Well-posedness and general decay for a Lamé viscoelastic system with logarithmic damping

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Abstract

Purpose – We study a Lamé-type viscoelastic system with logarithmic velocity damping and quantify its stabilizing effect. The dissipation is weak near zero ($\sigma(v)\cdot v \sim |v|^4$ as $|v|\to 0$) yet becomes stronger than linear for large speeds. Our goal is to prove global well-posedness and quantify the long-time decay of the natural energy.
Design/methodology/approach – We combine the Faedo–Galerkin scheme, Aubin–Lions compactness, and Minty's monotonicity to construct weak solutions and derive an energy identity. A Lyapunov function adapted to the logarithmic dissipation links the decay of the relaxation kernel g to the mechanical energy, assuming $g \geq 0$ is nonincreasing and $-g'(t) \geq \xi(t)g(t)$ with nonincreasing ξ.
Findings – We obtain global existence and uniqueness of weak solutions and a general decay estimate $E(t) \leq C\exp\left(-\kappa\int_0^t \xi(s)\,ds\right)$. Hence, exponential decay holds when $\inf_{t\geq 0}\xi(t) > 0$, whereas polynomial rates follow if $\xi(t)\sim c(1+t)^{-1}$. The results clarify the combined role of Lamé ellipticity, hereditary memory, and logarithmic damping in stabilization.
Originality/value – Prior Lamé–viscoelastic studies used logarithmic terms mainly as sources; here the damping itself is logarithmic. This reveals a distinct stabilization mechanism and unifies exponential and polynomial regimes.

Keywords Lamé system, Viscoelasticity, Logarithmic damping, General decay, Lyapunov method, Relaxation kernel, Weak solutions, Energy decay
Paper type Research article

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ with smooth boundary $\partial\Omega$. This paper investigates the well-posedness and asymptotic behavior of a viscoelastic Lamé system with logarithmic damping, governed by the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) + \sigma(u_t) + \displaystyle\int_0^t g(t-s)\Delta u(s)\,ds = 0, & x\in\Omega,\ t>0,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x\in\Omega,\\ u(x,t) = 0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

where the unknown $u = (u_1,\ldots,u_n) : \Omega\times\mathbb{R}^+ \to \mathbb{R}^n$ is the displacement vector field, μ and λ are the Lamé constants satisfying $\mu > 0$ and $\lambda+\mu > 0$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a relaxation kernel

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modeling the viscoelastic memory, and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is the logarithmic damping operator defined by $\sigma(v) = v \log\left(1 + |v|^2\right)$ for $v \in \mathbb{R}^n$.

The Lamé system (1.1) constitutes the fundamental linear model of isotropic, homogeneous elasticity. It describes wave propagation in elastic solids where the stress tensor depends linearly on both the strain (via the instantaneous Lamé operator $(\mu\Delta u + (\lambda + \mu)\bigtriangledown(\mathrm{div}\ u))$ and its history (through the Volterra integral term). From a physical viewpoint, this system governs the dynamics of numerous engineering and geophysical materials; we refer to the classical monograph [1] for a comprehensive derivation and discussion of the Lamé equations. The mathematical treatment of such vector-valued boundary value problems is covered in detail in Ref. [2].

The logarithmic damping term $\sigma(u_t)$ introduces a nonlinear frictional resistance that exhibits distinctive qualitative features: it is weaker than linear damping near the origin ($\sigma(v) \cdot v \sim |v|^4$ as $|v| \to 0$) yet becomes slightly stronger at large velocities ($\sigma(v) \cdot v \sim |v|^2 \log(1 + |v|^2)$ as $|v| \to \infty$). Such behavior is relevant for modeling complex viscoelastic materials, including certain polymers and biological tissues, where the dissipative mechanism deviates from classical linear or polynomial laws.

Mathematically, the analysis of Lamé systems presents unique challenges compared to scalar wave equations due to the vectorial nature of the displacement field and the coupling of components through the term $\bigtriangledown(\mathrm{div}\ u)$. These challenges are compounded when memory effects and non-standard damping mechanisms are introduced. While the stability of scalar viscoelastic wave equations has been extensively studied (see, e.g. Ref. [3] for polynomial decay rates), the literature on vectorial Lamé systems with memory is less developed but growing rapidly.

Recent contributions on Lamé systems include [4], who examined a purely viscoelastic Lamé model, and [5], which studies attractors for coupled Lamé systems. The work of [6] investigates synchronization phenomena in coupled Lamé systems, establishing conditions for exponential synchronization. Various damping mechanisms have been considered in the context of Lamé systems, including frictional damping [7], viscoelastic damping with infinite memories [8], and nonlinear damping with source terms [9]. Logarithmic nonlinearities have been investigated mainly as source terms in scalar wave and plate models [10–12]. However, their role as *damping mechanisms* in vector-valued elasticity systems governed by the Lamé operator has not yet been addressed in the literature. This distinction is essential, since the Lamé system involves component coupling through the operator $\bigtriangledown(\bigtriangledown \cdot u)$, which fundamentally alters both the functional framework and the energy analysis.

The present work aims to bridge this significant gap by providing a rigorous analysis of the Lamé system (1.1) with logarithmic damping and viscoelastic memory. Our main contributions are twofold. First, we prove the existence and uniqueness of global weak solutions using Faedo–Galerkin approximation combined with monotonicity arguments tailored to the logarithmic damping. Second, by constructing a suitable Lyapunov functional,we establish a general decay result of the form

$$E(t) \le C \exp\left(-\kappa \int_0^t \xi(s)\, ds\right), \qquad t \ge 0,$$

where $E(t)$ is the total energy and $\xi(t)$ is a function determined by the relaxation kernel g. This result recovers exponential and polynomial decay rates as special cases, depending on the behavior of g. Our analysis handles the interplay between three challenging features: the vectorial structure of the Lamé operator, the hereditary memory term, and the non-standard logarithmic damping.

The paper is organized as follows. Section 2 introduces the functional framework and assumptions. Section 3 proves the well-posedness of weak solutions. Section 4 establishes the general decay result via a Lyapunov functional. Finally, Section 5 offers concluding remarks.

2. Preliminaries

In this section, we collect some materials and assumptions needed to establish our main results. Throughout the paper, C denotes a positive constant which may change from line to line.

2.1 Function spaces

Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be a bounded Lipschitz domain. We define

$$H := L^2(\Omega)^n, \qquad V := H_0^1(\Omega)^n, \qquad V^* := \text{dual of } V.$$

For $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in H, we define the inner product and norm by

$$\langle u, v\rangle_H := \sum_{i=1}^{n} \int_\Omega u_i v_i \, dx, \qquad \|u\|_H := \sqrt{\langle u, u\rangle_H}.$$

Similarly, in $V = H_0^1(\Omega)^n$, we use

$$\langle \nabla u, \nabla v\rangle := \sum_{i=1}^{n} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx,$$

and define the H^1-seminorm

$$\|\nabla u\| := \sqrt{\langle \nabla u, \nabla u\rangle}.$$

For vector fields $z : \Omega \to \mathbb{R}^n$, we also use the notation

$$\|z\| := \|z\|_H = \left(\int_\Omega |z(x)|^2 \, dx\right)^{1/2}.$$

By Poincaré's inequality, $\|\nabla u\|$ is equivalent to the usual H^1-norm on V.

2.2 Problem setting

The unknown $u(x,t) \in \mathbb{R}^n$ represents the displacement vector field of an elastic body occupying the domain Ω.

We study the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla(\operatorname{div} u) + \sigma(u_t) + \int_0^t g(t-s)\Delta u(s)\, ds = 0, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.1}$$

Here $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ denotes the logarithmic damping operator,

$$\sigma(v) = v \log\left(1 + |v|^2\right), \qquad v \in \mathbb{R}^n,$$

and $g : [0, \infty) \to \mathbb{R}_+$ is a relaxation kernel.

2.3 Lamé operator and associated bilinear form

We consider the vector Lamé operator

$$\mathcal{L}u := \mu\,\Delta u + (\lambda+\mu)\,\nabla(\operatorname{div} u), \qquad u:\Omega\to\mathbb{R}^n,$$

with Lamé parameters $\mu > 0$, $\lambda \in \mathbb{R}$.

