CHAPTER 1

Theoretical Foundations for the Rotterdam Model[1]

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1.1. INTRODUCTION

A now widespread critique of the Rotterdam Model correctly observes that the model's properties and theoretical implications are known exactly only in a highly restrictive and uninteresting special case. Hence, the model currently can be related exactly to *available* theory only if that unacceptably restrictive special case is maintained. In this paper, we shall fill the gap in our knowledge, which has been observed to exist by the model's critics. We shall derive the model's theoretical properties at the aggregate level over a much larger region of the macroparameter space than the tiny region within which the currently understood special case is defined.

If the aggregated (over consumers) Rotterdam Model's parameters take values such that the model is integrable (in an origin-closed sense to be defined below), then the resulting community utility function is Cobb–Douglas. Dropping the origin-closed assumption results in only minor generalization. If the model is not integrable in the aggregate, then its theoretical properties are

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not known.[2] But Cobb–Douglas restrictions have never been imposed in the model's applications. Hence, the model—as used—has no rigorous exact link with currently available theory. Furthermore, the model has never convincingly been shown to approximate any well-defined theoretical construct that need exist at the aggregate level without Cobb–Douglas preferences. Hence, no relationship has been established successfully between the model's existing applications and the currently available theory.

Yet it is now widely recognized by theoreticians that integrability of *any* aggregate demand system is an unacceptably strong assumption. Hence, the region of the parameter space (the non-integrable subset) over which the Rotterdam Model's properties are not known is precisely that subset, which is of theoretical interest. In this paper, we shall begin by deriving a general and highly informative theoretical construct, which exists under assumptions substantially weaker than those necessary for aggregate integrability. We then shall derive strong restrictions implied by theory throughout the region on which our theoretical construct is defined. This provides a very general theoretical solution to the problem of demand aggregation, which increasingly has hindered demand studies and has been the subject of intensive research in the recent general equilibrium literature. Then we shall show that the Rotterdam Model provides a Taylor series local approximation to our new theoretical construct throughout the Rotterdam Model's feasible parameter set.

Our current knowledge of the Rotterdam Model depends upon constancy of the model's coefficients. But we shall prove that the assumed constancy of the model's coefficients at the aggregate (macro) level does not imply constancy of the model's coefficients at the consumer (micro) level. Hence the available results are applicable only at the aggregate level. At the macro level, the model is integrable only on a negligible Lebesgue measure zero subspace of the parameter space, and it is that negligible subspace on which the model's critics have explored the model's properties. But integrability of any model at the aggregate level obtains only if a community utility function exists, and such aggregate utility functions exist only under extremely restrictive and implausible conditions. Hence, we currently only know that the model has highly restrictive properties on

[2] An unnecessarily pessimistic implication has sometimes been read into this valid critique. It has been asserted (without support) that in fact no theoretical foundations for the Rotterdam Model could possibly exist without aggregate integrability and hence without Cobb–Douglas preferences. See, e.g., Yoshihara (1969), Phlips (1974), Christensen et al. (1975), Jorgenson and Lau (1975), Christensen and Manser (1977) or Jorgenson and Lau (1977), who thereby directly impute to the Rotterdam Model itself (as opposed to the investigated subset of its macroparameter space) the properties of a Cobb–Douglas system. We shall *provide* the missing theoretical foundations for the Rotterdam Model's non-integrable case.

a negligible parameter subspace on which theory dictates that such properties *should* be restrictive. All applications of the model have been based upon its properties on the *rest* of the parameter space, and we shall prove, under weak assumptions that useful and highly informative theoretical restrictions can be tested for or imposed *everywhere* on the model's parameter space, without necessarily depending upon or implying aggregate integrability.

As is now well known, few of the microeconomic properties of consumer demand systems carry over to aggregate commodity demand systems. By deriving a general *limiting stochastic transformation* of aggregate economic theory, we shall demonstrate, under clearly weak assumptions that conditions necessary for integrability of *micro* demand systems imply specific theoretical restrictions on that limiting transformation. Far more will be proved about our aggregate stochastic transformation than is known about aggregate demand systems themselves under any comparably weak assumptions. Hence a solution to the aggregation problem in demand theory lies in passing to a new space of limiting functions. Since our results are most easily acquired in terms of continuous time stochastic processes, we shall derive our model in terms of a continuous time consumption decision, rather than the usual discrete time finite period expenditure allocation decision.

Two closely related versions of the Rotterdam Model exist: the "relative price" version and the "absolute price" version. We shall derive the properties of the absolute price version, since its linearity in the parameters simplifies our proofs considerably. Our derivations and our results will differ from those currently available. Furthermore, we shall avoid approximations having unknown properties, and we shall minimize assumptions that are not necessary to the derivations.

1.2. THE INDIVIDUAL CONSUMER'S DECISION

Let N be the number of consumers, and n the number of goods, and define $\mathbf{m} = (m_1, \ldots, m_N)'$, where $m_c = m_c(t) > 0$ is consumer c's rate of total consumption expenditure at time t. Consumption is viewed as proceeding continuously over time. Let $\mathbf{q}_c(t) = (q_{1c}(t), \ldots, q_{nc}(t))'$ be consumer c's consumption flow at time t, where $q_{ic}(t) \in S_c$ is consumer c's instantaneous rate of consumption of good i and $S_c \subset \mathrm{R}^n$ is consumer c's consumption set, which we assume to be a subset of the non-negative orthant. Let $\mathbf{p}(t) = (p_1(t), \ldots, p_n(t))' > \underline{0}$ be the vector of corresponding prices.

Let T be the time interval of interest (perhaps unbounded above). We shall assume that at each instant of time, $t \in T$, consumer c selects $\mathbf{q}_c \in S_c$ to maximize $u_c(\mathbf{q}_c)$ subject to $\mathbf{q}_c'\mathbf{p}(t) = m_c(t)$, where u_c is an instantaneous utility function reflecting unchanging consumer preferences over instantaneous consumption flows, $\mathbf{q}_c(t)$, at any $t \in T$.[3] We assume that u_c has all of the usual neoclassical properties. The solution to the consumer's current expenditure flow allocation decision can be written as $\mathbf{q}_c = \mathbf{q}_c(m_c(t), \mathbf{p}(t))$.[4] We shall follow convention in referring to $m_c(t)$ as consumer c's instantaneous "income" at time t, and we shall assume for each consumer and for all $t \in T$ that

$$\mathbf{q}_c(m_c(t), \mathbf{p}(t)) \tag{2.1}$$

lies strictly within the interior of S_c.

We shall now assume that each consumer's instantaneous utility function can be written as $u_c(\mathbf{q}_c) = u(\mathbf{q}_c, \mathbf{s}_c)$ where $\mathbf{s}_c$ is a finite dimensional vector of taste determining factors (environmental, physiological, genetic, etc.) experienced by consumer c. The function u is fixed, and the vector $\mathbf{s}_c$ depends upon c but not upon t. We could view $\mathbf{s}_c$ (and thereby tastes) as fixed at birth. Observe that we can now introduce a function $\mathbf{q}$ such that

$$\mathbf{q}_c = \mathbf{q}(m_c(t), \mathbf{p}(t), \mathbf{s}_c). \tag{2.2}$$

Define consumer c's value (expenditure) share of the ith good by $w_{ic} = p_i q_{ic}/m_c$. Now differentiate the logarithm of Equation 2.2 with respect to t and multiply through by w_{ic}. We can then determine that

$$w_{ic}\mathrm{d}\log q_{ic}/\mathrm{d}t = \mu_i(m_c, \mathbf{p}, \mathbf{s}_c)\mathrm{d}\log \tilde{m}_c/\mathrm{d}t + \textstyle\sum_{j=1}^{n} \pi_{ij}(m_c, \mathbf{p}, \mathbf{s}_c)\mathrm{d}\log p_j/\mathrm{d}t, \tag{2.3}$$

where the consumer's marginal propensity to consume good i is

$$\mu_{ic} = \mu_i(m_c, \mathbf{p}, \mathbf{s}_c) = p_i \partial q_{ic}/\partial m_c,$$

[3] This is the continuous time instantaneous expenditure flow analogue of the usual discrete time single period expenditure allocation decision. The continuous time version follows from intertemporal preference separability in a manner similar to that of the discrete time version. By intertemporal preference separability we mean that at time t the consumer's intertemporal utility function is of the form

$$\int_t^{\infty} \mathrm{e}^{-\delta_c(t,\tau)\tau} u_c[\mathbf{x}_c(t,\tau)]\mathrm{d}\tau$$

where $\{\mathbf{x}_c(t,\tau) : t \le \tau < \infty\}$ is consumer c's future intertemporal consumption plan at time t. Note that $\mathbf{q}_c(t) = \mathbf{x}_c(t,\tau)$. We assume that the consumer replans continuously in accordance with his latest price expectations and wealth. We could hold the rate of time preference, $\delta_c(t,\tau)$, constant, if we sought intertemporarily consistent planning in the Strotz sense. Our result on instantaneous current expenditure flow allocation is shown easily through a proof by contradiction. Also see Lluch (1973) and Phlips (1974, Chapter 10). In its finite change form, the Rotterdam Model's conditional single period allocation stage is a block in a recursive system, and thereby is empirically as well as theoretically separable. For this remarkably strong result, see Chapter 8 of Theil (1976).

