 + I$. Then, $t^2 - t \in I$ and $t \notin I$. Hence, $t \notin TX_i$ for some $i \in \{1,2,\ldots,n\}$. As $t(t-1) \in I \subset TX_i$, it follows that $t - 1 \in TX_i$. Let $j \in \{1,2,\ldots,n\} \setminus \{i\}$. Since $TX_i + TX_j \neq T$, we obtain that $t \notin TX_j$. Hence, from $t(t-1) \in TX_j$, it follows that $t - 1 \in TX_j$. Therefore, $t - 1 \in \bigcap_{k=1}^{n} TX_k = I$. Therefore, $t + I = 1 + I$. Thus, R has no non-trivial idempotent element. So, R cannot be expressed as the direct product of two non-trivial rings. □

The following example illustrates Propositions 3.2-3.4.

Example 3.7.

(1) If $R = F_1 \times F_2$, where F_i is a field for each $i \in \{1,2\}$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$.

(2) If $R = \mathbb{Z} \times F$, where F is a field, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$.

(3) If $R = \mathbb{Z} \times \mathbb{Z}$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

(4) Let $n = 2$, and let T, R be as in the statement of Lemma 3.6. Then, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

Proof.

(1) We obtain from Proposition 3.3 that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$.

(2) If $R = \mathbb{Z} \times F$, where F is a field, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2$ by Proposition 3.3.

(3) By hypothesis, $R = \mathbb{Z} \times \mathbb{Z}$. Note that $Min(R) = \{\mathfrak{p}_1 = (0) \times \mathbb{Z}, \mathfrak{p}_2 = \mathbb{Z} \times (0)\}$. Since $\mathbb{Z}$ is an integral domain but not a field, we obtain that $\mathfrak{p}_i \notin Max(R)$ for each $i \in \{1,2\}$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ by Remark 3.5.

(4) Using the same notations as in the statement and proof of Lemma 3.6, $R = \frac{T}{I}$, where $T = K[X_1, X_2]$ and $I = T(X_1X_2)$. We know from the proof of Lemma 3.6 that R is a reduced ring, $Min(R) = \{Rx_i | \ i \in \{1,2\}\}$, where $x_i = X_i + I$ for each $i \in \{1,2\}$, and R has no non-trivial idempotent element. As $Rx_i \subset Rx_1 + Rx_2 \subset R$ for each $i \in \{1,2\}$, we obtain that $\mathfrak{p}_i \notin Max(R)$ for each $i \in \{1,2\}$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ by Remark 3.5. $\square$

Remark 3.8. Assume that $|Min(R)| = n$ for some $n \in \mathbb{N}$ with $n \geq 3$. Let $Min(R) = \{\mathfrak{p}_i | \ i \in \{1,2,3,\ldots,n\}\}$. We have noted in Remark 3.1 that $\gamma((\Omega(R))^c) = n$. We next try to determine $\gamma_{\mathcal{R}}((\Omega(R))^c)$. We obtain from Remark 2.2 that $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 2n$. We know from Lemma 2.9 that $\mathfrak{p}_i \in \mathbb{A}(R)^*$ for each $i \in \{1,2,3,\ldots,n\}$. Let $i, j \in \{1,2,3,\ldots,n\}$ be distinct. Since $\mathfrak{p}_i + \mathfrak{p}_j$ cannot be a subset of any member of $Min(R)$, $\mathfrak{p}_i + \mathfrak{p}_j \notin \mathbb{A}(R)$ by Lemma 2.8. Hence, $\mathfrak{p}_i - \mathfrak{p}_j$ is an edge of $(\Omega(R))^c$. Thus, $(\Omega(R))^c$ has at least one edge. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq \gamma((\Omega(R))^c) + 1 = n + 1$ by Remark 2.3.

If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, then we determine $\gamma_{\mathcal{R}}((\Omega(R))^c)$ in Theorem 3.22. We first state and prove some results that are used in the proof of Theorem 3.22. Khojasteh and Heydari [24] have computed the Roman domination number of the comaximal ideal graph of a commutative ring. The following proposition is motivated by [24, Theorem 1.6(i) and (ii)].

Proposition 3.9. Let $n \in \mathbb{N}$ be such that $n \geq 3$. If $|Min(R)| = n$, and if $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$, then the following statements hold.

(1) If I and J are distinct members of V_2, then I and J are not comparable under inclusion.

(2) If $\mathfrak{p} \in Min(R)$, then $\mathfrak{p}$ can contain at most one member of V_2.

Proof.