We use the notation

$$\nabla u : \nabla v := \sum_{i=1}^{n} \nabla u_i \cdot \nabla v_i.$$

Define, for $u, v \in V$,

$$a(u,v) := \mu \int_\Omega \nabla u : \nabla v\,dx + (\lambda+\mu)\int_\Omega (\operatorname{div} u)(\operatorname{div} v)\,dx. \tag{2.2}$$

In particular,

$$a(u,u) = \mu\|\nabla u\|^2 + (\lambda+\mu)\|\operatorname{div} u\|^2.$$

Ellipticity condition. We impose the following assumption on the Lamé parameters:
(A1) The Lamé parameters satisfy

$$\mu > 0 \quad \text{and} \quad \lambda+\mu \geq 0.$$

This condition guarantees that the bilinear form $a(\cdot,\cdot)$ is coercive on V.

Lemma 2.1. (Coercivity and boundedness of a). *Under Assumption* (A1), *there exist constants $K_1, K_2 > 0$ (depending on Ω, n, λ, μ) such that for all $u \in V$,*

$$K_1\|\nabla u\|^2 \leq a(u,u) \leq K_2\|\nabla u\|^2. \tag{2.3}$$

Proof. **Upper bound.** For a.e. $x \in \Omega$, write $\operatorname{div} u(x) = \sum_{i=1}^n \partial_i u_i(x)$. By Cauchy–Schwarz,

$$(\operatorname{div} u(x))^2 = \left(\sum_{i=1}^n \partial_i u_i(x)\right)^2 \leq n\sum_{i=1}^n (\partial_i u_i(x))^2 \leq n\,|\nabla u(x)|^2.$$

Integrating gives $\int_\Omega |\operatorname{div} u|^2 dx \leq n\int_\Omega |\nabla u|^2 dx$. Hence

$$a(u,u) = \mu\int_\Omega |\nabla u|^2 dx + (\lambda+\mu)\int_\Omega |\operatorname{div} u|^2 dx \leq (\mu + n(\lambda+\mu))\int_\Omega |\nabla u|^2 dx,$$

which yields the right inequality in (2.3) with $K_2 = \mu + n(\lambda+\mu)$.

Lower bound. From Assumption (A1), $\mu > 0$ and $(\lambda+\mu) \geq 0$. Therefore both terms in $a(u, u)$ are nonnegative and

$$a(u,u) \geq \mu\int_\Omega |\nabla u|^2 dx = \mu\|\nabla u\|^2,$$

which gives the left inequality in (2.3) with $K_1 = \mu$.

Finally, since $V = H_0^1(\Omega)^n$ satisfies Poincaré's inequality $\|u\| \leq C_P\|\nabla u\|$, the bounds (2.3) imply that $a(\cdot,\cdot)$ is equivalent to the H^1-seminorm (and to the H^1-norm) on V.

Consequently, we equip V with the norm

$$\|u\|_V := \sqrt{a(u,u)},$$

which is equivalent to the standard $H_0^1(\Omega)^n$ norm. □

2.4 Phase space

The natural phase space for problem (2.1) is

$$\mathcal{H} := V \times H.$$

For $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$ in $\mathcal{H}$, we define the inner product

$$\left\langle U, \tilde{U} \right\rangle_{\mathcal{H}} := a\left(u, \tilde{u}\right) + \left\langle v, \tilde{v} \right\rangle_H,$$

with associated norm

$$\|U\|_{\mathcal{H}}^2 = a(u,u) + \|v\|_H^2.$$

2.5 Relaxation kernel and memory functional

Kernel hypotheses. We impose the following assumptions on the relaxation kernel g:

(A2) (Kernel regularity) $g \in C^1([0,\infty))$, $g(t) \geq 0$, $g'(t) \leq 0$ for all $t \geq 0$, with $g(0) > 0$, and

$$\mu - \int_0^\infty g(s)\, ds =: \ell > 0.$$

This ensures positivity of the elastic part and well-posedness of the associated energy.

(A3) There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t)\, g(t), \qquad t \geq 0.$$

Moreover, ξ satisfies, for some constant $L > 0$,

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq L, \qquad \xi'(t) \leq 0, \qquad \int_0^\infty \xi(s)\, ds = \infty.$$

These conditions guarantee that the decay of g transfers to the energy, and allow explicit decay rates.

For w: $[0, T] \to V$, define the memory functional

$$(g \circ \nabla w)(t) := \frac{1}{2}\int_0^t g(t-s)\|\nabla w(t) - \nabla w(s)\|^2\, ds. \tag{2.4}$$

2.6 Logarithmic damping: definition and lemmas

(A4) (Logarithmic damping) The damping function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\sigma(v) = v \log\left(1 + |v|^2\right), \quad v \in \mathbb{R}^n.$$

It satisfies the following properties:

(1) *Monotonicity:* $(\sigma(v) - \sigma(w)) \cdot (v - w) \geq 0$, for all $v, w \in \mathbb{R}^n$.

(2) *Dissipativity:* $\sigma(v)\cdot v = |v|^2\log(1+|v|^2) \geq c_0 \min\{|v|^4, |v|^2\}$, for some universal $c_0 > 0$.

(3) *Local Lipschitz continuity:* For every $R > 0$, there exists $C_R > 0$ such that $|\sigma(v) - \sigma(w)| \leq C_R|v-w|$ whenever $|v|, |w| \leq R$.

Proof.

(1) *Monotonicity.* Since $\sigma = \nabla\Psi$ with the convex potential

$$\Psi(v) = \frac{1}{2}\left(1+|v|^2\right)\log\left(1+|v|^2\right) - \frac{1}{2}|v|^2,$$

it follows that $\nabla\Psi$ is monotone.

(2) *Dissipativity.* By direct computation,

$$\sigma(v)\cdot v = |v|^2\log\left(1+|v|^2\right) \geq 0.$$

For $|v|\leq 1, \log(1+|v|^2) \geq |v|^2/2$, hence $\sigma(v)\cdot v \geq \frac{1}{2}|v|^4$. For $|v| \geq \sqrt{e-1}, \log(1+|v|^2) \geq 1$, hence $\sigma(v)\cdot v \geq |v|^2$. Thus the unified bound $\sigma(v)\cdot v \geq c_0 \min\{|v|^4, |v|^2\}$ holds.

(3) *Local Lipschitz continuity.* The Jacobian matrix is

$$D\sigma(v) = \log\left(1+|v|^2\right)I + \frac{2}{1+|v|^2}\, v\otimes v,$$

which is bounded on the ball $\{|v| \leq R\}$. Therefore σ is Lipschitz continuous on bounded sets, which yields the stated estimate. □

Assumption (A4) gathers the main properties of the logarithmic damping, which will be used later in the Galerkin approximation and in the decay analysis.

2.7 Energy functional and dissipation

We define, for a (weak) solution u,

$$E(t) := \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\, a(u(t), u(t)) + (g\circ\nabla u)(t),$$

where $(g\circ\nabla u)(t)$ is given by (2.4).

Lemma 2.2. (Energy identity and monotonicity). *Under* (A1)–(A4), *and for every weak solution* $u \in L^\infty(0,T;V)$ *with* $u_t \in L^\infty(0,T;H)$*, the energy satisfies* $E \in W^{1,1}(0,T)$ *and for a.e.* $t \in (0,T)$,

$$E'(t) = -\int_\Omega \sigma(u_t(t))\cdot u_t(t)\,dx - \frac{g(t)}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla u(t) - \nabla u(s)\|^2\,ds \leq 0. \tag{2.5}$$

In particular, E is nonincreasing on $[0,T]$.

Proof. The derivation is first established for smooth solutions; the extension to weak solutions follows from the Galerkin approximation together with compactness and monotonicity arguments.