[4] The elements of $\mathbf{q}_c$ are variables rather than sets, since we have assumed strict quasiconcavity of u_c. If we had not done so, then in the next section q_{ic} would have become a random set. The mathematics of random sets is founded on sophisticated concepts in measure theory and topology. Those complications are considered in Barnett (1979b) but are not considered below. Observe that we have rather inelegantly used $\mathbf{q}_c$ also to designate the composite function of time $\mathbf{q}_c(t)$ so that $\mathbf{q}_c(t) = \mathbf{q}_c(m_c(t), \mathbf{p}(t))$.

and his Slutsky coefficients are defined for $i,j = 1,\dots,n$ by

$$\pi_{ijc} = \pi_{ij}(m_c, \mathbf{p}, \mathbf{s}_c) = \frac{p_i p_j}{m_c} \frac{\partial q_{ic}}{\partial p_j}\bigg|_{u_c = \text{constant}}.$$

The rate of change in real income flow $\mathrm{d}\log \bar{m}_c/\mathrm{d}t$ is defined to equal

$$\mathrm{d}\log m_c/\mathrm{d}t - \sum_{k=1}^{n} w_{kc}\mathrm{d}\log p_k/\mathrm{d}t.$$[5]

From p. 49 of Theil (1975) we also can determine that for any $t \in T$

$$\sum_{i=1}^{n} \mu_i(m_c, \mathbf{p}, \mathbf{s}_c) = 1$$
$$\sum_{j=1}^{n} \pi_{ij}(m_c, \mathbf{p}, \mathbf{s}_c) = 0 \qquad (2.4)$$

and $[\pi_{ij}]$ is a symmetric negative semi definite n by n matrix of rank $n-1$. Define the collections of variables = $\boldsymbol{\mu}_c = (\mu_{1c}, \dots, \mu_{nc})'$ and $\boldsymbol{\pi}_c = [\pi_{ijc}]$ and the functions $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)'$ and $\boldsymbol{\pi} = [\pi_{ij}]$. Subject to assumption 2.1, results 2.3 and 2.4 are completely general implications of neoclassical demand theory.

1.3. THE RANDOM MICROCOEFFICIENTS

Taste determining factors $\mathbf{s}_c$ are likely to vary over consumers, and we cannot reasonably expect to capture even the major components of $\mathbf{s}_c$ as explanatory variables in an estimable model. Hence we shall view Equation 2.3 as having random coefficients. We shall treat the existing finite population of consumers as a random sample of size N from an infinite population of "potential consumers" consistent with the current state (environmental, economic, etc.) of the world. Thus $\mathbf{s}_c$, $c = 1,\dots,N$, are N independent and identically distributed random vectors.[6] We shall assume that the income time path $\langle m_c(t) : t \in T\rangle$ assigned to the cth drawn consumer is sampled randomly from an infinite population of potential income paths. The simplest case occurs when each consumer in the infinite population of potential consumers has a predetermined income time path. Then when we select consumers at random from that population, both $m_c(t)$, $t \in T$, and $\mathbf{s}_c$ become random simultaneously through their joint dependency upon c. Alternatively, we could sample $\mathbf{s}_c$ in one stage and then randomly select

[5] The motivation for this definition can be found on pp. 27 and 129 of Theil (1975).

[6] Observe that the randomness is across consumers. Once the N consumers have been drawn, they remain the same for all $t \in T$; the sample of consumers is not redrawn at each t. Hence the random vectors $\mathbf{s}_c$ do not vary over time. We implicitly treat the N drawn consumers as having infinite lifetimes, although one could derive a finite lifetime analogue depending upon demographic variables.

$m_c(t)$, $t \in T$, in a second stage so that $\mathbf{s}_c$ and $m_c(t)$ become independently distributed for all $t \in T$. We shall not restrict the properties of the joint distribution of $(\mathbf{s}_c, m_c(t))$ for fixed $t \in T$ in any manner. They may be correlated.

We shall accept the following very weak assumption on the existence of our theoretical populations.

Assumption 1. For each $c = 1, \ldots, N$, $\langle m_c(t) : t \in T \rangle$ is a continuous time, differentiable, positive stochastic process.[7] At any fixed $t \in T$, the N random vectors $(m_c(t), \mathbf{s}_c')'$, $c = 1, \ldots, N$, are independently and identically distributed (i.i.d.) with distribution function H_t. The marginal distribution of $\mathbf{s}_c$ has distribution function G.

It follows from Assumption 1 that at any $t \in T$, $m_c(t)$, $c = 1, \ldots, N$, are i.i.d. We shall denote the distribution function of the marginal distribution of $m_c(t)$ by F_t. The function F_t is the theoretical income distribution function at t, which can be approximated by the observable empirical income distribution function. Observe that income distribution, by either measure, is free to vary over time. We shall refer to the induced stochastic processes $\boldsymbol{\mu}_c = \boldsymbol{\mu}(m_c(t), \mathbf{p}(t), \mathbf{s}_c)$ and $\boldsymbol{\pi}_c = \boldsymbol{\pi}(m_c(t), \mathbf{p}(t), \mathbf{s}_c)$ as the model's micro-coefficients.[8]

1.4. A GENERAL RESULT ON AGGREGATION OVER CONSUMERS

We now shall aggregate over the random coefficient micro equations 2.3 using Theil's (1971, pp. 570–573) convergence approach to aggregation. This section will provide the purely theoretical results we shall need in exploring the more restrictive results of succeeding sections. Theil's (1975) aggregation of the relative price version of the Rotterdam Model implicitly accepts the Rotterdam Model's parameterization of Equation 2.3. We here seek a general theoretical result requiring no such assumption.

Define $\bar{\boldsymbol{\mu}} = (\bar{\mu}_1, \ldots, \bar{\mu}_n)'$ and $\bar{\boldsymbol{\pi}} = [\bar{\pi}_{ij}]$ such that for $i, j = 1, \ldots, n$

$$\bar{\mu}_i = E(m_c \mu_{ic}) / E(m_c) \tag{4.1}$$

and

$$\bar{\pi}_{ij} = E(m_c \pi_{ijc}) / E(m_c) \tag{4.2}$$

[7] Strictly speaking we should say that $m_c(t)$ is [a.s.] positive for all $t \in T \cap A^c$ where A has Lebesgue measure zero. We use the notation [a.s.] to designate "almost surely" in the conventional measure theoretic sense. We shall be rather casual in our treatment of such subtleties.

[8] Since prices are assumed to be the same for all consumers, we treat them as non-stochastic. Our assumption can be weakened to proportionality of prices over consumers. See p. 150 of Theil (1975).

We shall call $(\bar{\mu}, \bar{\pi})$ the macrocoefficients. They vary over time and are population versions of weighted average microcoefficients, with weights proportional to the corresponding incomes.

Theil (1975) has treated the macrocoefficients as the simple expectation of the microcoefficients. We would acquire that result as a special case if $m_c$were uncorrelated with the random microcoefficients. But such an assumption could be accepted only as an approximation, since the microcoefficients are themselves functions of m_c. We shall never assume the lack of such a functional relationship, even when we introduce the Rotterdam Model's parameterization of our general theoretical results. However, it should be observed that Theil's derivation relates to the model's relative price version, for which the assumptions must be stronger to permit necessary simplifications and to assure invariance of block independence to aggregation.

Let $v_c = \mathrm{d}\log \bar{m}_c/\mathrm{d}t$, and let $k_{ic} = m_c(\mu_{ic} - \bar{\mu}_i)$. Define the aggregated per-capita variables $Q_i = (1/N)\sum_{c=1}^{N} q_{ic}, M = (1/N)\sum_{c=1}^{N} m_c$ and $W_i = p_i Q_i/M$. We shall need the following weak assumptions on the finiteness of certain moments. In considering the plausibility of Assumption 2, observe that finiteness of the first two moments of v_c and k_{ic} is sufficient for finiteness of $E(v_c k_{ic})$. Also observe that π_{ijc} and μ_{ijc} will typically be less than one in absolute value.

Assumption 2. For all $t \in T$ and $c = 1,\ldots,N$, the values of $\bar{\mu}$, $\bar{\pi}$, $E(m_c(t)), E(v_c)$, and $E(v_c k_{ic})$, $i = 1,\ldots,n$, are finite.

Defining $\mathrm{d}\log \bar{M}/\mathrm{d}t$ to equal $\mathrm{d}\log M/\mathrm{d}t - \sum_{k=1}^{n} W_k \mathrm{d}\log p_k/\mathrm{d}t$, we now can prove the following theorem. We use the conventional notation $o_p(1)$ to designate a random variable that converges in probability to 0 as $N \rightarrow \infty$. We use cov$(\cdot, \cdot)$ to designate a covariance.

Theorem 1. Except for a term of stochastic order $o_p(1)$, Assumptions 1 and 2 imply that for $i = 1,\ldots,n$,

$$W_i \mathrm{d}\log Q_i/\mathrm{d}t = \bar{\mu}_i \mathrm{d}\log \bar{M}/\mathrm{d}t + \sum\nolimits_{j=1}^{n} \bar{\pi}_{ij} \mathrm{d}\log p_j/\mathrm{d}t + \mathrm{cov}(k_{ic}, v_c)/E(m_c) \tag{4.3}$$

Proof. Multiply Equation 2.3 through by the cth drawn consumer's income share, m_c/NM, and sum over $c = 1,\ldots,N$. Following Theil (1975, p. 150), we find that the left-hand side becomes $W_i \mathrm{d}\log Q_i/\mathrm{d}t$.

The right-hand side of the aggregated equation can be grouped into two terms. As shown by Theil (1975, p. 154), the first term on the right-hand side can be rearranged to equal

$$\bar{\mu}_i \mathrm{d}\log \bar{M}/\mathrm{d}t + z(t) \tag{4.4}$$

where $z(t) = [\sum_{c=1}^{N}(m_c/NM)]$. We now shall seek the stochastic limit of $z(t)$ as N goes to infinity.