(1) Let I and J be distinct members of V_2. Suppose that I and J are comparable under inclusion. We can assume without loss of generality that $I \subset J$. If $A \in \mathbb{A}(R)^*$ is such that $A + I \notin \mathbb{A}(R)$, then $A + J \notin \mathbb{A}(R)$. Hence, $g = (V_0, V_1 \cup \{I\}, V_2 \setminus \{I\})$ is an RDF on $(\Omega(R))^c$. As the weight of g equals $|V_1| + 1 + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 <$ the weight of f, we arrive at a contradiction, since f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ by hypothesis. Therefore, I and J are not comparable under inclusion.

(2) Since $|Min(R)| < \infty$, $Min(R) \subseteq \mathbb{A}(R)^*$ by Lemma 2.9. Let $\mathfrak{p} \in Min(R)$. We claim that $\mathfrak{p}$ can contain at most one member of V_2.

First, we verify that $\mathfrak{p}$ can contain at most two members of V_2. Suppose that $\mathfrak{p}$ contains at least three members of V_2. Let I_1, I_2, I_3 be three pairwise distinct members of V_2 such that $I_i \subseteq \mathfrak{p}$ for each $i \in \{1,2,3\}$. We obtain from (1) that $\mathfrak{p} \notin V_2$. As $\mathbb{A}(R)^* = \bigcup_{i=0}^{2} V_i$, we get that $\mathfrak{p} \in V_0 \cup V_1$. If $\mathfrak{p} \in V_0$, then $g = (V_0 \setminus \{\mathfrak{p}\}, V_1 \cup \{I_i | \, i \in \{1,2,3\}\}, (V_2 \cup \{\mathfrak{p}\}) \setminus \{I_i \mid i \in \{1,2,3\}\})$ is an RDF on $(\Omega(R))^c$ with the weight of g equals $|V_1| + 3 + 2(|V_2| + 1 - 3) = |V_1| + 2|V_2| - 1 <$ the weight of f, a contradiction, since f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ by hypothesis. Hence, $\mathfrak{p} \notin V_0$. If $\mathfrak{p} \in V_1$, then $g' = (V_0, (V_1 \cup \{I_i | \, i \in \{1,2,3\}\}) \setminus \{\mathfrak{p}\}, (V_2 \cup \{\mathfrak{p}\}) \setminus \{I_i | \, i \in \{1,2,3\}\})$ is an RDF on $(\Omega(R))^c$ with the weight of g' equals $|V_1| + 2 + 2\ (|V_2| + 1 - 3) = |V_1| + 2|V_2| - 2 <$ the weight of f, a contradiction, since f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$. Therefore, $\mathfrak{p}$ can contain at most two members of V_2.

We next show that $\mathfrak{p}$ can contain at most one member of V_2. Suppose that $\mathfrak{p}$ contains exactly two members of V_2. Let $I, J \in V_2$ be distinct such that $\mathfrak{p}$ contains both I and J but $\mathfrak{p}$ does not contain any other member of V_2. We obtain from (1) that $\mathfrak{p} \notin V_2$. Thus, $\mathfrak{p} \in V_0 \cup V_1$. Either $|V_2| = 2$ or $|V_2| > 2$. If $|V_2| = 2$, then $V_2 = \{I, J\}$. As $\mathfrak{p} + I = \mathfrak{p} = \mathfrak{p} + J \in \mathbb{A}(R)$, $\mathfrak{p}$ is not adjacent to I (respectively, J) in $(\Omega(R))^c$. Hence, $\mathfrak{p}$ cannot be in V_0, so $\mathfrak{p} \in V_1$. Observe that $g'' = (V_0, (V_1 \cup \{I, J\}) \setminus \{\mathfrak{p}\}, \{\mathfrak{p}\})$ is an RDF on $(\Omega(R))^c$ with the weight of g'' equals $|V_1| + 1 + 2 < |V_1| + 4$. As the weight of f equals $|V_1| + 4$, we obtain that the weight of $g'' <$ the weight of f, contradicting the hypothesis that f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$. Suppose that $|V_2| \geq 3$. Let $A \in V_2 \setminus \{I, J\}$. There exists $\mathfrak{p}' \in Min(R)$ such that $A \subseteq \mathfrak{p}'$ by Lemma 2.8. Note that $\mathfrak{p} \neq \mathfrak{p}'$. As any member of $Min(R)$ can contain at most two members of V_2, we get that either $I \not\subseteq \mathfrak{p}'$ or $J \not\subseteq \mathfrak{p}'$. Without loss of generality, we can assume that $J \not\subseteq \mathfrak{p}'$. Since $\mathfrak{p}' \in Min(R)$ and $J \not\subseteq \mathfrak{p}'$, it follows from Lemma 2.8 that $J + \mathfrak{p}' \notin \mathbb{A}(R)$. So, J and $\mathfrak{p}'$ are adjacent in $(\Omega(R))^c$. Hence, $\mathfrak{p}'$ cannot belong to V_1 by [21, Proposition 3(b)]. Either $A = \mathfrak{p}'$ or $A \neq \mathfrak{p}'$. Assume that $A \neq \mathfrak{p}'$. Then, $\mathfrak{p}'$ cannot belong to V_2 by (1). Therefore, $\mathfrak{p}' \in V_0$. Since any two distinct members of V_2 are not comparable under inclusion, we get that $\mathfrak{p}$ cannot belong to V_2. As $A \not\subseteq \mathfrak{p}$, A and $\mathfrak{p}$ are adjacent in $(\Omega(R))^c$. Hence, $\mathfrak{p}$ cannot belong to V_1 by [21, Proposition 3(b)]. Therefore, $\mathfrak{p} \in V_0$. Let $g''' = ((V_0 \cup \{J, A\}) \setminus \{\mathfrak{p}, \mathfrak{p}'\}, V_1 \cup \{I\}, (V_2 \cup \{\mathfrak{p}, \mathfrak{p}'\}) \setminus \{I, J, A\})$. Observe that g''' is an RDF on $(\Omega(R))^c$ and the weight of g''' equals $|V_1| + 1 + 2\ (|V_2| + 2 - 3) = |V_1| + 2|V_2| - 1$. As the weight of g''' is strictly less than the weight of f, we arrive at a contradiction to the hypothesis that f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$. Suppose that $A = \mathfrak{p}'$. If $g'''' = ((V_0 \cup \{J\}) \setminus \{\mathfrak{p}\}, V_1 \cup \{I\}, (V_2 \cup \{\mathfrak{p}\}) \setminus \{I, J\})$, then g'''' is an RDF on $(\Omega(R))^c$ and the weight of g'''' equals $|V_1| + 1 + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1$. Thus, the weight of g'''' is strictly less than the weight of f, a contradiction to the hypothesis that f is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$.