Step 1: Test (2.1) with $u_t(t)$ in H and integrate over Ω:

$$(u_{tt}, u_t) - \mu(\Delta u, u_t) - (\lambda + \mu)(\nabla(\nabla \cdot u), u_t) + \int_{\Omega} \sigma(u_t) \cdot u_t \, dx + \left(\int_0^t g(t-s)\Delta u(s)\, ds,\ u_t \right) = 0.$$

Boundary terms vanish by $u|_{\partial\Omega} = 0$. Using the definition of $a(\cdot, \cdot)$ and symmetry (guaranteed by (A1)),

$$(u_{tt}, u_t) = \frac{d}{dt}\frac{1}{2}\|u_t\|^2, \qquad -\mu(\Delta u, u_t) - (\lambda+\mu)(\nabla(\nabla\cdot u), u_t) = a(u, u_t) = \frac{d}{dt}\frac{1}{2}\, a(u,u).$$

The damping contribution satisfies, by dissipativity in (A4), $\int_\Omega \sigma(u_t) \cdot u_t \, dx \geq 0$. For the memory term, integrate by parts in space:

$$\left(\int_0^t g(t-s)\Delta u(s)\, ds,\ u_t(t) \right) = -\int_0^t g(t-s)\, (\nabla u(s), \nabla u_t(t))\, ds.$$

Summing up,

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2} a(u,u) \right) + \int_{\Omega} \sigma(u_t) \cdot u_t \, dx - \int_0^t g(t-s)\, (\nabla u(s), \nabla u_t(t))\, ds = 0. \quad (2.6)$$

Step 2: Dafermos identity for $(g \circ \nabla u)\,(t)$. By (A2)–(A3), $(g \circ \nabla u)\,(t)$ is absolutely continuous and

$$\begin{aligned} \frac{d}{dt}(g \circ \nabla u)(t) &= -\frac{g(t)}{2}\|\nabla u(t)\|^2 + \int_0^t g(t-s)\, (\nabla u_t(t),\ \nabla u(t) - \nabla u(s))\, ds \\ &\quad - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla u(t) - \nabla u(s)\|^2\, ds. \end{aligned} \quad (2.7)$$

Split the mixed integral:

$$\int_0^t g(t-s)\, (\nabla u_t(t), \nabla u(t) - \nabla u(s))\, ds = \underbrace{\int_0^t g(t-s)\, (\nabla u_t(t), \nabla u(t))\, ds}_{\mathcal{I}_1} - \underbrace{\int_0^t g(t-s)\, (\nabla u_t(t), \nabla u(s))\, ds}_{\mathcal{I}_2}.$$

Note that the memory term in (2.6) equals $+\mathcal{I}_2$:

$$-\int_0^t g(t-s)\, (\nabla u(s), \nabla u_t(t))\, ds = +\mathcal{I}_2.$$

Step 3: Combining identities. Add (2.7) to (2.6). The terms $\pm\mathcal{I}_2$ cancel. We obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}a(u,u) + (g\circ\nabla u)(t)\right) + \int_\Omega \sigma(u_t)\cdot u_t\,dx$$
$$= -\frac{g(t)}{2}\|\nabla u(t)\|^2 + \mathcal{I}_1 - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla u(t) - \nabla u(s)\|^2\,ds.$$

But $\mathcal{I}_1 = \int_0^t g(t-s)\,(\nabla u_t(t), \nabla u(t))\,ds$ is already part of $\frac{d}{dt}(g\circ\nabla u)(t)$ in (2.7), hence the whole right-hand side is exactly the one displayed in (2.5), i.e.

$$E'(t) = -\int_\Omega \sigma(u_t(t))\cdot u_t(t)\,dx - \frac{g(t)}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla u(t) - \nabla u(s)\|^2\,ds.$$

Step 4: Sign of $E'(t)$. We check the sign of each contribution in (2.5):

(1) By dissipativity in (A4):

$$\int_\Omega \sigma(u_t)\cdot u_t\,dx \geq 0.$$

(2) By (A2):

$$g(t) \geq 0.$$

(3) By (A3):

$$-g'(t-s) \geq 0 \quad \text{for all } s \in [0,t]\,.$$

Hence every term on the right-hand side of (2.5) is nonpositive, and therefore

$$E'(t) \leq 0.$$

Integrability and absolute continuity. Integrating (2.5) over $(0, t)$ gives

$$E(0) - E(t) = \int_0^t \int_\Omega \sigma(u_\tau)\cdot u_\tau\,dx\,d\tau + \frac{1}{2}\int_0^t g(\tau)\|\nabla u(\tau)\|^2\,d\tau$$
$$+ \frac{1}{2}\int_0^t \int_0^\tau (-g'(\tau-s))\|\nabla u(\tau) - \nabla u(s)\|^2\,ds\,d\tau.$$

The three terms on the right are nonnegative and finite on $(0, T)$, hence the right-hand side of (2.5) belongs to $L^1(0, T)$. Therefore $E \in W^{1,1}(0, T)$ and (2.5) holds for a.e. $t \in (0, T)$.

Remark on weak solutions. The above derivation was carried out formally for smooth functions. For the weak solution constructed via the Galerkin scheme in Section 3, all steps can be rigorously justified: uniform a priori bounds from (A1)–(A4) yield compactness (Aubin–Lions), while the monotonicity of σ and the Dafermos identity for $(g \circ \nabla u)(t)$ ensure that (2.5) also holds for the weak solution. □

3. Existence and uniqueness of solutions

3.1 Weak formulation

Let $T > 0$. A function

$$u \in L^\infty(0, T; V), \qquad u_t \in L^\infty(0, T; H), \qquad u_{tt} \in L^2\left(0, T; V^*\right)$$

is called a *weak solution* to (2.1) on $(0, T)$ if $u(0) = u_0$, $u_t(0) = u_1$, and for a.e. $t \in (0, T)$,

$$\langle u_{tt}(t), v\rangle + a(u(t), v) + \int_\Omega \sigma(u_t(t))\cdot v\,dx - \int_0^t g(t-s)\,(\nabla u(s), \nabla v)\,ds = 0, \qquad \forall\, v \in V, \tag{3.1}$$

where $(\cdot, \cdot)$ denotes the H-inner product and $\langle \cdot, \cdot\rangle$ the duality between V^* and V.

Theorem 3.1. (Existence and uniqueness). *Assume* (A1)–(A4) *and let* $(u_0, u_1) \in V \times H$. *Then there exists a unique weak solution*

$$u \in L^\infty(0, T; V), \qquad u_t \in L^\infty(0, T; H), \qquad u_{tt} \in L^2\left(0, T; V^*\right)$$

to problem (2.1) *in the weak sense* (3.1). *Moreover, the energy identity* (2.5) *holds on* $(0, T)$ *and the energy* $E(t)$ *is nonincreasing.*

Proof.

Step 1. Weak formulation. We recall that u is a weak solution if $u(0) = u_0$, $u_t(0) = u_1$, and (3.1) holds for all $v \in V$.

Step 2. Galerkin approximation. Let $\{w_j\}_{j\geq 1} \subset V$ be an H-orthonormal basis of eigenfunctions of the Lamé operator, and set $V_m := \text{span}\{w_1, \ldots, w_m\}$. We seek $u^m(t) = \sum_{j=1}^m d_j^m(t) w_j$ solving, for all $v \in V_m$,

$$\left(u_{tt}^m(t), v\right) + a(u^m(t), v) + \int_\Omega \sigma\left(u_t^m(t)\right)\cdot v\,dx + \int_0^t g(t-s)\,(\nabla u^m(s), \nabla v)\,ds = 0, \tag{3.2}$$

with $u^m(0) = P_m u_0$, $u_t^m(0) = P_m u_1$, where $P_m\colon H \to V_m$ is the H-orthogonal projector. By the Carathéodory existence theorem (see, e.g. Amann [13]), the Galerkin system (3.2) admits a local solution.

Step 3. A priori estimates. Testing (3.2) with $v = u_t^m(t)$ in H and repeating the energy calculation of Lemma 2.2 (which is fully justified here since u^m is smooth in time) yields the discrete energy identity

$$\begin{aligned}\frac{d}{dt}E_m(t) = -\int_\Omega \sigma\left(u_t^m(t)\right)\cdot u_t^m(t)\,dx - \frac{g(t)}{2}\|\nabla u^m(t)\|^2 - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla u^m(t)\\ - \nabla u^m(s)\|^2\,ds \leq 0,\end{aligned} \tag{3.3}$$

where

$$E_m(t) = \frac{1}{2}\|u_t^m(t)\|^2 + \frac{1}{2}\,a(u^m(t), u^m(t)) + (g\circ\nabla u^m)(t).$$

By the coercivity of $a(\cdot,\ \cdot)$ (A1) together with Poincaré's inequality, $a(u^m, u^m) \simeq \|u^m\|_V^2$, so (3.3) implies the uniform bound

$$\sup_{t\in[0,T]}\left(\|u_t^m(t)\|^2 + \|u^m(t)\|_V^2\right) + \sup_{t\in[0,T]}(g\circ\nabla u^m)(t) \leq C, \tag{3.4}$$

with a constant $C > 0$ depending only on the initial energy $E_m(0) \leq E(0)$ and independent of m.

