Letter to the Editor: Considering the Whitham-Broer-Kaup system in the shallow water inspired by HFF 33, 3801; 34, 1189 and 34, 2197

Recently, Ramos and Garcia Lopez (2024) and Gao (2023a, 2023b) have conducted several comprehensive shallow-water studies in *International Journal of Numerical Methods for Heat and Fluid Flow*, that is, blowup in finite time of the solutions to a one-dimensional, bidirectional, nonlinear wave model equation for the propagation of small-amplitude waves in the shallow water (Ramos and Garcia Lopez, 2024), novel hetero-Bäcklund transformations and similarity reductions for a $(2+1)$-dimensional generalized dispersive long-wave system that models certain dispersive and nonlinear long gravity waves on the oceanic shallow water (Gao, 2023a), and novel similarity reductions for a $(2+1)$-dimensional generalized modified dispersive water-wave system for the nonlinear and dispersive long gravity waves in the shallow water of uniform depth (Gao, 2023b).

Those studies (Gao, 2023a, 2023b; Ramos and Garcia Lopez, 2024) have sparked renewed interest in shallow-water research. Consequently, this Letter is dedicated to the Whitham-Broer-Kaup system for the dispersive long waves in the shallow oceanic water (Ahmad *et al.*, 2015; Fan and Bao, 2024; Sabawi and Hamad, 2024; Fan and Bao, 2022; Gao *et al.*, 2020; Cao *et al.*, 2020; Arshad *et al.*, 2017; Ren and Lin, 2018; Kuo, 2017; Fei *et al.*, 2017; Zhou and Lu, 2017; Xu *et al.*, 2017; Lin *et al.*, 2011a, 2011b; Bhrawy *et al.*, 2014; Shan *et al.*, 2012; Wang *et al.*, 2010, 2009; Zhang *et al.*, 2008) (and references therein), i.e.:

$$\begin{aligned} &p_t + pp_x + q_x + \beta p_{xx} = 0, \\ &q_t + (pq)_x + \alpha p_{xxx} - \beta q_{xx} = 0, \end{aligned} \tag{1}$$

where x and t are the scaled space and time coordinates, the subscripts represent the partial derivatives, $p(x, t)$ and $q(x, t)$ are the real functions denoting the horizontal velocity field and height of the deviation from the equilibrium position, respectively, α is the real dispersion coefficient and β is the real diffusion coefficient.

For System (1), Bäcklund transformation, soliton and similar wave solutions have been studied (Fan and Bao, 2024); quartic B-spline collocation method has been used to obtain the numerical solutions (Sabawi and Hamad, 2024); Painlevé integrability, bilinear forms and Bäcklund transformations have been offered (Gao *et al.*, 2020; Zhang *et al.*, 2008; Fan and Bao, 2022); two different groups of variational principles have been constructed

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Statements and Declarations

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(Cao *et al.*, 2020); travelling wave solutions in the form of solitons, dark solitons, bell and anti-bell periodic waves have been obtained (Ahmad *et al.*, 2015; Arshad *et al.*, 2017); nonlocal symmetry, consistent tanh-expansion solvability and power-series solutions have been studied (Ren and Lin, 2018; Zhou and Lu, 2017); modified simplest equation method has been used to derive some soliton solutions (Kuo, 2017); residual symmetries and interaction solutions have been investigated (Fei *et al.*, 2017); double Wronskian solutions and soliton interactions have been studied via the Wronskian technique (Xu *et al.*, 2017; Lin *et al.*, 2011a, 2011b; Wang *et al.*, 2009); solitons, cnoidal waves and snoidal waves have been presented (Bhrawy *et al.*, 2014); as well as Darboux transformation and some soliton solutions have been investigated (Shan *et al.*, 2012; Wang *et al.*, 2010). Other contributions on System (1) have appeared as the references in the aforementioned papers.

However, to the best of our knowledge, for System (1), generalized Darboux transformation (GDT), which can be used to derive the rogue wave, multi-pole and mixed solutions, has not yet been reported. Using symbolic computation (Gao, 2023a, 2024), we aim to construct a GDT for System (1). With respect to the horizontal velocity field and the height of the deviation from the equilibrium position, the GDT supports the derivation of the multi-pole solitons, multi-pole breathers, higher-order rogue waves and mixed waves. We hope this study contributes to a deeper understanding of the physical dynamics of the nonlinear waves in shallow water environments.