Therefore, if $\mathfrak{p}$ is any element of $Min(R)$, then $\mathfrak{p}$ can contain at most one member of V_2. □

If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, then each $\mathfrak{p} \in Min(R)$ has degree at least two in $(\Omega(R))^c$. Hence, if $f = (V_0, V_1, V_2)$ is any $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$, then $V_2 \neq \emptyset$ by [21, Proposition 3(a)]. The following corollary is motivated by [24, Theorem 1.6 (iii)].

Corollary 3.10. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, then there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on $(\Omega(R))^c$ such that $|V_1|$ is minimum and $V_2 \subseteq Min(R)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $|V_1|$ is minimum. Note that $V_2 \neq \emptyset$. Let $V_2 = \{I_i | \, i \in \{1, \ldots, t\}\}$. Let $i \in \{1, \ldots, t\}$. As $I_i \in \mathbb{A}(R)^*$, there exists $\mathfrak{p}_i \in Min(R)$ such that $I_i \subseteq \mathfrak{p}_i$ by Lemma 2.8. As any member of $Min(R)$ can contain at most one member of V_2 by Proposition 3.9(2), we get that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all distinct $i, j \in \{1, \ldots, t\}$. Thus, for each $i \in \{1, \ldots, t\}$, $I_i \subseteq \mathfrak{p}_i$ but $I_i \not\subseteq \mathfrak{p}_j$ for all $j \in \{1, \ldots, t\} \setminus \{i\}$. We consider the following cases.

Case(1). $t = 1$.

If $I_1 = \mathfrak{p}_1$, then $f = (V_0, V_1, V_2 = \{\mathfrak{p}_1\})$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $|V_1|$ is minimum and $V_2 \subset Min(R)$. Suppose that $I_1 \neq \mathfrak{p}_1$. As $\mathfrak{p}_1 + I_1 = \mathfrak{p}_1 \in \mathbb{A}(R)$ and each member of V_0 must be adjacent to I_1 in $(\Omega(R))^c$, we obtain that $\mathfrak{p}_1 \in V_1$. Let $f' = (V_0, (V_1 \cup \{I_1\}) \setminus \{\mathfrak{p}_1\}, \{\mathfrak{p}_1\})$ is an RDF on $(\Omega(R))^c$ with the weight of f' equals the weight of f. Note that $|(V_1 \cup \{I_1\}) \setminus \{\mathfrak{p}_1\}| = |V_1|$.

Case(2). $t \geq 2$.