Step 4. Compactness and passage to the limit. From the uniform bound (3.4), we extract a subsequence (still denoted by u^m) such that

$$u^m \rightharpoonup^* u \quad \text{in } L^\infty(0,T;V), \qquad u_t^m \rightharpoonup^* u_t \quad \text{in } L^\infty(0,T;H).$$

In addition, testing (3.2) with arbitrary $v \in V_m$ and using assumptions (A1)–(A4) provides a uniform estimate for u_{tt}^m in $L^2(0,\ T;\ V^*)$. Together with (3.4), the Aubin–Lions lemma (see Simon [14]) yields the compactness

$$u^m \to u \quad \text{in } L^2(0,T;H), \qquad u_t^m \to u_t \quad \text{in } L^2\left(0,T;V^*\right).$$

Linear terms. Passing to the limit in all linear contributions of (3.2) is straightforward by weak convergence.

Nonlinear damping. For the damping operator σ, monotonicity (A4) and Minty's method (see Minty [15] and Brézis [16]) give the identification of the weak limit:

$$\int_0^T \int_\Omega \left(\sigma\left(u_t^m\right) - \sigma(v)\right)\cdot\left(u_t^m - v\right) dx\, dt \geq 0, \qquad \forall v \in L^2(0,T;H),$$

which implies

$$\sigma\left(u_t^m\right) \rightharpoonup \sigma(u_t) \quad \text{in } L^1_{\text{loc}}((0,T)\times\Omega).$$

Memory term. The Dafermos identity for $(g \circ \nabla\cdot)$, valid under (A2)–(A3), combined with the strong convergence of u^m in $L^2(0,\ T;\ H)$, yields

$$\int_0^t g(t-s)\,(\nabla u^m(s), \nabla v)\, ds \to \int_0^t g(t-s)\,(\nabla u(s), \nabla v)\, ds, \qquad \forall v \in V,\ \text{a.e. } t \in (0,T).$$

Limit identification. Hence the limit function u satisfies the weak formulation (3.1). The initial conditions follow from $u^m(0) = P_m u_0$, $u_t^m(0) = P_m u_1$, together with weak* continuity in $L^\infty(0,\ T;\ V \times H)$.

Step 5. Uniqueness. Let u and v be two weak solutions corresponding to the same initial data, and set $w = u - v$. Then w satisfies (3.1) with homogeneous initial conditions:

$$\begin{aligned}\langle w_{tt}, \phi\rangle + a(w,\phi) + \int_\Omega (\sigma(u_t) - \sigma(v_t))\cdot\phi\, dx + \int_0^t g(t-s)\,(\nabla w(s), \nabla\phi)\, ds \\ = 0, \qquad \forall \phi \in V.\end{aligned}$$

Choosing $\phi = w_t$ and repeating the energy calculation of Lemma 2.2, we obtain

$$\frac{d}{dt}E_w(t) = -\int_{\Omega}(\sigma(u_t) - \sigma(v_t))\cdot(u_t - v_t)\,dx - \frac{g(t)}{2}\|\nabla w(t)\|^2 - \frac{1}{2}\int_0^t (-g'(t-s))\|\nabla w(t) - \nabla w(s)\|^2\,ds,$$

where

$$E_w(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\,a(w(t), w(t)) + (g\circ\nabla w)(t).$$

By monotonicity of σ (A4) and the positivity of the memory terms (A2)–(A3), the right-hand side is nonpositive, hence

$$\frac{d}{dt}E_w(t) \le 0.$$

Since $E_w(0) = 0$ (zero initial data), it follows that $E_w(t) \equiv 0$ for all $t \ge 0$, and thus $w \equiv 0$. This proves uniqueness of weak solutions. □

4. Energy decay

In this section we establish the asymptotic stability of the system. Recall from Lemma 2.2 that the total energy

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\,a(u(t), u(t)) + (g\circ\nabla u)(t)$$

is nonincreasing. Our goal is to quantify its decay rate under the structural assumptions (A2)–(A3) on the relaxation kernel g.

4.1 Lyapunov functional

For a parameter $\varepsilon > 0$ define

$$\mathcal{V}(t) := E(t) + \varepsilon\,\mathcal{J}(t), \qquad \mathcal{J}(t) := \int_0^t g(t-s)\,(\nabla u(t),\ \nabla u(t) - \nabla u(s))\,ds. \tag{4.1}$$

Lemma 4.1. (Equivalence with the energy). *Assume* (A1)–(A2). *Then there exist* $\varepsilon_0, c_1, c_2 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and all* $t \ge 0$,

$$c_1\,E(t) \le \mathcal{V}(t) \le c_2\,E(t). \tag{4.2}$$

Proof. Recall

$$(g\circ\nabla u)(t) = \frac{1}{2}\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds.$$

To estimate $\mathcal{J}(t)$, rewrite it as a weighted inner product. Define

$$f(s) := \sqrt{g(t-s)}\,\nabla u(t), \qquad h(s) := \sqrt{g(t-s)}\,(\nabla u(t) - \nabla u(s)).$$

Then $\mathcal{J}(t) = \int_0^t f(s) \cdot h(s)\, ds$, and by Cauchy–Schwarz,

$$|\mathcal{J}(t)| \leq \left(\int_0^t |f(s)|^2\, ds \right)^{1/2} \left(\int_0^t |h(s)|^2\, ds \right)^{1/2}.$$

A direct computation gives

$$\int_0^t |f(s)|^2\, ds = \int_0^t g(t-s) \|\nabla u(t)\|^2\, ds = \|\nabla u(t)\|^2\, G(t), \qquad G(t) := \int_0^t g(t-s)\, ds,$$

and

$$\int_0^t |h(s)|^2\, ds = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|^2\, ds = 2\, (g \circ \nabla u)(t).$$

Therefore

$$|\mathcal{J}(t)| \leq \|\nabla u(t)\| \sqrt{G(t)}\, \left(2\, (g \circ \nabla u)(t)\right)^{1/2}.$$

By (A2) we have $G(t) \leq G_\infty := \int_0^\infty g(s)\, ds < \infty$, hence

$$|\mathcal{J}(t)| \leq \|\nabla u(t)\| \sqrt{G_\infty}\, \left(2\, (g \circ \nabla u)(t)\right)^{1/2}.$$

Finally, apply Young's inequality to the product

$$a := \|\nabla u(t)\| \sqrt{G_\infty}, \qquad b := \left(2\, (g \circ \nabla u)(t)\right)^{1/2},$$

which yields, for any $\alpha > 0$,

$$ab \leq \frac{a^2}{2\alpha} + \frac{\alpha b^2}{2} = \frac{G_\infty}{2\alpha} \|\nabla u(t)\|^2 + \alpha\, (g \circ \nabla u)(t).$$

This proves

$$|\mathcal{J}(t)| \leq \frac{G_\infty}{2\alpha} \|\nabla u(t)\|^2 + \alpha\, (g \circ \nabla u)(t). \tag{4.3}$$

To further estimate $\mathcal{J}(t)$, we use the coercivity of the bilinear form $a(\cdot,\ \cdot)$. By Lemma 2.1, there exist constants $K_1, K_2 > 0$ such that

$$K_1 \|\nabla u(t)\|^2 \leq a(u(t), u(t)) \leq K_2 \|\nabla u(t)\|^2.$$

In particular this implies

$$\|\nabla u(t)\|^2 \leq \frac{1}{K_1}\, a(u(t), u(t)).$$

Substituting this bound into inequality (4.3) we obtain

$$|\mathcal{J}(t)| \leq \frac{G_\infty}{2\alpha K_1}\, a(u(t), u(t)) + \alpha\, (g \circ \nabla u)(t). \tag{4.4}$$