First observe that

$$z(t) = \left[\sum_{c=1}^{N} m_c/N\right]^{-1}(1/N)\sum_{c=1}^{N} m_c(\mu_{ic} - \bar{\mu}_i)\mathrm{d}\log m_c/\mathrm{d}t. \quad (4.5)$$

Under Assumption 1, $m_c(t)$, $c = 1,\ldots,N$, are i.i.d. at time t. Then by Assumption 2, we find from Khinchine's Theorem that $(1/N)\sum_{c=1}^{N} m_c = E(m_c) + o_p(1)$. From Assumption 1, we know that $E(m_c) > 0$. Hence by Slutsky's Theorem, it follows that

$$\left[\sum_{c=1}^{N} m_c/N\right]^{-1} = \left(1/E(m_c)\right) + o_p(1). \quad (4.6)$$

Now $k_{ic}\ v_c$, $c = 1,\ldots,N$, are i.i.d. Hence by Assumption 2 and Khinchine's Theorem, we see that

$$(1/N)\sum_{c=1}^{N} k_{ic}v_c = E(k_{ic}v_c) + o_p(1) \quad (4.7)$$

So by Equations 4.5–4.7, it follows that

$$z(t) = E(k_{ic}v_c)/E(m_c) + o_p(1) \quad (4.8)$$

Now $E(k_{ic}) = E(m_c\mu_{ic}) - \bar{\mu}_i E(m_c) = 0$ by the definition of $\bar{\mu}_i$. Hence we find that $E(k_{ic}v_c) = \mathrm{cov}(k_{ic}, v_c)$. Thus by Equation 4.8, it follows that $z(t) = \mathrm{cov}(k_{ic}, v_c)/E(m_c) + o_p(1)$. So by Equation 4.4, the first term on the right-hand side of the aggregate equation is

$$\bar{\mu}_i \mathrm{d}\log \bar{M}/\mathrm{d}t + \mathrm{cov}(k_{ic}, v_c)/E(m_c) + o_p(1).$$

Similarly, the second term on the right-hand side of the aggregate equation can be written as $\sum_{c=1}^{N} a_{ijc}$ where

$$a_{ijc} = (\mathrm{d}/\mathrm{d}t)\log p_j \sum_{c=1}^{N}(m_c/NM)\pi_{ijc}$$

$$= (\mathrm{d}/\mathrm{d}t)\log p_j\left[\sum_{c=1}^{N}(m_c/N)^{-1}\left(\sum_{c=1}^{N} m_c\pi_{ijc}/N\right)\right].$$

Now by Assumption 1 and Khinchine's Theorem, we know that

$$(1/N)\sum_{c=1}^{N} m_c\pi_{ijc} = E(m_c\pi_{ijc}) + o_p(1).$$

So by Equation 4.6 and Slutsky's Theorem, we have that $a_{ijc} = \pi_{ij}\mathrm{d}\log p_j/\mathrm{d}t + o_p(1)$. Hence the second term on the right-hand side of the aggregate equation is

$$\sum_{j=1}^{n} \pi_{ij}\mathrm{d}\log p_j/\mathrm{d}t + o_p(1).$$ □

We have deleted the $o_p(1)$ term in Equation 4.3, since in applications N typically will be very large. With the exception of the last term, which we shall call the global (or globally small) remainder term, our aggregate system of Equations 4.3 is the direct aggregate analogue of our micro system (Equation 2.3). Observe that we still are considering a very general transformation of economic theory, since Assumptions 1 and 2 are very weak.[9]

We now shall explore implications of economic theory as reflected in our limiting stochastic transformation of economic theory Equation 4.3. Observe that the proofs of both Theorems 1 and 2 lean heavily upon the particular functional structure of Equation 2.3, especially upon its linearity in the microcoefficients. Also observe that by Theorems 2 and Equation 2.4, we have that $\sum_i k_{ic} = 0$.

Theorem 2. If Assumption 1 and 2 obtain, then for all $t \in T : \sum_{i=1}^{n} \bar{\mu}_i = 1$, $\bar{\boldsymbol{\pi}}$ is symmetric negative semidefinite of rank $n - 1$, and $\sum_{j=1}^{n} \bar{\pi}_{ij} = 0$ for $i = 1, \ldots, n$.

Proof. By Equation 2.4, we can find that $\sum_{i=1}^{n} m_c \mu_{ic} / E(m_c) = m_c / E(m_c)$. Taking the expectation of each side, we get that $\sum_{i=1}^{n} \bar{\mu}_i = 1$. The other results of Theorem 2 are easily proved in the analogous manner. □

Thus we see that the macrocoefficients $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\pi}})$ have properties analogous to those of the microcoefficients $(\boldsymbol{\mu}_c, \boldsymbol{\pi}_c)$. Observe that Theorem 2 is a general result in aggregation theory, since it has been derived under clearly weak assumptions. Observe that Assumption 2 has not been used, and Assumption 1 was accepted largely as a convenience. Our proofs have used Khinchine's Weak Law of Large Numbers. If we had used Chebychev's Weak Law of Large Numbers (Rao, 1973, p. 112), the random variables assumed to be stochastically independent in Assumption 1 could have been assumed to be only uncorrelated. The macroparameters then would have been limiting averages of expectations rather than just expectations.

By contrast, let us see what has happened in the space of aggregate demand functions as we have let N go to infinity. For finite N, we have that $Q = (1/N)\sum_{c=1}^{N} \mathbf{q}(m_c(t), \mathbf{p}(t), \mathbf{s}_c)$, where $\mathbf{Q} = (Q_1, \ldots, Q_n)'$. Now under our Assumption 1, we find from Khinchine's Theorem that $\mathbf{Q} = E\mathbf{q}_c + o_p(1)$. Hence for large N, we can treat $E\mathbf{q}_c$ as our per-capita aggregate demand functions. But observe that we know very little about those functions other than a version of the budget constraint, which obtains for even finite N. In fact micro theory is only distantly related to the

[9] There is nothing local about the "approximation" we acquire by dropping the $o_p(1)$ term for large N. We have not expanded any function about some single point. The $o_p(1)$ term is arbitrarily small *everywhere* with arbitrarily high probability for sufficiently large N.

properties of $E\mathbf{q}_c$, which does not even lie in the same function space as $\mathbf{q}_c(m_c, \mathbf{p})$. Observe, for example, that $E\mathbf{q}_c$ is not a function of income, m_c, but rather is a *functional* depending upon the distribution function H_t. In a somewhat different context, Mossin (1968) has found conditions under which his "mean demand function" depends upon $\mathbf{p}$ and $E(m_c)$. But in general, passing to the limit as N goes to infinity provides no new information in the space of demand functions.

We maintain that Equation 4.3 is itself a more powerful *fundamental* theoretical construct than an aggregate demand system, since far more is known about Equation 4.3 than about aggregate demand systems. Acquisition of comparably strong results on aggregate demand functions requires substantially stronger assumptions than our Assumptions 1 and 2. Observe that Theorem 2 was *not* acquired from any implicit or explicit assumption of integrability of Equation 4.3. We have not aggregated over utility functions, and our results are not dependent upon or implicitly induced by any community utility function. The properties of the macrocoefficients provided by Theorem 2 are necessary conditions for integrability of each *individual*'s demand functions. Although those properties are defined in terms of the macrocoefficients, the properties are neither necessary nor sufficient for integrability of the aggregated system Equation 4.3 itself. The relative price version of the Rotterdam Model, not considered in this paper, does aggregate over certain stochastic properties of preferences, but not over complete utility functions.

1.5. INTEGRABILITY THEORY

As we have seen, the theoretical properties of Equation 4.3 provided by Theorem 2 do not depend for their validity upon integrability of the aggregate system Equation 4.3. Nevertheless, we shall find it useful to explore the integrability properties of Equation 4.3.

The following definitions will be required.

Definition 1. We shall say that Equation 4.3 "corresponds with" an aggregate demand system, $\mathbf{a}(\mathbf{m}, \mathbf{p})$, if $\mathbf{a}(\mathbf{m}, \mathbf{p}) = (1/N)\sum_{c=1}^{N} \mathbf{q}_c(m_c(t), \mathbf{p}(t))$ for some collection of neoclassical demand systems $\mathbf{q}_c(\mathbf{m}_c, \mathbf{p})$, $c = 1, \ldots, N$, and if the system of differential Equations 4.3 can be solved for $\mathbf{Q} = (Q_1, \ldots Q_n)'$ such that $\mathbf{Q} = \mathbf{a}(\mathbf{m}(t), \mathbf{p}(t))$ for $t \in T$.

Definition 2. We shall say that Equation 4.3 corresponds with the aggregate demand system of a "representative consumer", if Equation 4.3 corresponds with an aggregate demand system $\mathbf{a}(\mathbf{m}, \mathbf{p})$, and if there exists a function $\mathbf{b}$ such that $\mathbf{b}(M(t), \mathbf{p}(t)) = \mathbf{a}(\mathbf{m}(t), \mathbf{p}(t))$ for every $t \in T$.

Definition 3. We shall say that Equation 4.3 is integrable, if Equation 4.3 corresponds with the aggregate demand system of a representative consumer, and if there exists a strictly quasi-concave, monotonically increasing (community) utility function U, such that at every $t \in T$, $\mathbf{Q} = \mathbf{b}(\mathbf{m}(t), \mathbf{p}(t))$ maximizes $U(\mathbf{Q})$ subject to $\mathbf{Q}'\mathbf{p} \leq M$.

We have *no* prior reason to believe that Equation 4.3 necessarily must *or should* correspond with any aggregate demand system. It is well known that limits of sequences of functions can have fundamentally different properties from those of any of the functions in the sequence and commonly will not be bijectively related to any element or finite collection of elements of the function sequence. In fact Equation 4.3 is a fundamentally different sort of construct from an aggregate demand function system, and in general *none* of the relationships in Definitions 1, 2, or 3 need obtain. Observe the nature of Equation 4.3 itself. It has hybrid properties, in the sense that some of its factors were introduced into the equation only in the limit as N goes to infinity, while others are defined for finite N. For example, $\bar{M}$, and W_i depend upon N, while the macrocoefficients appeared in the limit as N goes to infinity. But functions depending upon N tend to depend upon income through the finite sample of incomes $\mathbf{m}$, while the functionals that appeared in the limit depend upon income through the distribution *functions* F_t and H_t. With Equation 4.3 depending upon income through an income distribution function *as well as* upon $\mathbf{m}$ or $\bar{M}$, it is clear that Equation 4.3 is a fundamentally different sort of construct from an aggregate demand function system. None of the relationships in Definitions 1, 2, or 3 can be expected to obtain; this is immediately evident from the dependence of Equation 4.3 upon t through F_t and H_t as well as through $(\mathbf{m}(t), \mathbf{p}(t))$. This form of time dependence is not permitted in Definitions 1, 2, or 3.