If $I_i = \mathfrak{p}_i$ for each $i \in \{1, 2, \ldots, t\}$, then $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $V_2 \subseteq Min(R)$. So, we can assume that $I_i \neq \mathfrak{p}_i$ for at least one $i \in \{1, 2, \ldots, t\}$. Suppose that $I_i \neq \mathfrak{p}_i$ for each $i \in \{1, 2, \ldots, t\}$. Let $i \in \{1, 2, \ldots, t\}$. As $\mathfrak{p}_i$ and I_j are adjacent in $(\Omega(R))^c$ for all $j \in \{1, 2, \ldots, t\} \setminus \{i\}$, $\mathfrak{p}_i$ cannot belong to V_1 by [21, Proposition 3(b)]. Therefore, $\mathfrak{p}_i \in V_0$. Let $f'' = (V_0 \cup \{I_i | \, i \in \{1, 2, \ldots, t\}\}, V_1, \{\mathfrak{p}_i | \, i \in \{1, 2, \ldots, t\}\})$ is an RDF on $(\Omega(R))^c$ and the weight of f'' equals the weight of f. Suppose that there exists at least one $i \in \{1, 2, \ldots, t\}$ such that $I_i = \mathfrak{p}_i$ and there exists at least one $j \in \{1, 2, \ldots, t\}$ such that $I_j \neq \mathfrak{p}_j$. We can assume without loss of generality that there exists s with $1 \leq s < t$ such that $I_i = \mathfrak{p}_i$ for each $i \in \{1, \ldots, s\}$ and $I_j \subset \mathfrak{p}_j$ for each $j \in \{s+1, \ldots, t\}$. Observe that $\mathfrak{p}_j$ cannot be in V_1 for each $j \in \{s+1, \ldots, t\}$. Therefore, $\mathfrak{p}_j \in V_0$ for each $j \in \{s+1, \ldots, t\}$. If $f''' = ((V_0 \cup \{I_{s+1}, \ldots, I_t\}) \setminus \{\mathfrak{p}_j | \, j \in \{s+1, \ldots, t\}\}, V_1, \{\mathfrak{p}_i | \, i \in \{1, 2, \ldots, t\}\})$, then f''' is an RDF on $(\Omega(R))^c$ and the weight of f''' equals the weight of f.

Therefore, there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on $(\Omega(R))^c$ such that $|V_1|$ is minimum and $V_2 \subseteq Min(R)$. □

Lemma 3.11. Let $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$. If $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $V_2 \subseteq Min(R)$, and if $|V_2| = t < n$, then $\mathfrak{p} \in Max(R)$ for any $\mathfrak{p} \in Min(R) \setminus V_2$.

Proof. Let $Min(R) = \{\mathfrak{p}_i | \, i \in \{1, 2, 3, \ldots, n\}\}$. Assume that $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $V_2 \subseteq Min(R)$ and $|V_2| = t < n$. Observe that $t \geq 1$. Without loss of generality, we can assume that $V_2 = \{\mathfrak{p}_k | \, k \in \{1, \ldots, t\}\}$. Then, $Min(R) \setminus V_2 = \{\mathfrak{p}_{t+j} | \, j \in \{1, \ldots, n-t\}\}$. Either $t = 1$ or $t \geq 2$. If $t = 1$, then, $V_2 = \{\mathfrak{p}_1\}$. Let $I \in \mathbb{A}(R)^*$ be such that $V(I) \cap Min(R) = \{\mathfrak{p}_1\}$ and $I \neq \mathfrak{p}_1$. As I and $\mathfrak{p}_1$ are adjacent in $\Omega(R)$ and each member of V_0 is adjacent to $\mathfrak{p}_1$ in $(\Omega(R))^c$, it follows that $I \in V_1$. As V_1 is finite, we get that $|\{A \in \mathbb{A}(R)^* | \, V(A) \cap Min(R) = \{\mathfrak{p}_1\}\}| \leq |V_1| + 1 < \infty$. Let $t \geq 2$, and let $I \in \mathbb{A}(R)^*$ be such that $V(I) \cap Min(R) = \{\mathfrak{p}_k | \, k \in \{1, 2, \ldots, t\}\}$. As $t \geq 2$, it follows that I cannot belong to V_2. Since I is adjacent to each member of V_2 in $\Omega(R)$, it follows that I cannot belong to V_0. Hence, $I \in V_1$. Therefore, $\left\{I \in \mathbb{A}(R)^* | \, V(I) \cap Min(R) = \{\mathfrak{p}_k | \, k \in \{1, 2, \ldots, t\}\right\} \subseteq V_1$, so $\left|\left\{I \in \mathbb{A}(R)^* | \, V(I) \cap Min(R) = \{\mathfrak{p}_k | \, k \in \{1, 2, \ldots, t\}\right\}\right| < \infty$. Therefore, $\mathfrak{p}_{t+j} \in Max(R)$ for each $j \in \{1, \ldots, n-t\}$ by Lemma 2.10. □

Corollary 3.12. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, and if $Min(R) \cap Max(R) = \emptyset$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$.