Step 3: Lower bound for $\mathcal{V}$. Starting from $\mathcal{V} = E(t) + \varepsilon \mathcal{J}(t)$ and using the triangle inequality,

$$\mathcal{V}(t) = E(t) + \varepsilon \mathcal{J}(t) \geq E(t) - \varepsilon \, |\mathcal{J}(t)|.$$

Invoking (4.4), we obtain

$$\mathcal{V}(t) \geq \frac{1}{2}\|u_t(t)\|^2 + \left(\frac{1}{2} - \varepsilon \, \frac{G_\infty}{2\alpha K_1}\right) a(u(t), u(t)) + (1 - \varepsilon\alpha) \, (g \circ \nabla u)(t).$$

It is convenient to write the elastic coefficient relatively to $\frac{1}{2}$, namely

$$\frac{1}{2} - \varepsilon \, \frac{G_\infty}{2\alpha K_1} = \frac{1}{2}\left(1 - \varepsilon \, \frac{G_\infty}{\alpha K_1}\right),$$

so that the three coefficients in front of the energy components are, respectively,

$$\underbrace{1}_{\text{for } \frac{1}{2}\|u_t\|^2}, \qquad \underbrace{1 - \varepsilon \, \frac{G_\infty}{\alpha K_1}}_{\text{for } \frac{1}{2}a(u,u)}, \qquad \underbrace{1 - \varepsilon\alpha}_{\text{for } (g \circ \nabla u)}.$$

Choose $\alpha = 1$ to simplify the expressions. Then select $\varepsilon_0 > 0$ so small that

$$\varepsilon_0 \, \frac{G_\infty}{2K_1} < \frac{1}{4} \quad \text{and} \quad \varepsilon_0 < \frac{1}{2}.$$

These two conditions imply, for every $\varepsilon \in (0, \varepsilon_0]$,

$$1 - \varepsilon \, \frac{G_\infty}{K_1} \geq \frac{1}{2} \qquad \text{and} \qquad 1 - \varepsilon \geq \frac{1}{2},$$

hence all three coefficients above are positive and uniformly bounded away from zero.

Now define

$$m(\varepsilon) := \min\left\{ 1, \ 1 - \varepsilon \, \frac{G_\infty}{K_1}, \ 1 - \varepsilon \right\}.$$

With this choice we have, componentwise,

$$\frac{1}{2}\|u_t\|^2 \geq m(\varepsilon) \cdot \frac{1}{2}\|u_t\|^2, \qquad \frac{1}{2}\left(1 - \varepsilon \, \frac{G_\infty}{K_1}\right) a(u,u) \geq m(\varepsilon) \cdot \frac{1}{2} a(u,u),$$

$$(1 - \varepsilon) \, (g \circ \nabla u)(t) \geq m(\varepsilon) \cdot (g \circ \nabla u)(t).$$

Summing the three inequalities yields

$$\mathcal{V}(t) \geq m(\varepsilon) \left(\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2} a(u(t), u(t)) + (g \circ \nabla u)(t)\right) = m(\varepsilon) \, E(t).$$

Since $m(\varepsilon) \geq \frac{1}{2}$ for all $\varepsilon \in (0, \varepsilon_0]$, we may set $c_1 := \frac{1}{2}$ (or any $c_1 \in \left(0, \frac{1}{2}\right]\right)$, independent of t, and conclude

$$\mathcal{V}(t) \geq c_1 E(t), \qquad \forall\, t \geq 0.$$

Step 4: Upper bound for $\mathcal{V}$. Starting from $\mathcal{V} = E(t) + \varepsilon \mathcal{J}(t)$ and using (4.4) with the triangle inequality,

$$\begin{aligned}\mathcal{V}(t) &= E(t) + \varepsilon\, \mathcal{J}(t) \leq E(t) + \varepsilon\, |\mathcal{J}(t)| \\ &\leq \frac{1}{2}\|u_t(t)\|^2 + \left(\frac{1}{2} + \varepsilon\, \frac{G_\infty}{2\alpha K_1}\right) a(u(t), u(t)) + (1 + \varepsilon\alpha)\, (g \circ \nabla u)(t).\end{aligned}$$

Choose $\alpha = 1$ for simplicity. Then

$$\mathcal{V}(t) \leq \frac{1}{2}\|u_t(t)\|^2 + \left(\frac{1}{2} + \varepsilon\, \frac{G_\infty}{2K_1}\right) a(u(t), u(t)) + (1 + \varepsilon)\, (g \circ \nabla u)(t).$$

We now compare the coefficients with those of $E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2} a(u, u) + (g \circ \nabla u)(t)$. Let

$$c_2(\varepsilon) \;:=\; \max\left\{ 1,\; 1 + \varepsilon\, \frac{G_\infty}{K_1},\; 1 + \varepsilon \right\}.$$

Then, componentwise,

$$\frac{1}{2} \leq c_2(\varepsilon) \cdot \frac{1}{2}, \qquad \frac{1}{2} + \varepsilon\, \frac{G_\infty}{2K_1} \leq c_2(\varepsilon) \cdot \frac{1}{2}, \qquad 1 + \varepsilon \leq c_2(\varepsilon) \cdot 1.$$

Therefore

$$\mathcal{V}(t) \leq c_2(\varepsilon) \left(\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2} a(u(t), u(t)) + (g \circ \nabla u)(t) \right) = c_2(\varepsilon)\, E(t).$$

Finally, since $\varepsilon \in (0, \varepsilon_0]$, we may fix

$$c_2 \;:=\; \max\left\{ 1,\; 1 + \varepsilon_0\, \frac{G_\infty}{K_1},\; 1 + \varepsilon_0 \right\} > 1,$$

which is independent of t. Hence

$$\mathcal{V}(t) \leq c_2\, E(t), \qquad \forall t \geq 0.$$

Combining the lower and upper bounds obtained in Steps 3 and 4, we conclude (4.2) with constants c_1, $c_2 > 0$ independent of t. □

Proposition 4.2. (Differential inequality for $\mathcal{V}$). *Assume* (A1)–(A4). *There exist* $\varepsilon_0 \in (0, 1)$ *and* c_*, $c^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and a.e.* $t > 0$,

$$\mathcal{V}'(t) \leq -\int_\Omega \sigma(u_t)\cdot u_t\,dx - c_*\,g(t)\|\nabla u(t)\|^2 - c^*\int_0^t(-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds. \tag{4.5}$$

Proof. We argue first for smooth solutions and then pass to weak solutions by Galerkin approximation and lower semicontinuity, which preserves the differential inequality.

Recall $\mathcal{V}(t) = E(t) + \varepsilon\,\mathcal{J}(t)$ with $\varepsilon > 0$. Differentiating,

$$\mathcal{V}'(t) = E'(t) + \varepsilon\,\mathcal{J}'(t).$$

From the energy identity (2.5), for a.e. $t > 0$,

$$E'(t) = -\int_\Omega \sigma(u_t(t))\cdot u_t(t)\,dx - \frac{g(t)}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t(-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2\,ds.$$

Moreover, using Dafermos' calculus under (A2)–(A3),

$$\mathcal{J}'(t) = \frac{g(t)}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t(-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2\,ds + \int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds.$$

Combining,

$$\begin{aligned}\mathcal{V}'(t) \quad &\leq -\int_\Omega \sigma(u_t(t))\cdot u_t(t)\,dx \\ &- \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)g(t)\|\nabla u(t)\|^2 - \left(\frac{1}{2}+\frac{\varepsilon}{2}\right)\int_0^t(-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2\,ds \\ &+\varepsilon\int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds.\end{aligned}$$

We now estimate the mixed term. By Cauchy–Schwarz with weight $g(t-s)\,ds$ and Young's inequality, for any $\delta > 0$,

$$\begin{aligned}&\left|\int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds\right| \\ &\qquad\leq \left(\int_0^t g(t-s)\,ds\right)^{1/2}\|\nabla u_t(t)\|\left(\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds\right)^{1/2} \\ &\qquad\leq \frac{\delta}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2\delta}\left(\int_0^t g(t-s)\,ds\right)\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds.\end{aligned}$$

Multiplying by ε and keeping all terms, we have, for any $\delta > 0$,

$$\varepsilon\int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds \le \frac{\varepsilon\delta}{2}\|\nabla u_t(t)\|^2+\frac{\varepsilon}{2\delta}\,G(t)\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds,$$

with $G(t)=\int_0^t g(t-s)\,ds\le G_\infty$ by (A2). Using (A3), we also have

$$\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds \le \frac{1}{\xi(t)}\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds.$$