Furthermore, it was our intent to *assure* that no such correspondences (as those in Definitions 1, 2, or 3) could exist, since little is known about the properties of aggregate demand systems (except under extremely strong assumptions) and hence of *any* system of equations directly derivable from an aggregate demand system. To escape from the unacceptably restrictive implications of available theorems on aggregate demand systems, we must break the links that exist when the properties in Definitions 1, 2, or 3 obtain, and we must pass to a fundamentally different function space. We have done *precisely* that. The strong results in Theorems 1 and 2 were rendered logically possible by the lack of any need for "correspondence" (in the sense of Definitions 1, 2, or 3) between Equation 4.3 and an aggregate demand system, whether or not integrable.

Although we have no reason to impose upon Equation 4.3 restrictions sufficient for *aggregate* integrability, we now consider what would happen if we did. First we presume that we have imposed some set of restrictions sufficient for Equation 4.3 to correspond with an aggregate demand system. That would require, at the least, that the dependency of Equation 4.3 upon F_t and H_t be eliminated. We now seek *further* conditions sufficient for Equation 4.3 to be integrable. Gorman (1953) has proved that the aggregate demand system defined in Definition 1 is integrable in the sense of Definition 3 if and only if each consumer has linear Engel curves, which are parallel across consumers. As observed by Phlips (1974, p. 19), Gorman's result obtains "under totally unrealistic conditions". Such a severe restriction on consumer preferences cannot be accepted, and we have induced randomness into the taste-determining factors $\mathbf{s}_c$ specifically to account for more general preference variability. Nevertheless we now shall follow Yoshihara in introducing even greater restrictiveness.

We shall refer to a function property as being origin-closed, if it would obtain even if each consumer's consumption set, $\mathbf{s}_c$, contained the origin (so that the consumer can survive with consumption even at the origin). Although the origin-closed case is restrictive, the sole generalization that has found widespread empirical use is the translation of the origin of R^n to a subsistence consumption bundle; the translated non-negative orthant then is used as the consumption set. The generalization of origin-closed properties to the resulting affine space is obvious (Barnett, 1977a,b). A further, but less common, generalization is provided by Gorman's polar form. The following result is well known in various forms.

Theorem 3. If Equation 4.3 is origin-closed integrable then all consumers' demand functions for all goods are identical and have unitary income elasticities.

Proof. As we have seen, all consumers must have linear Engel curves that are parallel across consumers if Equation 4.3 is to be integrable. But demand systems with linear Engel curves are origin-closed integrable if and only if they have unitary income elasticities.[10] This is easily seen as follows. The demand function $q_{ic}(m_c, \mathbf{p})$ has linear Engel curves if and only if $q_{ic}(m_c, \mathbf{p})$ is of the form $a_{ic}(\mathbf{p}) + b_i(\mathbf{p})m_c$. That function is origin-closed integrable if and only if $a_{ic}(\mathbf{p}) = 0$. Hence we find that $q_{ic}(m_c, \mathbf{p})$ has linear Engel curves and also is origin-closed integrable if and only if $q_{ic}(m_c, \mathbf{p})$ is of the form $b_i(\mathbf{p})m_c$, which has unitary income elasticities. □

As we shall see in section 1.9, an analogous result, derived under the *same origin-closed assumption*, is precisely that for which Yoshihara criticized the Rotterdam Model's particular parameterization of Equation 4.3. In fact, we see that under this

[10] Detailed consideration of the origin-closed case in aggregation can be found in Eisenberg (1961), Green (1964, pp. 44–50), Katzner (1970, p. 139), and Chipman (1974).

assumption, *no* aggregate demand system and *no* theoretical system Equation 4.3 will be integrable unless each consumer's demand function has unitary income elasticity. The proper conclusion (now widely accepted by theorists) is that integrability of any aggregate demand system (whether or not its domain includes the origin) is extremely unlikely and should not be assumed exclusively on theoretical grounds.[11] We do believe, for *empirical* reasons, that aggregate

[11] Much recent literature has appeared on the theoretical implications of aggregation over consumers. Since Gorman's conditions are both necessary and sufficient for aggregate integrability, the recent literature can say nothing further on the issues we have defined above, which are the relevant issues in considering the critique of the Rotterdam Model. However, the recent literature explores results weaker than aggregate integrability, and such results may be useful in motivating the construction of future models. Hence we now discuss that literature briefly. In the general equilibrium literature, Sonnenschein (1973), Debreu (1974), McFadden et al. (1974), and Mantel (1974, 1976) have demonstrated that very little is known about aggregate demand functions and aggregate excess demand functions. We have seen that such functions generally are not integrable. In considering potentially weaker results, Sonnenschein (1973, p. 404) observed that his proofs provide "a striking indication that the budget and homogeneity restrictions largely exhaust the empirical implications of the utility hypothesis for market demand functions, even under the strong hypothesis that community income is shared equally". Sonnenschein (1973, p. 406) concluded that at the aggregate level "there is little left of demand theory beyond homogeneity and balance ...it remains an empty (empirical) box". Muellbauer (1975, 1976) has considered the possibility of acquiring weaker results than Gorman's, but under stronger conditions than those accepted in the above general equilibrium literature. This possibility also has been considered by Diewert (1976), but only under the empirically impractical assumption that the number of consumers is less than the number of goods. Muellbauer seeks existence of a shadow income level such that aggregate demand would be integrable if the representative consumer were allocated the shadow level of income. Since the existing empirical literature allocates measured aggregate per-capita income to the representative consumer, Muellbauer's promising results cannot be used to rationalize existing models postulating integrable aggregate demand. In section 1.4, we observed that the random sampling approach to modelling income and taste differences could be used directly with aggregate demand functions rather than with our transformation of demand theory. The resulting "stratification" approach has been used by Green (1964, p. 67) and Diewert (1976). But to capture variations in tastes and income distribution, the approach would require knowledge of the joint distribution of income and of all of the taste determining stratifying variables $\mathbf{s}_c$, which generally cannot themselves reasonably be specified, although there has been some progress in the direct modelling of some elements of $\mathbf{s}_c$ (Muellbauer, 1977) and in the related attributes approach to exact aggregation of the translog model. If preferences are viewed to be the same for all consumers, as assumed in Diewert's (1976) version, then the stratification approach reduces to a model of a representative consumer faced with random income. But it is the existence of that representative consumer which is at issue. In fact Sonnenschein (1973) and Mantel (1976) have shown that the problem exists even when all incomes vary proportionately, so that income distribution remains constant. The source of the problem is non-trivial differences in tastes over consumers rather than variations in income distribution over time (or highly simplified linear effects of a few easily identified aggregate "attribute" distribution statistics). Barnett (1977b, 1981a, Chapter 8) has provided a rigorous approach to the related problem of incorporating household characteristics explicitly into household demand, but he assumed the existence of a "representative household" in order to aggregate over households.

integrability can be a useful and entirely justifiable *functional regularity* condition, and we ourselves have maintained aggregate integrability for that purpose with our g-hypo model in Barnett (1977a). But in the current theoretical paper we shall not and *need not* impose restrictions sufficient for aggregate integrability.

1.6. THE COMPONENTS OF THE REMAINDER TERM

By the definition of v_c, we know that the global remainder term of Equation 4.3 is

$$(1/E(m_c))\,\text{cov}(v_c, k_{ic}) = \alpha_i(t) - \beta_i(t), \tag{6.1}$$

where

$$\alpha_i(t) = \frac{1}{E(m_c)}\text{cov}(k_{ic}, \text{d}\log m_c/\text{d}t), \tag{6.2}$$

and

$$\beta_i(t) = \frac{1}{E(m_c)}\text{cov}\Big(k_{ic}, \sum_{k=1}^{n} w_{kc}\text{d}\log p_k/\text{d}t\Big). \tag{6.3}$$

As we shall see in this section, the potential exists for confounding the term $\beta_i(t)$ with other terms in Equation 4.3. However, this problem does not exist with the term $\alpha_i(t)$, which is an independent function of time.

The theoretical issues that we are considering in this paper relate to the properties of the macrocoefficients, which appear only in the other terms of Equation 4.3; hence the properties of $\alpha_i(t)$ are not related to our objectives. However, the empirical implementation of our results would require some consideration of $\alpha_i(t)$. Hence we now briefly shall consider the properties of $\alpha_i(t)$, which we shall argue typically is negligibly small (except perhaps during periods of revolutionary shifts in income distribution).

Observe that

$$\alpha_i(t) = \text{cov}\left[\frac{\text{d}\log m_c}{\text{d}t}, \frac{m_c}{E(m_c)}(\mu_{ic} - \bar{\mu}_i)\right] = \theta_i\rho_i\left(\text{var}\frac{\text{d}\log m_c}{\text{d}t}\right)^{1/2},$$

where ρ_i is the correlation coefficient between $m_c(\mu_{ic} - \bar{\mu}_i)/E(m_c)$ and $\text{d}\log m_c/\text{d}t$ in the consumer population and where θ_i is the positive square root of

$$\theta_i^2 = \left[1/(E(m_c))^2\right] E\left[m_c^2(\mu_{ic} - \bar{\mu}_i)^2\right].$$

The coefficient θ_i is a dispersion measure of the ith marginal budget share across consumers, in the sense that it is a non-negative number, which vanishes if each μ_{ic} for this i is equal to $\bar{\mu}_i$ with unit probability. The dispersion measure is weighted towards the rich through the squared income weighting of the squared discrepancies between μ_{ic} and $\bar{\mu}_i$. Hence $\alpha_i(t)$ is a non-random function of time, which can be expressed as a multiple $\theta_i\rho_i$ of the standard deviation of the logarithmic income rates of change across consumers. It could be observed at this point that the sums over i of $\alpha_i(t)$, $\beta_i(t)$, $\gamma_{ij}(t)$, $\rho_{ij}\theta_i$, and $\rho_i\theta_i$ are all zero, while the sum of $\gamma_{ij}(t)$ over j also is zero.