Proof. By hypothesis, $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$ and $Min(R) \cap Max(R) = \emptyset$. We know from Corollary 3.10 that there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ such that $V_2 \subseteq Min(R)$. Let $|V_2| = t$. Note that $t \geq 1$. Since $Min(R) \cap Max(R) = \emptyset$ by assumption, $t = n$ by Lemma 3.11. As $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 2n$ by Remark 3.8, we obtain that $V_1 = \emptyset$ and the weight of f equals $2|V_2| = 2n$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$. □

Lemma 3.13. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, and if $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $|V_1|$ is minimum and $V_2 \subseteq Min(R)$, then $|V_2| \geq n-2$.

Proof. Let $Min(R) = \{\mathfrak{p}_i | \, i \in \{1, 2, 3, \ldots, n\}\}$. Assume that $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $|V_1|$ is minimum and $V_2 \subseteq Min(R)$. Let $|V_2| = t$. Then, $t \geq 1$. Thus, if $n = 3$, then $|V_2| \geq n-2$. Assume that $n \geq 4$. Without loss of generality, we can assume that

$V_2 = \{\mathfrak{p}_k |\, k \in \{1, \ldots, t\}\}$. Suppose that $t < n-2$. Then, $n-t \geq 3$. Observe that $I_1 = \bigcap_{k=1}^{t+1} \mathfrak{p}_k, I_2 = I_1 \cap \mathfrak{p}_{t+2}$ are distinct members of $\mathbb{A}(R)^*$, and I_j is adjacent to each member of V_2 in $\Omega(R)$ for each $j \in \{1, 2\}$. Hence, I_j must belong to V_1 for each $j \in \{1, 2\}$. Observe that $\mathfrak{p}_{t+3} \in \mathbb{A}(R)^*$ and $\mathfrak{p}_{t+3} - \mathfrak{p}_1$ is an edge of $(\Omega(R))^c$. Since there cannot be any edge of $(\Omega(R))^c$ which joins one vertex in V_1 and the other in V_2 by [21, Proposition 3(b)], we obtain that $\mathfrak{p}_{t+3}$ must belong to V_0. As $\mathfrak{p}_{t+3}$ is adjacent to I_j in $(\Omega(R))^c$ for each $j \in \{1, 2\}$, we arrive at a contradiction, since any member of V_0 can be adjacent to at most one member of V_1 in $(\Omega(R))^c$ by [21, Proposition 4(c)]. Hence, $t \geq n-2$. $\square$

Corollary 3.14. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ with $n \geq 4$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 2n-1$.

Proof. By hypothesis, $|Min(R)| = n$ for some $n \in \mathbb{N}$ with $n \geq 4$. We know from Corollary 3.10 that there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ with $|V_1|$ is minimum and $V_2 \subseteq Min(R)$. Let $Min(R) = \{\mathfrak{p}_i |\, i \in \{1, 2, 3, 4, \ldots, n\}\}$. Let $|V_2| = t$. Then, $t \geq n-2$ by Lemma 3.13. Without loss of generality, we can assume that $V_2 = \{\mathfrak{p}_k |\, k \in \{1, \ldots, t\}\}$. Assume that $t = n$. Then, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$ follows as in the proof of Corollary 3.12. Thus, if $t = n$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) > 2n-1$. Suppose that $t = n-1$. As $\bigcap_{j=1}^{n-1} \mathfrak{p}_j \in \mathbb{A}(R)^*$ is adjacent to each member of V_2 in $\Omega(R)$, $\bigcap_{j=1}^{n-1} \mathfrak{p}_j$ must belong to V_1. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) =$ the weight of $f = |V_1| + 2|V_2| \geq 1 + 2(n-1) = 2n-1$. Suppose that $t = n-2$. By hypothesis, $n \geq 4$. So, $t = n-2 \geq 2$. Observe that $I_1 = \bigcap_{k=1}^{n-2} \mathfrak{p}_k, I_2 = I_1 \cap \mathfrak{p}_{n-1}$, and $I_3 = I_1 \cap \mathfrak{p}_n$ are pairwise distinct members of $\mathbb{A}(R)^*$. Since I_j is adjacent to each member of V_2 in $\Omega(R)$ for each $j \in \{1, 2, 3\}$, we get that I_j must belong to V_1 for each $j \in \{1, 2, 3\}$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) =$ the weight of $f = |V_1| + 2|V_2| \geq 3 + 2(n-2) = 2n-1$. Thus, if $n \geq 4$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 2n-1$. $\square$

Lemma 3.15. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, and if $|Min(R) \cap Max(R)| \geq 2$, then R is ring-isomorphic to $T \times F_1 \times F_2$, where T is a reduced ring with $|Min(T)| = n-2$ and F_i is a field for each $i \in \{1, 2\}$.