Hence,

$$\varepsilon\int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds \le \frac{\varepsilon\delta}{2}\|\nabla u_t(t)\|^2 + \frac{\varepsilon}{2\delta}\,\frac{G(t)}{\xi(t)}\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds.$$

Choosing $\delta=1$ and $G(t)\le G_\infty$ yields

$$\varepsilon\int_0^t g(t-s)\,(\nabla u_t(t),\,\nabla u(t)-\nabla u(s))\,ds \le \frac{\varepsilon}{2}\|\nabla u_t(t)\|^2 + \varepsilon\,\frac{G_\infty}{2\,\xi(t)}\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds.$$

Inserting this bound into the differential inequality for $\mathcal{V}'(t)$ and grouping like terms yields

$$\mathcal{V}'(t)\le -\int_\Omega \sigma(u_t)\cdot u_t\,dx-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)g(t)\|\nabla u(t)\|^2 -\left(\frac{1}{2}+\frac{\varepsilon}{2}-\varepsilon\,\frac{G_\infty}{2\,\xi(t)}\right)\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds+\frac{\varepsilon}{2}\|\nabla u_t(t)\|^2.$$

Then

$$\mathcal{V}'(t)-\frac{\varepsilon}{2}\|\nabla u_t(t)\|^2\le -\int_\Omega \sigma(u_t)\cdot u_t\,dx-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)g(t)\|\nabla u(t)\|^2 -\left(\frac{1}{2}+\frac{\varepsilon}{2}-\varepsilon\,\frac{G_\infty}{2\,\xi(t)}\right)\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds.$$

Since $-\frac{\varepsilon}{2}\|\nabla u_t(t)\|^2\le 0$, dropping it from the left-hand side weakens the inequality, hence

$$\mathcal{V}'(t)\le -\int_\Omega \sigma(u_t)\cdot u_t\,dx-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)g(t)\|\nabla u(t)\|^2 -\left(\frac{1}{2}+\frac{\varepsilon}{2}-\varepsilon\,\frac{G_\infty}{2\,\xi(t)}\right)\int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds.$$

Fix $T > 0$ and set $\xi_T := \inf_{0\le s\le T}\xi(s) = \xi(T)$ (since ξ is nonincreasing). Choose

$$\varepsilon_0 := \min\left\{\frac{1}{2}, \frac{\xi_T}{4\,G_\infty}\right\}, \qquad \text{and take } \varepsilon \in (0, \varepsilon_0].$$

Then, for all $t \in (0, T]$,

$$\frac{1}{2} - \frac{\varepsilon}{2} \ge \frac{1}{4} \qquad \text{and} \qquad \frac{1}{2} + \frac{\varepsilon}{2} - \varepsilon\,\frac{G_\infty}{2\,\xi(t)} \ge \frac{1}{2} - \varepsilon\,\frac{G_\infty}{2\,\xi_T} \ge \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

Therefore we can take the explicit constants

$$c_* := \frac{1}{4}, \qquad c^* := \frac{3}{8},$$

which depend only on T through ξ_T (and on G_∞) but are independent of $t \in (0, T]$. Hence,

$$\mathcal{V}'(t) \le -\int_\Omega \sigma(u_t(t))\cdot u_t(t)\,dx - c_*\,g(t)\|\nabla u(t)\|^2 - c^*\int_0^t (-g'(t-s))\|\nabla u(t) - \nabla u(s)\|^2 ds,$$

which is exactly (4.5).

Finally, the above computations are rigorous for Galerkin approximants. Passing to the limit uses the weak lower–semicontinuity of norms together with the monotonicity of σ (A4) and the Dafermos calculus under (A2)–(A3), preserving the differential inequality for weak solutions. □

Theorem 4.3. (General decay estimate). *Let assumptions* (A1)–(A4) *hold. Suppose further that the relaxation kernel g satisfies* (A2)–(A3) *with a nonincreasing function ξ such that*

$$-g'(t) \ge \xi(t)\,g(t), \qquad t \ge 0,$$

and ξ fulfills the integrability conditions in (A3)*. Then there exist constants $C > 0$ and $\kappa \in (0, 1)$, depending only on the structural parameters in* (A1)–(A4)*, such that*

$$E(t) \le C\exp\left(-\kappa\int_0^t \xi(s)\,ds\right), \qquad t \ge 0. \tag{4.6}$$

In particular:

(1) *if $\xi(t) \equiv \xi_0 > 0$, then $E(t) \le Ce^{-\kappa\xi_0 t}$ (exponential decay);*

(2) *if $\xi(t) \sim c(1+t)^{-1}$, then $E(t) \le C(1+t)^{-\kappa c}$ (polynomial decay).*

Proof of Theorem 4.3. The proof combines Proposition 4.2 with the kernel structure (A3) and the equivalence Lemma 4.1.

Step 1: Differential inequality for $\mathcal{V}'$. By Proposition 4.2, for all $\varepsilon \in (0, \varepsilon_0]$ and a.e. $t > 0$,

$$\mathcal{V}'(t) \le -\underbrace{\int_\Omega \sigma(u_t)\cdot u_t\,dx}_{\mathcal{D}_1(t)} - \underbrace{c_* \, g(t)\|\nabla u(t)\|^2}_{\mathcal{D}_2(t)}$$

$$-\underbrace{c^* \int_0^t (-g'(t-s))\|\nabla u(t)-\nabla u(s)\|^2 ds}_{\tilde{\mathcal{D}}_3(t).}.$$

Step 2: From $\tilde{\mathcal{D}}_3$ to a $\xi(t)$–weighted memory dissipation. Using (A3) that $-g'(r) \ge \xi(r)\, g(r)$ with ξ nonincreasing, for each fixed $t > 0$,

$$\tilde{\mathcal{D}}_3(t) \ge c^*\, \xi(t) \int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds = 2c^*\,\xi(t)\,(g\circ\nabla u)(t) =: \mathcal{D}_3(t).$$

Hence

$$\mathcal{V}'(t) \le -\mathcal{D}_1(t) - \mathcal{D}_2(t) - \mathcal{D}_3(t) \qquad \text{with} \quad \mathcal{D}_3(t) = 2c^*\,\xi(t)\,(g\circ\nabla u)(t). \tag{4.7}$$

Step 3: Absorbing the elastic energy through $\mathcal{D}_2(t)$. By Lemma 2.1 there is $K_2 > 0$ such that $a(u, u) \le K_2\|\nabla u\|^2$. Fix an arbitrary $T > 0$. Since g is nonincreasing and $g(0) > 0$, we have $g(t) \ge g(T) > 0$ for $t \in [0, T]$. Therefore,

$$\mathcal{D}_2(t) = c_*\, g(t)\|\nabla u(t)\|^2 \ge c_*\, g(T)\|\nabla u(t)\|^2 \ge \frac{c_*\, g(T)}{K_2}\, a(u(t), u(t)) \qquad \forall\, t \in [0, T]. \tag{4.8}$$

Step 4: Splitting the memory dissipation. Fix $\theta \in (0, 1)$. Split $\mathcal{D}_3$ as

$$\mathcal{D}_3(t) = \underbrace{2\theta c^*\,\xi(t)\,(g\circ\nabla u)(t)}_{\mathcal{D}_{3a}(t)} + \underbrace{2(1-\theta)c^*\,\xi(t)\,(g\circ\nabla u)(t)}_{\mathcal{D}_{3b}(t).}.$$

We keep $\mathcal{D}_{3a}$ to manufacture the variable rate $\xi(t)$ in the final inequality, while $\mathcal{D}_{3b}$ remains an *additional* (helpful) negative term.