The nature of the connection between $\alpha_i(t)$ and income distribution is clear from the fact that var(d log $m_c/\mathrm{d}t$) and therefore $\alpha_i(t)$ vanish when all incomes change proportionately. As Sonnenschein (1973) has shown, proportional income distribution is a weaker assumption than aggregate demand integrability. An alternative manner in which $\alpha_i(t)$ could be zero is if either θ_i or ρ_i were zero. We now consider assumptions under which ρ_i is zero. Those assumptions (which are unrelated to income proportionality) can be used in cases in which income proportionality is an inappropriate assumption. The intent of this discussion of potential assumption structures is to suggest that the term $\alpha_i(t)$ will typically be small. In fact we suspect that even if none of the assumptions discussed below were applicable, the *complete* global remainder term Equation 6.1 would still commonly be small. By Schwartz's inequality, we know that $\mathrm{cov}(v_c, k_{ic})$ is bounded by $[\mathrm{var}(v_c)\mathrm{var}(k_{ic})]^{1/2}$. We would expect this bound to be small relative to $E(m_c)$ in a developed economy in which the variability that induces randomness into (v_c, k_{ic}) is small relative to $E(m_c)$. Hence $\mathrm{cov}(v_c, k_{ic})/E(m_c)$ will typically be small.

Although Assumption 1 will be accepted throughout our analysis, a further more restrictive assumption will be used solely in the discussion below. That assumption can be viewed as consistent with the two-stage sampling procedure (discussed earlier) in which income paths are sampled independently of tastes.

Assumption 3. For each $t \in T$ and each $c = 1,\ldots,N$, the random vector $\mathbf{s}_c$ is stochastically independent of $m_c(t)$.

Assumption 3 is a stochastic analogue to the conventional assumption that income does not appear in the consumer's utility function. This stochastic assumption also is used by Green (1964, pp. 66–67). Nevertheless, one would expect that some of the factors affecting the consumer's intertemporal income prospects may also affect his tastes. Consider, for example, hereditary and environmental factors. Hence Assumption 3 (which we shall not maintain) is not weak.

We now consider an additional assumption.

Assumption 4. At any $t \in T$, the logarithmic rate of change of the cth randomly drawn consumer's income, d log $m_c/\mathrm{d}t$, is stochastically independent of his logarithmic level of income $\log m_c(t)$. We also assume that $\log m_c(t)$ is a differentiable second-order stochastic process. That is, we assume that $E(\log m_c(t))^2$ is finite for all $t \in T$. The value of this assumption can be seen from the following theorem.

Theorem 4. If Assumptions 3 and 4 obtain, then $\alpha_i(t)$ is uniformly zero for all $t \in T$.

Proof. Recall that $k_{ic} = m_c(\mu_i(m_c, \mathbf{p}, \mathbf{s}_c) - \bar{\mu}_i)$. But under Assumptions 3 and 4, $(\mathbf{s}_c, \mathbf{m}_c)$ is stochastically independent of $\mathrm{d}\log m_c/\mathrm{d}t$. Thus, k_{ic} is stochastically independent of $\mathrm{d}\log m_c/\mathrm{d}t$, and our theorem follows from Equation 6.2. □

The interpretation of Assumption 3 is transparent, and has been discussed above. Assumption 4 is a regularity condition on the stochastic process $\log m_c(t)$. In Chapter 3 of Barnett (1981a), we investigate in depth the class of stochastic processes consistent with that regularity condition, and we argue that the class of admissible processes is large and plausible. In non-linear models with fixed coefficients, similar restrictions on the sequences of explanatory variables are required simply to assure consistency of estimators (Malinvaud, 1970a,b, p. 331; Barnett, 1976).

The result of Theorem 4 actually is stronger than necessary. Since an intercept is commonly used with the Rotterdam Model, we need only argue that $\alpha_i(t)$ can be approximated by a constant over all t ∈ T. That constant need not be zero. In fact we even could acquire our results if $\alpha_i(t)$ could be approximated by

$$a_{0i} + a_{1i}\mathrm{d}\log \bar{M}/\mathrm{d}t + \sum\nolimits_{j=1}^{n} a_{2ij}\mathrm{d}\log p_j/\mathrm{d}t + u_{it}$$

where u_{it} is random and where the a's are constants adding up over i to zero and satisfying $a_{2ij} = a_{2ji}$ for all $i, j = 1, \ldots, n$.

One should recognize that the theoretical arguments in the body of this paper are not dependent upon acceptance of the assumptions used above. Our practice of dropping the $\alpha_i(t)$ terms in the analysis below is a simplification of little theoretical consequence. Even the *empirical* implementation of our results is not, in principle, dependent upon our ability to drop the $\alpha_i(t)$ terms. Since our interest is in inferences solely about the model's other terms, the $\alpha_i(t)$ nuisance terms could be approximated (somewhat inelegantly) uniformly and arbitrarily well by the polynomial time trend dictated by the Weierstrass Approximation Theorem. But available empirical evidence suggests that when leisure is admitted as one of the n goods, not even the zeroth order (intercept) term of that trend is statistically significant. As Barnett's (1979a, 1981a) empirical results indicate, intercepts appear to arise in the model solely as proxies for apparent taste change over goods, and only when non-weakly-separable leisure consumption is ignored. In addition, he has shown that absorption of the statistically insignificant term $\alpha_i(t)$ into the error structure does not contaminate the error structure. Using a Kolmogorov–Smirnov test applied to orthogonally transformed residuals, he strongly accepted normality. All other hypotheses on the error structure have been accepted by Barnett (1979a, 1981a), Theil (1976), and Paulus (1972) using differing data.

We now have considered two alternative assumption structures that are sufficient for $\alpha_i(t)$ to be zero. Necessary conditions would be much weaker. We believe that $\alpha_i(t)$ typically will be small.

We now investigate the more important and potentially troublesome term $\beta_i(t)$. Let ρ_{ij} be the correlation coefficient between $m_c(\mu_{ic} - \bar{\mu}_i/E(m_c))$ and w_{jc} in the consumer population. Then it follows that $\beta_i(t) = \sum_{j=1}^{n} \gamma_{ij}(t)(\mathrm{d} \log p_j/\mathrm{d}t)$ where

$$\gamma_{ij}(t) = \rho_{it}\theta_i(\mathrm{var}\, w_{jc})^{1/2}, \tag{6.4}$$

and where θ_i is as defined above. Substituting this expression for $\beta_i(t)$ into Equation 4.3 with the $\alpha_i(t)$ component of the global remainder dropped, we get that

$$W_i \frac{\mathrm{d} \log Q_i}{\mathrm{d}t} = \bar{\mu}_i \frac{\mathrm{d} \log \bar{M}}{\mathrm{d}t} + \sum_{j=1}^{n} \left[\bar{\pi}_{ij} - \gamma_{ij}(t) \right] \frac{\mathrm{d} \log p_j}{\mathrm{d}t}. \tag{6.5}$$

Hence we see that $\gamma_{ij}(t)$ can be viewed as the *asymptotic aggregation bias* of the (i,j)th Slutsky macrocoefficient $\bar{\pi}_{ij}$.[12]

Let us consider the goods and estimates in the tables on pp. 188–189 of Theil (1975). As has been pointed out to me by Theil, we can expect, for his data, that both θ_i and $(\mathrm{var}\, w_{jc})^{1/2}$ will be well below 0.1. Hence the aggregation bias of the $(i,j)^{\mathrm{th}}$ Slutsky coefficient will be well below $|\rho_{ij}/100|$ in absolute value. Now we may view ρ_{ij} as the correlation coefficient of $m_c\mu_{ic} = \partial(p_i q_{ic})/\partial \log m_c$ and w_{jc} in the consumer population. There appears to be no reason whatsoever to believe that this quantity should differ appreciably from zero, and it cannot exceed 1. Hence for $i \neq j$, it follows that $\gamma_{ij}(t)$ is likely to fall well below 0.001 uniformly in $t \in T$. This is very modest by comparison with the Slutsky coefficient estimates presented by Theil in those tables. The unconstrained off-diagonal estimates in those tables average 0.032 in absolute value.

Nevertheless, the potential complications that could result from this previously unrecognized (although perhaps commonly small) off-diagonal aggregation bias may merit further theoretical and empirical exploration. Whenever our previous speculations are not applicable, the powerful Slutsky symmetry result of

[12] I am indebted to Henri Theil for pointing out this interpretation to me.

Theorem 2 could be swamped by non-symmetric aggregation bias. But the desired symmetry condition is $\gamma_{ij}(t) - \gamma_{ji}(t) = 0$ for all $i \neq j$. By the triangle inequality, $|\gamma_{ij}(t) - \gamma_{ji}(t)| \leq |\gamma_{ij}(t)| + |\gamma_{ji}(t)|$. Hence smallness of $|\gamma_{ij}(t)|$ for all $i \neq j$ is sufficient, but not necessary, for $[\gamma_{ij}(t)]$ to be almost symmetric. Direct speculations on the magnitude of $|\gamma_{ij}(t) - \gamma_{ji}(t)|$ would provide a much lower upper bound. But we shall not consider such a direct bound, since the stronger condition of smallness of each individual $|\gamma_{ij}(t)|$ for all $i \neq j$ permits more informative interpretation of the macroparameters.

Similar consideration of the diagonal elements of $[\gamma_{ij}(t)]$ suggests that $\gamma_{ii}(t)$ typically will be less than 0.01 in absolute value for all $t \in T$ and probably will be positive. That upper bound on $|\gamma_{ii}(t)|$ follows from the same considerations presented in the previous discussion of $\gamma_{ij}(t)$ for $i \neq j$. Positivity results from the properties of ρ_{ii}, which relates $m_c\mu_{ic}$ and w_{ic} for the same good i. It appears plausible, on the average, that consumers whose tastes yield a relatively large w_{ic} also have a large μ_{ic}. If this speculation should be true, then ρ_{ii} and hence $\gamma_{ii}(t)$ would be positive. It follows that the coefficient of $\mathrm{d}\log p_j/\mathrm{d}t$ in Equation 6.5 may be slightly larger than $\bar{\pi}_{ij}$.