Proof. By hypothesis, $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$ and $|Min(R) \cap Max(R)| \geq 2$. We obtain from Lemma 2.12 that R is ring-isomorphic to $R_1 \times F_1$, where R_1 is a reduced ring with $|Min(R_1)| = n-1$ and F_1 is a field. As $|Min(R) \cap Max(R)| \geq 2$, it follows that $|Min(R_1) \cap Max(R_1)| \geq 1$. Hence, R_1 is ring-isomorphic to $R_2 \times F_2$, where R_2 is a reduced ring with $|Min(R_2)| = n-2$ and F_2 is a field by Lemma 2.12. Thus, R is ring-isomorphic to $R_2 \times F_2 \times F_1$. Let us denote R_2 by T. Therefore, R is ring-isomorphic to $T \times F_1 \times F_2$, where T is a reduced ring with $|Min(T)| = n-2$ and F_i is a field for each $i \in \{1, 2\}$. $\square$

Lemma 3.16. If $R = D \times F_1 \times F_2$, where D is an integral domain (D can be a field) and F_i is a field for each $i \in \{1, 2\}$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

Proof. The ring $R = D \times F_1 \times F_2$, where D is an integral domain and F_i is a field for each $i \in \{1, 2\}$ is reduced with $Min(R) = \{\mathfrak{p}_1 = (0) \times F_1 \times F_2, \mathfrak{p}_2 = D \times (0) \times F_2, \mathfrak{p}_3 = D \times F_1 \times (0)\}$. Thus, $|Min(R)| = 3$. We obtain from Remark 3.8 that $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 4$. Let $V_2 = \{\mathfrak{p}_1\}$, $V_1 = \{\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0) \times (0) \times F_2, \mathfrak{p}_1 \cap \mathfrak{p}_3 = (0) \times F_1 \times (0)\}$, and $V_0 = \mathbb{A}(R)^* \setminus (V_1 \cup V_2)$. Let $f = (V_0, V_1, V_2)$. If $J \in V_0$, then $J \not\subseteq \mathfrak{p}_1$. So, J is adjacent to $\mathfrak{p}_1$ in $(\Omega(R))^c$. Hence, f is an RDF on $(\Omega(R))^c$ and the weight of f equals $|V_1| + 2|V_2| = 4$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 4$, so $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$. $\square$

Corollary 3.17. If $|Min(R)| = 3$, and if $|Min(R) \cap Max(R)| \geq 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

Proof. By hypothesis, $|Min(R)| = 3$ and $|Min(R) \cap Max(R)| \geq 2$. As any reduced ring T with $|Min(T)| = 1$ is an integral domain, we obtain from Lemma 3.15 that R is ring-isomorphic to $D \times F_1 \times F_2$, where D is an integral domain (D can be a field) and F_i is a field for each $i \in \{1, 2\}$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ by Lemma 3.16. $\square$

Lemma 3.18. If $R = T\times F$, where T is a reduced ring such that $|Min(T)| = 2$ and $Min(T) \cap Max(T) = \varnothing$, and F is a field, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$.

Proof. By hypothesis, $R = T\times F$, where T is a reduced ring such that $|Min(T)| = 2$ and $Min(T) \cap Max(T) = \varnothing$, and F is a field. Let $Min(T) = \{\mathfrak{p}_k |\ k \in \{1,2\}\}$. By assumption, $\mathfrak{p}_k \notin Max(T)$ for each $k \in \{1,2\}$. As $Min(R) = \{\mathfrak{P}_k = \mathfrak{p}_k\times F |\ k \in \{1,2\}\} \cup \{\mathfrak{P}_3 = T\times(0)\}$, we obtain that $|Min(R)| = 3$ and $\mathfrak{P}_k \notin Max(R)$ for each $k \in \{1,2\}$. If $f = (V_0, V_1, V_2)$ is any $\gamma_{\mathcal{R}}$-function on $(\Omega(R))^c$ with $V_2 \subseteq Min(R)$, then $\{\mathfrak{P}_k |\ k \in \{1,2\}\} \subseteq V_2$ by Lemma 3.11. Note that $\bigcap_{k=1}^{2} \mathfrak{P}_k = (0)\times F$ is a simple R-module and $(0)\times F \notin V_2$. We know from Remark 3.8 that $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 4$. If $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$, then $V_2 = \{\mathfrak{P}_k |\ k \in \{1,2\}\}$ and $V_1 = \varnothing$. So, $(0)\times F$ must be in V_0, a contradiction, since $(0)\times F$ is not adjacent to any member of V_2 in $(\Omega(R))^c$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 5$. If $f' = (V_0, V_1, V_2)$ is given by $V_2 = \{\mathfrak{P}_k |\ k \in \{1,2\}\}$, $V_1 = \{\bigcap_{k=1}^{2} \mathfrak{P}_k = (0)\times F\}$, and $V_0 = \mathbb{A}(R)^* \setminus (V_1 \cup V_2)$, then for any $A \in V_0$ is adjacent to $\mathfrak{P}_k$ in $(\Omega(R))^c$ for some $k \in \{1,2\}$. So, f' is an RDF on $(\Omega(R))^c$. Since the weight of f' equals 5, we get that $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 5$. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$. □

