# Letter to the Editor: Singular-manifold view on a (3+1)-dimensional fourth-order nonlinear equation in a fluid via HFF 32, 1664 (2022)



Recently, Wazwaz (2022) and Meng *et al.* (2023) have made some outstanding contributions to a $(3+1)$-dimensional integrable fourth-order nonlinear equation in a fluid, which is:

$$v_{tt} - v_{xxxt} - 3(v_x v_t)_x + \alpha v_{xt} + \beta v_{yt} + \gamma v_{zt} = 0, \tag{1}$$

with $\gamma$, $\beta$ and $\alpha$ being the real nonzero constants, $v(x, y, z, t)$ denoting a real differentiable function of the independent variables $x, y, z$ and $t$, while the subscripts representing the partial derivatives (Meng *et al.*, 2023). For equation (1), Wazwaz (2022) has investigated the Painlevé integrability, lump and multiple soliton solutions, while Meng *et al.* (2023) has presented the special cases in fluid dynamics, bilinear auto-Bäcklund transformations, breather and mixed lump-kink solutions.

This Letter, based on the work in Wazwaz (2022) and Meng *et al.* (2023), aims to seek an auto-Bäcklund transformation for equation (1), which is different from those in Meng *et al.* (2023).

In equation (1) let us put the truncated Painlevé expansion, in a generalized Laurent series (Zhou and Tian, 2022; Zhou *et al.*, 2023; Gao, 2023a, 2023b, 2023c), around a noncharacteristic movable singular manifold conferred by an analytic function $\psi(x, y, z, t) = 0$, as:

$$v(x, y, z, t) = \psi^{-K}(x, y, z, t) \sum_{k=0}^{K} v_k(x, y, z, t)\, \psi^k(x, y, z, t), \tag{2}$$

where $v_k(x, y, z, t)$'s also represent the analytic functions, with $v_0(x, y, z, t) \neq 0$, $\psi_x(x, y, z, t) \neq 0$ and $\psi_t(x, y, z, t) \neq 0$, and if the powers of $\psi$ at the lowest orders cancel out, the positive integer:

$$K = 1. \tag{3}$$

Using symbolic computation (Wu *et al.*, 2022a, 2022b; Shen *et al.*, 2022, 2023; Gao and Tian, 2022; Gao *et al.*, 2021, 2022) and substituting formulae (2) and (3) into equation (1), we recommend that the coefficients of like powers of $\psi$ fade away, to obtain the Painlevé-Bäcklund equations:

$$\psi^{-5}: \; v_0 = 2\psi_x, \tag{4}$$

$$\psi^{-4}: \; \text{(satisfied)}$$

$$\psi^{-3}: \; \alpha\psi_x\psi_t + \beta\psi_y\psi_t + \gamma\psi_z\psi_t - 3\psi_x\psi_t v_{1,x} - 3\psi_x^2 v_{1,t} + \psi_t^2 - \psi_{xxx}\psi_t + 3\psi_{xt}\psi_{xx} - 3\psi_x\psi_{xxt} = 0, \tag{5}$$

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$$\psi^{-2}: \ 2\alpha\psi_{xt}\psi_x + \alpha\psi_t\psi_{xx} + \beta\psi_{yt}\psi_x + \beta\psi_y\psi_{xt} + \beta\psi_t\psi_{xy} + \gamma\psi_{zt}\psi_x + \gamma\psi_z\psi_{xt} + \gamma\psi_t\psi_{xz} - 3\psi_x^2 v_{1,xt} - 6\psi_{xt}\psi_x v_{1,x} - 9\psi_{xx}\psi_x v_{1,t} - 3\psi_t\psi_x v_{1,xx} - 3\psi_t\psi_{xx} v_{1,x} + \psi_{tt}\psi_x - 4\psi_{xxxt}\psi_x + 2\psi_t\psi_{xt} + 2\psi_{xt}\psi_{xxx} - \psi_t\psi_{xxxx} = 0, \tag{6}$$

$$\psi^{-1}: \ \alpha\psi_{xxt} + \beta\psi_{xyt} + \gamma\psi_{xzt} - 3\psi_{xx}v_{1,xt} - 3\psi_{xt}v_{1,xx} - 3\psi_{xxt}v_{1,x} - 3\psi_{xxx}v_{1,t} + \psi_{xtt} - \psi_{xxxxt} = 0, \tag{7}$$

$$\psi^0: \ v_{1,tt} - v_{1,xxxt} - 3(v_{1,x}v_{1,t})_x + \alpha v_{1,xt} + \beta v_{1,yt} + \gamma v_{1,zt} = 0. \tag{8}$$

Mutually consistent or as noticed below, explicitly solvable with respect to $\psi(x, y, z, t)$, $v_0(x, y, z, t)$ and $v_1(x, y, z, t)$, equations (2)–(8) fashion an auto-Bäcklund transformation for equation (1).

Next, the assumptions:

$$\psi(x, y, z, t) = e^{\eta_1 x + \eta_2 y + \eta_3 z + \eta_4 t + \eta_5} + 1,$$
$$v_1(x, y, z, t) = \eta_6 x + \eta_7 y + \eta_8 z + \eta_9 t + \eta_{10},$$

are substituted into auto-Bäcklund transformation (2)–(8) via symbolic computation, leading to:

$$\eta_9 = \frac{\eta_4(\alpha\eta_1 + \beta\eta_2 - \eta_1^3 - 3\eta_1\eta_6 + \eta_3\gamma + \eta_4)}{3\eta_1^2}$$

and the following explicit soliton solutions for equation (1):

$$v(x, y, z, t) = \eta_1 \tanh\left(\frac{\eta_1 x + \eta_2 y + \eta_3 z + \eta_4 t + \eta_5}{2}\right) + \eta_6 x + \eta_7 y + \eta_8 z + \frac{\eta_4(\alpha\eta_1 + \beta\eta_2 - \eta_1^3 - 3\eta_1\eta_6 + \eta_3\gamma + \eta_4)}{3\eta_1^2} t + \eta_1 + \eta_{10}, \tag{9}$$

where $\eta_1 \ldots \eta_{10}$ are the real constants with $\eta_1 \neq 0$ and $\eta_4 \neq 0$.

Our results are linked to $\gamma$, $\beta$ and $\alpha$, the coefficients in equation (1).

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