Although our *a priori* upper bound on the diagonal elements of $[\gamma_{ij}(t)]$ is larger than that on the off-diagonal elements, the off-diagonal bound is of more concern through its relationship with the empirically powerful Slutsky symmetry condition. Also observe that Theil's unconstrained estimates of the diagonal elements of the Slutsky matrix average 0.067, as opposed to only 0.032 for the off-diagonal elements. Hence greater diagonal aggregation bias is empirically tolerable.

1.7. THE FINITE CHANGE VERSION

We now shall proceed to operationalize our results. We shall begin by dropping the Slutsky aggregation biases, $[\gamma_{ij}(t)]$, in Equation 6.5. Observe that we previously have argued that both $\gamma_{ij}(t)$ and $\alpha_i(t)$ are small *uniformly* on $t \in T$. Hence we are basing our model on a *global* approximation, rather than on a local property applicable only at a single point. We, therefore, have referred to the remainder (last) term of Equation 4.3 as the global (or globally small) remainder term. The empirical problems (correlation with other terms, specification error, etc.) resemble those associated with dropping the remainder terms of the translog or generalized Leontief Taylor series approximations. Also observe that $\gamma_{ij}(t)$ and $\alpha_i(t)$ could average zero over $t \in T$ without being precisely zero everywhere on T, although we shall not explicitly pursue that possibility.

We now shall convert Equation 6.5 into a finite change form having a stochastic error term. We assume that our observations are evenly spaced over time at time intervals of size Δt. Define $\bar{t} = t + \Delta t$, and define the finite log change operator D such that

$$Dx_t = \log x(\bar{t}) - \log x(t)$$

Then define $W_{it}^* = \frac{1}{2}(W_i(\bar{t}) + W_i(t))$ and $D\bar{M}_t = DM_t - \sum_{k=1}^{n} W_{kt}^* Dp_{kt}$ and let ε_{it} be a stochastic error term assumed to be uncorrelated with $D\bar{M}_t$ and Dp_{jt} for all $t \in T$, $j = 1, \ldots, n$.[13] Then by adding the stochastic error term onto an approximate finite change analogue to Equation 6.5, we shall show that

$$W_{it}^* DQ_{it} = \bar{\mu}_i D\bar{M}_t + \sum_{j=1}^{n} \bar{\pi}_{ij} Dp_{jt} + \varepsilon_{it}, \;\; \mathrm{i} = 1, \ldots, \mathrm{n}. \qquad (7.1)$$

Derivation of Equation 7.1. Integrating Equation 6.5 (under our assumptions) with respect to time from t to $\bar{t}$, we get that for $i = 1, \ldots, n$

$$\int_t^{\bar{t}} W_i \frac{\mathrm{d} \log Q_i}{\mathrm{d}t} \mathrm{d}t = \int_t^{\bar{t}} \bar{\mu}_i \frac{\mathrm{d} \log \bar{M}}{\mathrm{d}t} + \sum_{j=1}^{n} \int_t^{\bar{t}} \bar{\pi}_{ij} \frac{\mathrm{d} \log p_j}{\mathrm{d}t} \mathrm{d}t. \qquad (7.2)$$

We now shall define $\bar{W}_i = 1, \ldots, n$, such that

$$\int_t^{\bar{t}} W_i \left(\frac{\mathrm{d} \log Q_i}{\mathrm{d}t} \mathrm{d}t \right) = \bar{W}_i \int_t^{\bar{t}} \left(\frac{\mathrm{d} \log Q_i}{\mathrm{d}t} \mathrm{d}t \right),$$

so that

$$\int_t^{\bar{t}} W_i \left(\frac{\mathrm{d} \log Q_i}{\mathrm{d}t} \mathrm{d}t \right) = \bar{W}_i DQ_{it} \qquad (7.3)$$

If $\mathrm{d} \log Q_i/\mathrm{d}t$ is positive over $(t, \bar{t})$, as is most common for normal goods, the mean value theorem for integrals (Buck, 1965, p. 106), assures us that there exists $t^* \in (t, \bar{t})$ such that $\bar{W}_i = W_i(t^*)$. In general, we can expect $\bar{W}_i$ to lie in a local neighbourhood of $W_i(t)$ and $W_i(\bar{t})$ regardless of the size of $\mathrm{d} \log Q_i/\mathrm{d}t$. Hence for small Δt, we can approximate $\bar{W}_i$ by W_{it}^*. Applying similar reasoning to the right-hand side of Equation 7.2, the right-hand side becomes

$$\bar{\mu}_i \int_t^{\bar{t}} (\mathrm{d} \log \bar{M}/\mathrm{d}t) + \sum_{j=1}^{n} \bar{\pi}_{ij} Dp_{jt},$$

[13] Theoretical support for this assumption of uncorrelated errors and explanatory variables is found in Theil's (1976, Chapters 7 and 8) block recursiveness result.

where $\bar{\mu}_i$ and $\bar{\pi}_{ij}$ are evaluated at t. Also by similar reasoning and by the results on p. 332 of Theil (1971), we find $D\bar{M}_t$ provides a finite change approximation to

$$\int_{t}^{\bar{t}} (\mathrm{d} \log \bar{M}/\mathrm{d}t)\mathrm{d}t.$$

Then 7.1 follows from this result and results 7.2 and 7.3. □

The finite change approximation introduces an approximation error, as shown in the above derivation, but the error will be *uniformly* small over all $t \in T$, if the finite changes are small. When we convert to finite changes, we usually approximate instantaneous flows by annual totals or annual averages. This tends to lead to the interpretation of t as a time in the interior of its year. Observe that this approximation is not local, since we do not restrict the variation in the levels of $(\mathbf{m}(t), \mathbf{p}(t))$ over $t \in T$. When the macroparameters are held constant (as discussed in the next section), the equation system 7.1 subject to the coefficient constraints of Theorem 2 is the absolute price version of the Rotterdam model.

1.8. CONSTANCY OF THE PARAMETERS

Theil (1967, pp. 203–204), Theil (1975, p. 105), and Barten (1974, pp. 13–14) have argued (under assumptions sometimes differing from ours) that variations in the macrocoefficients capture higher order effects than those otherwise inherent in the corresponding terms. Empirical tests of the constancy of the macrocoefficients are available in Barten (1974), Theil and Brooks (1970), Paulus (1972), and Theil (1976, Chapter 15). None of these studies detected explainable parameter variability, and none could reject the hypothesis of constant macroparameters.[14] In this section, we shall parameterize our theoretical system of Equations 4.3. A fundamental objective of the parameterization considered below is simplicity of estimation. Alternative parameterizations are easily constructed, although they necessarily become non-linear in the parameters. We could, for example, derive the macrocoefficients for a world of identical translog consumers (although identical preferences over consumers need not and will not be assumed with the parameterization presented below).

[14] Further evidence on this subject is available in Deaton (1974a). In fact the empirical problem generally is to restrict further the already large number of free parameters, rather than to increase them (Paulus, 1975).

We begin by considering plausible restrictions on the following stochastic processes

$$m_c\mu_c/E(m_c) \quad (8.1)$$

$$m_c\pi_c/E(m_c) \quad (8.2)$$

Consider the mean functions of the processes 8.1 and 8.2.[15] We would expect that the (stochastic) numerator of those expressions would be subject to trends biased upwards as the result of long run trends in nominal income, m_c. However, division by $E(m_c)$ tends, on the average, to deflate that particular source of trend; no reason remains to expect a bias necessarily towards positive (or negative) trends in the sample paths of Equations 8.1 and 8.2 for a randomly selected abstract good. In other words, we have no general theory to guide us in the specification of the macroparameter paths. We do *not* say that the macroparameter will not trend, or should not be expected actually to trend, in one direction or another. We merely state that we are unable to anticipate the direction or nature of such possible trends in advance. If we were considering a particular good, rather than a randomly selected abstract good, we might have prior subjective information, but we consider only theory at present. Hence the following simplifying assumption merits some consideration, although we shall not maintain that assumption.

Assumption 5. The stochastic processes Equations 8.1 and 8.2 have constant mean functions.

The following result is immediate.

Tautology 1. The macrocoefficients 4.1 and 4.2 are constant if and only if Assumption 5 obtains.

Assumption 5 does not exclude sample paths of Equations 8.1 and 8.2 exhibiting either increasing or decreasing trends or even exhibiting cycles in response to variations in $m_c(t)$ and $\mathbf{p}(t)$ over time. Consider, for example, the processes $z(t) = Zt + c$ and

$$x(t) = X\cos\lambda_1(t) + Y\sin\lambda_2(t) + k, \qquad t \in T,$$

where $(c, k, \lambda_1, \lambda_2) > 0$ are constants, and (X, Y, Z) are random variables with zero means. Furthermore, *all* stationary stochastic processes (including the widely used stationary Gaussian process) and many widely used non-stationary processes have constant mean functions (including the Wiener process or Brownian motion, martingales, and symmetric random walks). In fact *any* arbitrary function of time

[15] The mean function of a stochastic process $x(t)$, $t \in T$, is the function of time $f(t) = E(x(t))$, $t \in T$.

is a sample path of any of an infinite number of stochastic processes having constant mean function. Suppose, for example, we seek some arbitrary path $f(t)$. Then consider the process $x(t)$ having path $f(t)$ with probability $\frac{1}{2}$ and path $-f(t)$ with probability $\frac{1}{2}$, so that $E(x(t)) = 0$ for all $t \in T$.

Use of our Assumption 5 would be somewhat analogous to the use of uniform priors in Bayesian statistics, since it would impose no prior tendency in any predetermined direction. However, it should be recognized that if we were to accept Assumption 5, we would pass from the purely theoretical result Equation 4.3 to a parameterized special case. Although we shall not maintain Assumption 5, it is useful to consider the implications of its exact satisfaction. Since our approach is designed for use in the usual case in which a community utility function need not exist, we would hope that Assumption 5 need not imply Gorman's (necessary and sufficient) conditions for the existence of a community utility function. This possibility can be dispelled by the counter-example of a population of Cobb–Douglas consumers having different Cobb–Douglas utility functions (different parameters). Engel curves then would not be parallel, violating Gorman's conditions. But if all consumers had constant income shares in community income, the income weighted average of any subset of the consumers' microcoefficients would be constant over time. Taking the probability limit as the number of consumers in the subset go to infinity, we would find that the macrocoefficients would be constant, and Assumption 5 would be satisfied.