Corollary 3.19. If $|Min(R)| = 3$, and if $|Min(R) \cap Max(R)| = 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$.

Proof. By hypothesis, $|Min(R)| = 3$ and $|Min(R) \cap Max(R)| = 1$. We obtain from Lemma 2.12 that R is ring-isomorphic to $T\times F$, where T is a reduced ring with $|Min(T)| = 2$ and F is a field. As $|Min(R) \cap Max(R)| = 1$, $Min(T) \cap Max(T) = \varnothing$. Therefore, we obtain from Lemma 3.18 that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$. □

Lemma 3.20. Let $n \in \mathbb{N}$ be such that $n \geq 4$. If $R = T\times F$, where T is a reduced ring with $Min(T) = n-1$ and F is a field, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$.

Proof. Assume that $R = T\times F$, where T is a reduced ring with $|Min(T)| = n-1$ for some $n \in \mathbb{N}$ with $n \geq 4$ and F is a field. Note that R is reduced. Let $Min(T) = \{\mathfrak{p}_k |\ k \in \{1,2,3,\ldots,n-1\}\}$. Then, $Min(R) = \{\mathfrak{P}_k = \mathfrak{p}_k\times F |\ k \in \{1,2,3,\ldots,\ n-1\}\} \cup \{\mathfrak{P}_n = T\times(0)\}$. Thus, $|Min(R)| = n$. Let $V_2 = \{\mathfrak{P}_k |\ k \in \{1,2,\ldots,n-1\}\}$, $V_1 = \{\bigcap_{k=1}^{n-1} \mathfrak{P}_k\}$, and $V_0 = \mathbb{A}(R)^* \setminus (V_1 \cup V_2)$. Note that $\bigcap_{k=1}^{n-1} \mathfrak{P}_k = (0)\times F$ and $(0)\times F$ is a simple R-module. If $A \in \mathbb{A}(R)^*$ is such that $A \in V_0$, then $A \not\subseteq \mathfrak{P}_k$ for some $k \in \{1,2,\ldots,n-1\}$, so A and $\mathfrak{P}_k$ are adjacent in $(\Omega(R))^c$. Therefore, $f = (V_0, V_1, V_2)$ is an RDF on $(\Omega(R))^c$ and the weight of f equals $|V_1| + 2|V_2| = 1 + 2(n-1) = 2n-1$. So, $\gamma_{\mathcal{R}}((\Omega(R))^c) \leq 2n-1$. As $n \geq 4$ by hypothesis, $\gamma_{\mathcal{R}}((\Omega(R))^c) \geq 2n-1$ by Corollary 3.14. Therefore, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$. □

Corollary 3.21. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 4)$, and if $|Min(R) \cap Max(R)| \geq 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$.

Proof. By hypothesis, $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 4)$ and $|Min(R) \cap Max(R)| \geq 1$. We obtain from Lemma 2.12 that R is ring-isomorphic to $T\times F$, where T is a reduced ring with $|Min(T)| = n-1$ and F is a field. From Lemma 3.20, we obtain that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$. □

Theorem 3.22. If $|Min(R)| = n$ for some $n \in \mathbb{N}$ $(n \geq 3)$, then the following statements hold.

(1) If $Min(R) \cap Max(R) = \varnothing$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$.

(2) If $n = 3$, and if $|Min(R) \cap Max(R)| \geq 2$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$ and R is ring-isomorphic to $D\times F_1\times F_2$, where D is an integral domain (D can be a field) and F_i is a field for each $i \in \{1,2\}$. If $n = 3$, and if $|Min(R) \cap Max(R)| = 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$, and R is ring-isomorphic to $T\times F$, where T is a reduced ring with $|Min(T)| = 2$, $Min(T) \cap Max(T) = \varnothing$, and F is a field.

(3) If $n \geq 4$, and $|Min(R) \cap Max(R)| \geq 1$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$, and R is ring-isomorphic to $T\times F$, where T is a reduced ring with $|Min(T)| = n-1$, $Min(T) \cap Max(T)$ can be nonempty, and F is a field.