With (4.7) and (4.8), for $t \in [0, T]$,

$$\mathcal{V}'(t) \le -\mathcal{D}_1(t) - \frac{c_*\, g(T)}{K_2}\, a(u(t), u(t)) - \mathcal{D}_{3a}(t) - \mathcal{D}_{3b}(t). \tag{4.9}$$

Step 5: A time-interval Lyapunov functional. Define the time-interval functional

$$\Phi(t) := \int_t^{t+1} \mathcal{V}(s)\, ds, \qquad t \ge 0.$$

Then $\Phi'(t) = \mathcal{V}(t+1) - \mathcal{V}(t)$. Integrating (4.9) over $(t, t+1)$ gives, for $t \in [0, T-1]$,

$$\Phi'(t) \ \le \\ -\int_t^{t+1}\int_\Omega \sigma(u_s\Big)\cdot u_s\,dx\,ds - \frac{c_*\,g(T)}{K_2}\int_t^{t+1} a(u(s),u(s))\,ds - 2\theta c^*\int_t^{t+1}\xi(s)\,(g\circ\nabla u)(s)\,ds. \tag{4.10}$$

Since ξ is nonincreasing, for $s \in [t, t+1] \subset [0, T]$ we have $\xi(s) \ge \xi(t+1) \ge \xi(T)$; hence

$$\int_t^{t+1}\xi(s)\,(g\circ\nabla u)(s)\,ds \ge \xi(t+1)\int_t^{t+1}(g\circ\nabla u)(s)\,ds. \tag{4.11}$$

Combining (4.10)–(4.11) yields

$$\Phi'(t) \le -\int_t^{t+1}\int_\Omega \sigma(u_s)\cdot u_s\,dx\,ds - \frac{c_*\,g(T)}{K_2}\int_t^{t+1} a(u,u)\,ds \\ -2\theta c^*\,\xi(t+1)\int_t^{t+1}(g\circ\nabla u)(s)\,ds. \tag{4.12}$$

Step 6: Lower bounding $\Phi(t)$ *by the averaged dissipations.* By Lemma 4.1 (equivalence) there exist $c_1, c_2 > 0$ independent of t such that

$$c_1\,E(s) \le \mathcal{V}(s) \le c_2\,E(s) \qquad \text{for a.e. } s \ge 0,$$

with $E(s) = \frac{1}{2}\|u_s(s)\|^2 + \frac{1}{2}a(u(s),u(s)) + (g\circ\nabla u)(s)$. Therefore

$$\Phi(t) = \int_t^{t+1}\mathcal{V}(s)\,ds \le c_2\int_t^{t+1}\left(\frac{1}{2}\|u_s(s)\|^2 + \frac{1}{2}a(u,u) + (g\circ\nabla u)(s)\right)ds. \tag{4.13}$$

To compare the kinetic part with its dissipation, we use the pointwise inequality (valid for all $y \ge 0$):

$$y^2 \le y^2\log\left(1+y^2\right) + 1.$$

Integrating over Ω and then over $(t, t+1)$ gives

$$\int_t^{t+1}\|u_s(s)\|^2 ds \le \int_t^{t+1}\int_\Omega \sigma(u_s)\cdot u_s\,dx\,ds + |\Omega|. \tag{4.14}$$

Using (4.13) and (4.14), we deduce

$$\Phi(t) \ \le \frac{c_2}{2}\int_t^{t+1}\int_\Omega \sigma(u_s)\cdot u_s\,dx\,ds + \frac{c_2}{2}\int_t^{t+1}a(u,u)\,ds + c_2\int_t^{t+1}(g\circ\nabla u)(s)\,ds + \frac{c_2}{2}\,|\Omega|. \tag{4.15}$$

Step 7: Closing the differential inequality for Φ. Multiply (4.15) by

$$\eta_T := \min\left\{\frac{2}{c_2},\ \frac{2c_*\,g(T)}{c_2K_2},\ \frac{2\theta c^*\,\xi(T)}{c_2}\right\},$$

and compare with (4.12). Using $\xi(s) \geq \xi(t+1)$ for $s \in [t, t+1]$, we obtain

$$\eta_T\,\Phi(t) \leq \int_t^{t+1}\int_\Omega \sigma(u_s)\cdot u_s\,dx\,ds + \frac{c_*\,g(T)}{K_2}\int_t^{t+1} a(u,u)\,ds + 2\theta c^*\int_t^{t+1}(g\circ\nabla u)(s)\,ds + C_T,$$

with $C_T \coloneqq \eta_T\,\frac{c_2}{2}\,|\Omega|$. Hence, by (4.12),

$$\Phi'(t) \leq -\eta_T\,\Phi(t) + C_T \qquad (0 \leq t \leq T-1).$$

Moreover, using $\ell > 0$ in (A2) and the pointwise bound in (A4), there exist constants $C_1, C_2 > 0$ (independent of t) such that for a.e. s,

$$E(s) \leq c_1\int_\Omega \sigma(u_s)\cdot u_s\,dx + C_2\,(g\circ\nabla u)(s).$$

Averaging on $(t, t+1)$ and invoking $c_1E \leq V \leq c_2E$ (Lemma 4.1) yields

$$\int_t^{t+1}(g\circ\nabla u)(s)\,ds \geq \frac{1}{C_2}\,\frac{1}{c_2}\,\Phi(t) - \frac{C_1}{C_2}\int_t^{t+1}\int_\Omega \sigma(u_s)\cdot u_s\,dx\,ds - C_0,$$

for some $C_0 > 0$ depending only on structural data. Plugging this in (4.12) (and absorbing a small multiple of $\int\sigma$ into its negative coefficient) gives the **variable-rate** inequality

$$\Phi'(t) \leq -\kappa\,\xi(t+1)\,\Phi(t) + \overline{C}\,\xi(t+1) \qquad (0 \leq t \leq T-1),$$

for some $\kappa, \overline{C} > 0$ depending only on $(A1)$–$(A4)$. Applying Grönwall with variable rate and using that ξ is nonincreasing with $\int_0^\infty \xi = +\infty$, we get

$$\Phi(t) \leq \left(\Phi(0) + \frac{\overline{C}}{\kappa}\right)\exp\!\left(-\kappa\int_0^t \xi(s)\,ds\right) \qquad (0 \leq t \leq T-1).$$

Time-interval constant rate. Fix $k \in \mathbb{N}$ and set $T \coloneqq k+1$. Since ξ is nonincreasing, for every $t \in [k, k+1]$ we have $\xi(s) \geq \xi(t+1) \geq \xi(k+1)$ for all $s \in [t, t+1]$. Define

$$\lambda_k \coloneqq \eta_{k+1}, \qquad C_{k+1} \coloneqq \frac{c_2}{2}\,|\Omega|.$$

Then, for all $t \in [k, k+1]$, the differential inequality (4.12) together with (4.15) (and the definition of η_{k+1} in Step 7) yields

$$\Phi'(t) \leq -\lambda_k\,\Phi(t) + \lambda_k\,C_{k+1}. \tag{4.16}$$

Solving (4.16) on the interval $[k, k+1]$ gives

$$\Phi(t) \leq e^{-\lambda_k\,(t-k)}\,\Phi(k) + \left(1 - e^{-\lambda_k\,(t-k)}\right)C_{k+1}, \qquad t \in [k, k+1],$$

and, in particular,

$$\Phi(k+1) \leq e^{-\lambda_k}\,\Phi(k) + \left(1 - e^{-\lambda_k}\right)C_{k+1}. \tag{4.17}$$

Step 8: From Φ back to E and time-interval iteration. Iterating (4.17) over $k = 0, 1, \ldots, m-1$ yields

$$\Phi(m) \leq e^{-\sum_{j=0}^{m-1} \lambda_j} \Phi(0) + \sum_{j=0}^{m-1} \left(1 - e^{-\lambda_j}\right) C_{j+1}\, e^{-\sum_{i=j+1}^{m-1} \lambda_i}.$$

Using the equivalence $c_1 E \leq \mathcal{V} \leq c_2 E$ and the monotonicity of E, we get

$$E(m+1) \leq \frac{1}{c_1} \Phi(m) \leq \frac{\Phi(0)}{c_1} e^{-\sum_{j=0}^{m-1} \lambda_j} + \frac{1}{c_1} \sum_{j=0}^{m-1} \left(1 - e^{-\lambda_j}\right) C_{j+1}\, e^{-\sum_{i=j+1}^{m-1} \lambda_i}.$$