We also may wonder whether Assumption 5 implies homotheticity of preferences or any other such implausible restrictions on preferences. Again we provide a counterexample. Let all consumers have identical tastes with non-linear Engel curves. We immediately have contradicted homotheticity. We place no further restrictions on preferences. Let relative prices remain constant over time, but let price levels vary (non-stochastically and proportionately) over time such that $m_c(t)$ divided by a numeraire price is generated by any arbitrary strong-sense stationary stochastic process. The macroparameters again can be shown to be constants.[16] The purpose of this counter-example is to illustrate that constancy of the macroparameters does not depend exclusively upon preferences. The macroparameters are the mean functions of the stochastic processes 8.1 and 8.2, which depend *jointly* upon the price paths and the income stochastic process as well as upon the tastes of infinitely many consumers and upon the

[16] Recall that each consumer's marginal budget shares and Slutsky coefficients depend solely upon relative prices and numeraire-price-deflated income. Hence our macroparameters are the mean functions of stochastic processes having identical marginal distributions at all t. Constancy of the macroparameters follows immediately.

random vector s_c. Also observe that Assumption 5 depends upon these *actual* joint distributions and paths, not upon every possible such joint distribution and path.

As we have observed previously, we have no prior reason to believe (for an arbitrarily selected unknown abstract good) that the mean function of any arbitrary *one* of the stochastic processes 8.1 and 8.2 will trend in some predictable predetermined direction. But to assume that *all* of these, or of any other macroparameters, will in reality be *jointly* constant over time would be unreasonable. Such an assumption (as Assumption 5) would necessarily be very strong, since it would purport to define a large number of true constants in nature. Even more importantly, lack of knowledge of the form of a trend does *not* imply non-existence of a trend. Although we have no theory suggesting the form or direction of those possible trends, we may well speculate on the merits of further (theoretically unguided and unrestricted) flexibility in the specification of the macroparameter paths. Hence, we shall not accept Assumption 5. In fact in general, rich parameterizations frequently are acquired through local approximations by which functions become constants tautologically through evaluation at a fixed single point of approximation.[17] We now shall use a Taylor series approximation to expand the macroparameters about such a point.

To simplify our discussion, let us assume that H_t (the joint distribution of $m_c(t)$ and s_c at t) has finite moments, and let us stack those moments into the vector $\xi(t)$. Now let $\phi(t) = (\xi_t', p(t)')'$, and let ϕ_0 (perhaps corresponding to a midpoint year or to a centroid of $\{\phi(t) : t \in T\}$ be the value of ϕ_t about which we shall expand the macroparameters. Expand each of the macroparameters in a complete infinite order Taylor series approximation about ϕ_0, and substitute these expansions for the macroparameters in Equation 7.1. Finally let $\psi(t) = (D\bar{M}_t, Dp_{1t}, \ldots, Dp_{nt}, \phi_t' - \phi_0')'$, which is a vector of changes. Dropping terms of the second or higher order in ψ_t, we get back Equation 7.1 with parameters held constant through evaluation at ϕ_0. The class of so-called "flexible functional forms" similarly drops a second order remainder term from its demand system. We shall call our second order remainder term the local remainder term to distinguish it from the global remainder term introduced earlier.

We now shall treat our model as a local approximation of the first order in ψ_t. Macro-parameter constancy would obtain with a uniformly zero remainder term only if Assumption 5 were satisfied. Since we do not maintain Assumption 5, constancy of the macroparameters should be understood to imply the existence of a second order remainder term. As with any such Taylor series approximation, the

[17] Precisely the same procedure was used to acquire tautologically constant parameters for the so-called "flexible functional forms", such as the translog, generalized Leontief, and generalized Cobb–Douglas.

size of the remainder term will be small when we restrict ourselves to a local neighbourhood of the point of the approximation. In our case, the merits of the approximation would be greatest when ψ_t remains small for all $t \in T$.

Our unwillingness to maintain Assumption 5 should be no surprise. It is not our intention to argue that the model provides a perfect approximation (uniformly zero remainder term) under weak assumptions. The model does not. It *is* our intention to argue that the model approximates a theoretical construct Equation 4.3, which *exists* under weak assumptions. The issue that we raise is not the merits of the approximator,[18] but rather the existence of that which is being approximated. Engel curves do not become parallel by looking at them locally. Hence by Gorman's necessary and sufficient conditions for the existence of a community utility function, the available competing models need not approximate anything that exists at the aggregate level-even locally.

1.9. THE ROTTERDAM MODEL'S INTEGRABILITY PROPERTIES

As we have seen in section 1.5, we have no reason to believe that Equation 4.3 is or should be integrable.[19] Hence we have no reason to believe that any empirical specification of or approximation to Equation 4.3 is or should be integrable. Yet the integrability properties of the Rotterdam Model have been the subject of the existing theoretical work on the model. Hence we shall explore those properties in this section. We shall see that Equation 7.1 will be integrable only under extraordinary conditions; but in Theorem 3 we already have seen that *even the general theoretical construct* Equation 4.3 will be integrable only under similar extraordinary conditions.

The existing results on the model generally were deduced from its limit as the finite changes "approach" differentials. We delete ε_{it} from Equation 7.1 and

[18] The merits of the approximator depend upon whether the explanatory variables or the macroparameters vary more rapidly. The use of Taylor series approximations does raise statistical questions about the correlation between the remainder and the disturbance terms. But such problems with higher order terms are inevitable with any model, and if the remainder is small with high probability, then correlation problems are minimized by the Schwartz inequality.

[19] The lack of a correspondence between the model and an integrable aggregate demand system could severely limit the model's usefulness, if the model were prevented from forecasting $\mathbf{Q}_{t+1}$. But no such problem exists. The model is designed to forecast value share transitions (Theil, 1975, p. 39). Hence, if we know the value shares for period t, we can forecast next period's value shares and thereby $\mathbf{Q}_{t+1}$ for given $\mathbf{M}_{t+1}$ and $\mathbf{p}_{t+1}$.

replace the finite changes with differentials to get that

$$W_i \, \mathrm{d}\log Q_i = \bar{\mu}_i\Big[\mathrm{d}\log M - \sum_{k=1}^{n} W_k \mathrm{d}\log p_k\Big] + \sum_{j=1}^{n} \bar{\pi}_{ij}\mathrm{d}\log p_j, \; i = 1,\ldots,n. \quad (9.1)$$

Yoshihara (1969) proved the following theorem, which is central (in one form or another) to the existing theoretical results on the Rotterdam Model.[20]

> **Theorem 5.** The system of Equations 9.1 is origin-closed integrable to a community utility function only at those parameter values for which Equation 9.1 is origin-closed integrable to a Cobb–Douglas community utility function.[21]

We shall define the "feasible parameter set" to be the set of admissible parameter values satisfying the theoretical macrocoefficient restrictions of Theorem 2. The Cobb–Douglas demand system satisfies Equation 9.1 only on a proper subset of that feasible parameter set, although our theoretical foundations obtain *everywhere* on the feasible parameter set. In fact it is easily shown that Equation 9.1 is integrable (to a Cobb–Douglas demand system) only on a negligible *Lebesgue measure zero* subset of the feasible parameter set. We see from inspection of the feasible parameter set that the number of free parameters in that set far exceeds the number of free parameters of a Cobb–Douglas system. Hence it follows immediately that the Cobb–Douglas result can obtain only on a parameter subspace having lower dimension than that of the admissible parameter set itself. Thus by Theorem 5, the model is integrable only on that lower dimensional (and thereby Lebesgue product-measure zero) section of the feasible parameter set. Although Theorem 5 is correct, it informs us of the model's properties (such as unitary income elasticities) almost nowhere (in the language of measure theory). To impute the result of Theorem 5 to the model *in general* is analogous to basing a conclusion on an event which has probability measure zero. Furthermore, any model can be reduced to an absurdity by imposing additional severe restrictions on the parameters. For the Cobb–Douglas properties to relate to the model's applications, users of the Rotterdam Model would have had to

[20] Although, we shall consider the implications of this theorem in detail, perhaps it should be evident immediately that the widely discredited possibility of existence of a community utility function must provide a conspicuously poor criterion for judging a demand model. This fact now is widely recognized by theoreticians. See, e.g., Footnote 1 of Willig (1976).

[21] Yoshihara's proof (applied to his aggregate data) was derived in terms of the model's relative price version. But the proof is equally as applicable to the absolute price version. Although Yoshihara's was the first published proof, other proofs exist. McFadden (1964) has shown that the result can be weakened if the origin-closed condition is dropped.

impose parameter constraints sufficient to restrict the model to the Cobb–Douglas subset of the feasible parameter set. This never has been done.

It is far more instructive to consider the complement of the previously (and correctly) analysed, but negligible, Lebesgue measure zero Cobb–Douglas parameter subset. The model is not integrable on that complement, and hence the model is not integrable "almost everywhere".[22] But our theoretical macroparameter restrictions, provided by Theorem 2, are implied under Assumptions 1 and 2 by *micro*integrability. Those restrictions do not result from or imply aggregate integrability, and they were acquired under assumptions which are vastly weaker than those necessary and sufficient for aggregate integrability.

From the results and discussions of section 1.5, we see now that the properties of the Rotterdam Model are closely related to implications of highly general economic theory. In general, neither the *theoretical* construct Equation 4.3 nor *any* aggregate demand system will be origin-closed integrable except in a pathological case. As seen in Theorem 3, the exceptional case obtains when all consumers' income elasticities for each good are unity.[23] Analogously, the Rotterdam Model's aggregate "differential form" 4.1 will not be origin-closed integrable except on a Lebesgue measure zero subset of the model's feasible parameter set. As seen in Theorem 5, the exceptional case obtains when a Cobb–Douglas community utility function exists (in which case the aggregate income elasticity for each good is unity). But our theoretical system Equation 4.3, *unlike* an aggregate demand system, has the known global properties provided by Theorem 2 whenever our weak Assumptions 1 and 2 are maintained; and the Rotterdam Model locally approximates Equation 4.3 at *any* point in the model's feasible parameter set (not just within the Lebesgue measure zero Cobb–Douglas subset). We do not "approximate" aggregate integrability: we do not use it, need it, or accept it at all.