Proof. By hypothesis, $|Min(R)| = n$ for some $n \in \mathbb{N}$ with $n \geq 3$. Let $Min(R) = \{\mathfrak{p}_i | \, i \in \{1,2,3,\ldots,n\}\}$. We have noted in Remark 3.1 that $\gamma((\Omega(R))^c) = n$ and observed in Remark 3.8 that $n+1 \leq \gamma_{\mathcal{R}}((\Omega(R))^c) \leq 2n$.

(1) If $Min(R) \cap Max(R) = \emptyset$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$ by Corollary 3.12.

(2) Assume that $n = 3$. If $|Min(R) \cap Max(R)| \geq 2$, then we obtain from the proof of Corollary 3.17 that R is ring-isomorphic to $D \times F_1 \times F_2$, where D is an integral domain (D can be a field) and F_i is a field for each $i \in \{1,2\}$, and $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$. If $|Min(R) \cap Max(R)| = 1$, then we know from the proof of Corollary 3.19 that R is ring-isomorphic to $T \times F$, where T is a reduced ring with $|Min(T)| = 2$, $Min(T) \cap Max(T) = \emptyset$ and F is a field, and $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$.

(3) Assume that $n \geq 4$ and $|Min(R) \cap Max(R)| \geq 1$. Then, we know from the proof of Corollary 3.21 that R is ring-isomorphic to $T \times F$, where T is a reduced ring with $|Min(T)| = n-1$ ($Min(T) \cap Max(T)$ can be nonempty) and F is a field, and $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n-1$. □

The following example illustrates the results proved in this section.

Example 3.23.

(1) Let $n \in \mathbb{N}$ be such that $n \geq 3$. If $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times), then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$.

(2) Let n, T, R be as in the statement of Lemma 3.6 with $n \geq 3$. Then, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$ and R cannot be expressed as the direct product of two non-trivial rings.

(3) Let n, T, R be as in (2) with $n \geq 3$, and let F be a field. Then, $\gamma_{\mathcal{R}}((\Omega\ (R \times F))^c) = 2(n+1) - 1 = 2n+1$.

(4) If $R = \mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

(5) If $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3$, then $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$.

Proof.

(1) Note that $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times, $n \geq 3$) is a reduced ring. Let $i \in \{1,2,3,\ldots,n\}$. Let $\mathfrak{p}_i = I_{1i} \times I_{2i} \times I_{3i} \times \cdots \times I_{ni}$ with $I_{ii} = (0)$ and $I_{ji} = \mathbb{Z}$ for all $j \in \{1,2,3,\ldots,n\} \setminus \{i\}$. Then, $Min(R) = \{\mathfrak{p}_i | \, i \in \{1,2,3,\ldots,n\}\}$. Thus, $|Min(R)| = n$. As $\mathbb{Z}$ is not a field, we get that $\mathfrak{p}_i \notin Max(R)$ for each $i \in \{1,2,3,\ldots,n\}$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$ by Theorem 3.22(1).

(2) In the notation of the statement of Lemma 3.6, $R = \frac{T}{I}$, where $T = K[X_1, X_2, X_3, \ldots, X_n]$ and $I = T(\prod_{i=1}^{n} X_i)$. We have noted in the proof of Lemma 3.6 that R is reduced with $Min(R) = \{\mathfrak{p}_i = R(X_i + I) | \, i \in \{1,2,3,\ldots,n\}\}$ and R has no non-trivial idempotent element. Thus, $|Min(R)| = n$ and R cannot be expressed as the direct product of two non-trivial rings. As for each $i \in \{1,2,3,\ldots,n\}$, $\mathfrak{p}_i \subset \sum_{k=1}^{n} R(X_k + I) \subset R$, it follows that $\mathfrak{p}_i \notin Max(R)$. Hence, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2n$ by Theorem 3.22(1).

(3) Let n, T, R be as in the statement of Lemma 3.6. Let F be a field. Observe that $R \times F$ is a reduced ring with $|Min(R \times F)| = n+1 \geq 4$. We obtain from Theorem 3.22(3) that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 2(n+1) - 1 = 2n+1$.

(4) Assume that $R = \mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Since $\mathbb{Z}$ is an integral domain, and $\mathbb{Z}_3$ (respectively, $\mathbb{Z}_5$) is a field, we obtain from Lemma 3.16 that $\gamma_{\mathcal{R}}((\Omega(R))^c) = 4$.

(5) Assume that $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3$. Note that $T = \mathbb{Z} \times \mathbb{Z}$ is a reduced ring $|Min(T)| = 2$, $Min(T) \cap Max(T) = \emptyset$, and $\mathbb{Z}_3$ is a field. So, $\gamma_{\mathcal{R}}((\Omega(R))^c) = 5$ by Lemma 3.18. □

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