By construction,

$$\eta_{k+1} = \min\left\{ \frac{2}{c_2},\ \frac{2c^* g(k+1)}{c_2 K_2},\ \frac{2\theta c^*}{c_2} \xi(k+1) \right\} \geq 0,$$

and we set $\lambda_k := \eta_{k+1}$. Since $\xi(k+1) \downarrow 0$ while $g(k+1)$ and 1 are bounded, there exists k_0 such that, for all $k \geq k_0$,

$$\eta_{k+1} = \frac{2\theta c^*}{c_2} \xi(k+1).$$

Hence, for $k \geq k_0$,

$$\lambda_k \geq \kappa\, \xi(k+1), \qquad \kappa := \frac{2\theta c^*}{c_2} \in (0, 1).$$

Consequently,

$$\sum_{j=0}^{m-1} \lambda_j \geq \sum_{j=k_0}^{m-1} \lambda_j \geq \kappa \sum_{j=k_0}^{m-1} \xi(j+1) \geq \kappa \int_{k_0}^{m} \xi(s)\, ds.$$

Absorbing the finite window $[0, k_0]$ and the geometric tail into the constants yields, for all $t \geq 0$,

$$E(t) \leq c \exp\left(-\kappa \int_0^t \xi(s)\, ds \right).$$

and choosing $\theta = \frac{1}{2}$ yields the explicit value $\kappa = \frac{c^*}{c_2}$. This completes the proof. □

Corollary 4.4 (Explicit decay rates). *Under* (A1)–(A4) *and the structural inequality* $-g' \geq \xi g$ *with nonincreasing* ξ*, there exists* $\kappa \in (0, 1)$ *such that*

$$E(t) \leq c \exp\left(-\kappa \int_0^t \xi(s)\, ds \right), \qquad t \geq 0.$$

In particular:

(1) *If* $\xi(t) \geq \xi_0 > 0$*, then* $E(t) \leq C\, e^{-\kappa \xi_0 t}$.

(2) *If* $\xi(t) = \frac{p}{1+t}$ *with* $p > 1$*, then* $E(t) \leq C\, (1+t)^{-\kappa p}$.

Moreover, one can take the explicit value $\kappa = \frac{c^*}{c_2}$ *by choosing* $\theta = \frac{1}{2}$*, where* $c^* = \frac{3}{8}$ *and* $c_2 = \max\{1,\, 1 + \varepsilon_0\, G_\infty/K_1,\, 1 + \varepsilon_0\}$ *for any* $\varepsilon_0 \in \left(0, \min\left\{\frac{1}{2}, \frac{K_1}{2G_\infty}\right\}\right]$.

Remark: While Theorem 4.3 establishes the general form of the decay, this corollary provides explicit expressions for the decay rate constant κ in terms of the problem's parameters, which is valuable for quantitative analysis.

Remark 4.5 (Logarithmic damping with power $\theta > 0$). Consider the generalized damping

$$\sigma_\theta(v) \;:=\; v\left(\log\left(1 + |v|^2\right)\right)^\theta, \qquad v \in \mathbb{R}^n, \quad \theta > 0.$$

Then all the results proved above for the case $\theta = 1$ remain valid for any $\theta > 0$, with constants possibly depending on θ. In particular, assumptions (A4) hold for σ_θ. For completeness we collect the key properties and short proofs.

(1) *Convex potential and monotonicity.* Define $\Phi_\theta : [0,\infty) \to [0,\infty)$ by

$$\Phi_\theta(s) \;:=\; \int_0^s \left(\log(1+\tau)\right)^\theta d\tau, \qquad s \geq 0,$$

and set the convex potential

$$\Psi_\theta(v) \;:=\; \frac{1}{2}\,\Phi_\theta\left(|v|^2\right), \qquad v \in \mathbb{R}^n.$$

Then

$$\nabla_v \Psi_\theta(v) = \Phi_\theta'\left(|v|^2\right) v = \left(\log\left(1 + |v|^2\right)\right)^\theta v = \sigma_\theta(v).$$

Since $\Phi_\theta''(s) = \frac{\theta}{1+s}\left(\log(1+s)\right)^{\theta-1} \geq 0$, the map Φ_θ is convex, hence Ψ_θ is convex and $\sigma_\theta = \nabla \Psi_\theta$ is (maximal) monotone:

$$\left(\sigma_\theta(v) - \sigma_\theta(w)\right)\cdot(v - w) \geq 0, \qquad \forall\, v, w \in \mathbb{R}^n.$$

Equivalently, the Jacobian matrix reads

$$D\sigma_\theta(v) = \left(\log\left(1 + |v|^2\right)\right)^\theta I + \theta\left(\log\left(1 + |v|^2\right)\right)^{\theta-1} \frac{2}{1 + |v|^2}\, v \otimes v,$$

which is positive semidefinite for all v.

(2) *Dissipativity and pointwise lower bounds.* For all $v \in \mathbb{R}^n$,

$$\sigma_\theta(v)\cdot v = |v|^2\left(\log\left(1 + |v|^2\right)\right)^\theta \geq 0.$$

Moreover, we have the following explicit bounds:

$$\text{(near the origin)} \qquad \text{if } |v| \leq 1, \quad \log\left(1+|v|^2\right) \geq \frac{|v|^2}{2} \;\Rightarrow\; \sigma_\theta(v)\cdot v \geq 2^{-\theta}\,|v|^{2+2\theta};$$

$$\text{(at infinity)} \qquad \text{if } |v| \geq \sqrt{e-1}, \quad \log\left(1+|v|^2\right) \geq 1 \;\Rightarrow\; \sigma_\theta(v)\cdot v \geq |v|^2.$$

Consequently, there exists a universal $c_\theta > 0$ such that

$$\sigma_\theta(v)\cdot v \geq c_\theta \min\left\{|v|^{2+2\theta},\, |v|^2\right\}, \qquad \forall\, v \in \mathbb{R}^n.$$

(3) *Local Lipschitz continuity.* For any $R > 0$ there exists $L_{\theta,R} > 0$ with

$$|\sigma_\theta(v) - \sigma_\theta(w)| \leq L_{\theta,R}\,|v-w| \qquad \text{whenever } |v|, |w| \leq R.$$

Indeed, $\|D\sigma_\theta(v)\|$ is bounded on $\{|v| \leq R\}$ and one can take the explicit bound

$$L_{\theta,R} \;:=\; \left(\log\left(1+R^2\right)\right)^\theta + 2\theta\left(\log\left(1+R^2\right)\right)^{\theta-1}\frac{R^2}{1+R^2}.$$

(4) *Consequences for the PDE.* Replacing σ by σ_θ in the weak formulation (3.1) and in the Galerkin scheme, all arguments used for existence, uniqueness, energy dissipation, and decay estimates remain valid. The proof steps that rely on monotonicity, dissipativity, and local Lipschitz continuity carry over verbatim by (1)–(3) above. In particular, the Lyapunov analysis of Section 4 yields the same differential inequality for $\mathcal{V}$ and the same decay law

$$E(t) \leq c\exp\left(-\int_0^t \xi(s)\,ds\right),$$

so that exponential (resp. polynomial) decay occurs according to the kernel condition (A3). The exponent θ affects only the multiplicative constants (through local Lipschitz bounds and Young-type estimates), but not the qualitative type of decay, which is governed by g via ξ.

5. Conclusion

In this work, we analyzed a Lamé-type viscoelastic wave equation subject to a logarithmic damping term. The model differs from earlier contributions on Lamé systems where damping mechanisms were mainly polynomial, exponential, or incorporated as nonlinear sources. By introducing a logarithmic dissipation directly on the velocity, we highlighted a qualitatively different stabilization mechanism.

Through the Galerkin method and compactness arguments, we established the global well-posedness of weak solutions. The construction of a Lyapunov functional equivalent to the natural energy allowed us to derive a differential inequality, from which a general decay estimate was obtained. The resulting decay rates depend explicitly on the relaxation kernel: exponential when the kernel satisfies a uniform inequality, and polynomial otherwise.

Although the logarithmic damping is weaker than polynomial dissipation, our results demonstrate that it still guarantees stability in the presence of memory effects. This provides a complementary perspective to existing studies on viscoelastic systems, showing that logarithmic mechanisms can serve as effective alternatives in modeling energy decay.

Future work may focus on extending the analysis to stronger variants of logarithmic damping (e.g. powers of the logarithm), or to coupled systems with boundary feedback, in order to further clarify the interplay between different damping structures and viscoelastic memory.

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Further reading

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