Although, macrointegrability now has been shown to be irrelevant to our results, one still might ask whether microintegrability could imply implausible restrictions on the Rotterdam Model; at the micro level, integrability *is* a reasonable admissibility condition. To apply Theorem 5 or any of its variants to the micro system 2.3, its microcoefficients must have constant sample paths (independent of variations in $\mathrm{p}(t)$ and $m_c(t)$ over time). But the microcoefficient sample paths are not constant. They can vary with variations in both $\mathrm{p}(t)$ and $m_c(t)$ over time. As we have observed in Tautology 1, the macrocoefficients would be exactly constant (which we do not assume anyway) if and only if the stochastic

[22] This fact appears to have been recognized by Deaton (1974a,b).

[23] As we have seen in section 1.5, aggregate integrability of general theoretical demand systems (as well as of the Rotterdam Model) remains implausible when the origin-closed assumption is removed.

processes 8.1 and 8.2 had constant mean functions. But as we have seen in section 1.6, constancy of the mean functions of those processes does not imply constancy of the processes themselves, and it certainly does not imply constancy of the microcoefficients (either over consumers or over time). Hence the critique does not apply at the micro level. Alternatively one might attempt to apply the Rotterdam Model directly to the behaviour of a single consumer simply by setting $N = 1$ and viewing the single consumer as the total population. But the model was derived by taking stochastic limits as N goes to infinity. If one were to set $N = 1$, none of our stochastic convergence results would obtain. Hence we would be left with our original stochastically varying non-stationary microcoefficients and we therefore would not be able to apply the argument of the model's critics.

1.10. LOCALLY INTEGRABLE COMMUNITY UTILITY FUNCTIONS

Recently, a number of models have appeared having sufficient parameters to provide a second-order local approximation to an arbitrary community utility function.[24] In terms of the aggregate demand system (rather than an unobservable cardinal utility function), these models provide a first-order local approximation. But we have seen that the Rotterdam Model also has sufficient parameters for a local first-order approximation (although to a different theoretical construct). Hence claims of the superior "flexibility" of these models are not readily assessed in terms of the flexibility of the functional form itself. We shall have to consider the model's integrability properties.

The flexibility of these models is severely restricted and in some cases totally destroyed (as with the translog) if global integrability is imposed. Hence users of these models generally seek only local integrability. As a result, we shall refer to these models as the class of "locally integrable" models. The approximation properties of these models are known only if integrability is imposed *a priori* at no more than *a single infinitesimal point* of approximation. If we insist *in advance* (either through an understood agreement, to be enforced as an admissibility condition after the estimates become available, or through prior parameter restrictions) that such a model must be integrable over some predetermined *finite* region, then the model's abilities as an arbitrary first order approximator are lost. Yet in fact users of these models frequently (implicitly or explicitly) do seek integrability over the finite region (preferably the convex closure) of the observed data, since the model's behavior otherwise is suspect. Blackorby et al. (1977) have

[24] They include the translog, generalized Cobb–Douglas, and generalized Leontief models along with a number of other models based upon such quadratic transformations.

shown that if integrability is required *a priori* over any finite region, the models become subject to serious theoretical limitations in their ability to model the preferences even of the representative consumer. These limitations become especially troublesome if separability conditions are imposed, as is commonly desirable when the number of free parameters otherwise would be large. However, if integrability over a finite region is not required in advance, the functional behaviour of the locally integrable models can be troublesome. See Wales (1977). Alternatively Barnett's (1976) g-hypo model is both flexible and *globally* integrable. But g-hypo, like the locally integrable models, does assume the existence of a representative consumer, and hence is better suited to maintaining and *using* theory (for its functional regularity conditions) than to testing it.

Furthermore, we know that the representative consumer and his community utility function exist over any finite region only under entirely implausible conditions. Those who claim superior properties for the class of locally integrable functional forms do so on the grounds that such models permit aggregate integrability to a larger class of community utility functions over the region of the data. It is a strange convention which leads to selection of a model on the basis of its flexibility under entirely implausible conditions. The fact that the Rotterdam Model is restrictive on a Lebesgue measure zero subset of its parameter space (on which even theory dictates restrictiveness) is hardly a serious limitation. But in the usual and truly important case in which the representative consumer does *not* exist, the class of locally integrable models, unlike the Rotterdam Model, has only a distant and barely understood link with economic theory.

It has widely been asserted that empirical tests of the theoretical results of Theorem 2 with the Rotterdam Model implicitly test for the existence of a double log community utility function.[25] As we have seen, this is not true. However tests of theory with the locally integrable models *do* test for aggregate integrability, which we have seen is a theoretically pathological case. Hence the locally integrable models are well designed to test for inherently implausible theoretical restrictions which obtain only under far stronger conditions than our Assumptions 1 and 2. Furthermore, the *prior* imposition of integrability of the popular translog functional form over the local but finite region of the data *is* no easy task, if even possible, without maintaining Cobb–Douglas preferences, and cannot be expected to obtain if not imposed.

It is important to understand that tests for macro integrability of locally integrable models are tests for fundamentally different conditions from the necessary *micro* integrability conditions provided by our Theorem 2. We test for the results of Theorem 2 with the Rotterdam Model. In testing integrability with

[25] See, e.g., Philips (1974, pp. 56, 58, 89, and 94) and Christensen et al. (1975).

the locally integrable models, the maintained hypothesis is the existence of the models themselves. But here the necessary and sufficient conditions are Gorman's extremely strong conditions of linear Engel curves which are parallel across all consumers. The null hypothesis merely adds micro theory to Gorman's Engel curve restrictions. Before we even consider the mild transition from the maintained to the null hypothesis, we should consider the approximating properties of the patently unacceptable *maintained* hypothesis, on which the existence of the locally integrable models themselves depend. The alternative to the maintained hypothesis is rejection of the model itself.

Theoreticians accept micro-integrability and (strongly) question parallel linear Engel curves. Hence the interesting hypothesis test, relative to the necessary and sufficient conditions for macrointegrability, would be to maintain the weak assumption of micro-integrability and test the null hypothesis of parallel linear Engel curves. But that which is approximated by the locally integrable models exists if and only if parallel linear Engel curves obtain. Hence users of those models must *maintain* the interesting null hypothesis. In that case the above procedure would lead to equality of the null and maintained hypothesis, rendering rejection logically impossible. The locally integrable models *cannot test for macro integrability* at all.

The most that can be done in testing theory with such models is to maintain the strong assumption of parallel linear Engel curves and test the weak assumption of microintegrability. This, of course, is precisely the reverse of the interesting economic test and is methodologically convoluted. Under such circumstances, it is not surprising that tests of "theory" with these models have rejected the restrictions (such as symmetry) required to permit adequate precision of the model's parameter estimators. See, e.g., Christensen et al. (1975) and Berndt et al. (1977). By contrast the Rotterdam Model's symmetry restrictions have been accepted empirically by Theil (1971, 1975) and Deaton (1974a).

1.11. CONCLUSION

We have shown that far more is known about our general stochastically limiting differential Equation 4.3 than is known about conventional aggregate demand systems. We then have explored the properties of Equation 4.3 subject to the particular Rotterdam Model parameterization. Prior results on the Rotterdam Model's theoretical foundations related solely to a Lebesgue measure zero subset of the space of admissible (consistent with Assumptions 1 and 2) macrocoefficients. We have extended the existing results to apply throughout the much larger feasible set used in the model's empirical applications, and we have found the model's generality to be expanded considerably on the complement of the previously investigated Lebesgue measure zero subset. In addition, theory tells us that general aggregate demand properties on the negligible subset *should* be restrictive.

Continued analysis of the Rotterdam Model's specific parameterization of Equation 4.3 could profitably consider its potentially (although perhaps not typically) troublesome nonsymmetric asymptotic Slutsky aggregation bias. Acquisition of a more rigorously controllable Slutsky aggregation bias might be a fruitful objective in considering extensions of or alternatives to the Rotterdam Model's particular parameterization of our fundamental theoretical construct Equation 4.3. One potential generalization of the parameterization of the relative price version of the Rotterdam Model has been proposed by Theil (1975, pp. 108-112). Its usefulness is not yet known, although negative results have been reported in Section 15.4 of Theil (1976). An alternative generalization is proposed in Section 7.3 of Theil (1976) and is considered further in Sections 15.6 to 15.8 of the same source. An empirical rejection of Slutsky symmetry with the current parameterization would reflect the existence of non-negligible non-symmetric aggregation biases rather than any violations of theory. However available empirical evidence tends to support our conjecture that the Rotterdam Model's Slutsky aggregation bias is small.[26]

The class of so-called "flexible" functional forms (translog, etc.) are similar to the Rotterdam Model in providing first order Taylor series approximations to some theoretical system of equations. The Rotterdam Model has the advantage of approximating a more general theoretical construct (our system Equation 4.3) than that which is approximated by the flexible forms. That which is approximated by the Rotterdam Model exists under far weaker assumptions than that (the demand system of a representative consumer) which is approximated by the flexible forms. However the flexible forms have the advantage of providing a better understood approximation to that which is being approximated (when it exists). The flexible forms acquire their approximation exclusively through dropping the remainder term of a first order local Taylor series expansion. The Rotterdam Model acquires its approximation by dropping the remainder term of a first order local Taylor series expansion (our "locally small remainder term") *and* dropping an additional remainder term (our "globally small remainder term").

[26] See, e.g., Theil (1971, pp. 340–344) for a successful test of Slutsky symmetry. But such results cannot be viewed as conclusive for other potential data or goods.

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