Via the transformations:

$$p = 2\sqrt{\beta^2+\alpha}P,\ q = 4\left(\beta^2+\alpha\right)Q - 2\sqrt{\beta^2+\alpha}\left(\sqrt{\beta^2+\alpha}+\beta\right)P_x, \tag{2}$$

we transform System (1) into:

$$\begin{aligned} &P_t + \sqrt{\beta^2+\alpha}\left(2Q_x + 2PP_x - P_{xx}\right) = 0, \\ &Q_t + \sqrt{\beta^2+\alpha}\left[Q_{xx} + 2(PQ)_x\right] = 0. \end{aligned} \tag{3}$$

Wang *et al.* (2010) has offered a Lax pair for System (3):

$$\begin{aligned} &\Phi_x = U\Phi,\ U = \frac{1}{2}\begin{pmatrix} \lambda - P & -2Q \\ 2 & P-\lambda \end{pmatrix}, \\ &\Phi_t = V\Phi,\ V = \frac{\sqrt{\beta^2+\alpha}}{2}\begin{pmatrix} P^2 - P_x - \lambda^2 & 2Q(P+\lambda) + 2Q_x \\ -2(P+\lambda) & -\left(P^2 - P_x - \lambda^2\right) \end{pmatrix}, \end{aligned} \tag{4}$$

where $\Phi = (\phi, \psi)^{\mathrm{T}}$ is the eigenfunction, λ is a real spectral parameter and ϕ_1 and ϕ_2 are the complex functions of x and t. The compatibility condition $\Phi_{xt} = \Phi_{tx}$ leads to System (3).

We consider a*n* n-fold gauge transformation that transforms Lax Pair (4) into:

$$\begin{aligned} &\Phi^{[N]} = \Delta^{[N]}\Phi, \\ &\Phi_x^{[N]} = U^{[N]}\Phi^{[N]}, \\ &\Phi_t^{[N]} = V^{[N]}\Phi^{[N]}, \end{aligned} \tag{5}$$

where $[N]$ represents the N-fold solutions/matrices, $\Phi^{[N]} = (\phi^{[N]}, \psi^{[N]})^T$, $\Delta^{[N]}$ is the 2×2 gauge transformation matrix and N is a positive integer. $U^{[N]}$ and $V^{[N]}$ own the same forms as

U and V except that the initial potentials P and Q are replaced with their N-fold versions $P^{[N]}$ and $Q^{[N]}$. Combining Gauge Transformation (5) with Lax Pair (4), we derive:

$$\begin{aligned}&\left(\Delta^{[N]}\right)_x+\Delta^{[N]}U-U^{[N]}\Delta^{[N]}=0,\\&\left(\Delta^{[N]}\right)_t+\Delta^{[N]}V-V^{[N]}\Delta^{[N]}=0.\end{aligned}\tag{6}$$

Motivated by Wang *et al.* (2010), the n-fold gauge matrix $\Delta^{[n]}$ can be set as:

$$\Delta^{[N]}=\begin{pmatrix}\omega\left(\lambda^N+\sum\limits_{j=0}^{N-1}a_j\lambda^j\right) & \omega\left(\sum\limits_{j=0}^{N-1}b_j\lambda^j\right)\\ \omega^{-1}\left(\sum\limits_{j=0}^{N-1}c_j\lambda^j\right) & \omega^{-1}\left(\lambda^N+\sum\limits_{j=0}^{N-1}d_j\lambda^j\right)\end{pmatrix},\tag{7}$$

where $j=0,1,2,\ldots,N-1$, a_j's, b_j's, c_j's and d_j's are $4N$ functions of x and t, and $\omega^2=1+c_{N-1}$. To determine $4N$ unknown functions a_j's, b_j's, c_j's and d_j's, we suppose that λ_1, λ_2, …, λ_{2n} are the $2n$ roots of $\mathrm{Det}(\Delta^{[N]})$, i.e.:

$$\mathrm{Det}\left(\Delta^{[N]}\right)=\prod\nolimits_{k=1}^{n}(\lambda-\lambda_{2k-1})^{m_k+1}(\lambda-\lambda_{2k})^{m_k+1},\tag{8}$$

where Det denotes the determinant, $k=1,2,\ldots,n$, $n\le N$, m_k is a nonnegative integer and $n+\sum_{k=1}^{n}m_k=N$. We assume that $\Phi_{2k-1}=(\phi_{2k-1},\psi_{2k-1})^T$ and $\Phi_{2k}=(\phi_{2k},\psi_{2k})^T$ respectively are the eigenfunction of Lax Pair (4) at $\lambda=\lambda_{2k-1}$, and $\lambda=\lambda_{2k}$, and the superscript T denotes the transpose of the matrix.

According to $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k-1}}=0$ and $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k}}=0$, we give the following $4n$ equations:

$$\begin{aligned}&\phi_{2k-1}\sum_{j=0}^{N-1}a_j\lambda_{2k-1}^j+\psi_{2k-1}\sum_{j=0}^{N-1}b_j\lambda_{2k-1}^j=-\lambda_{2k-1}^N\phi_{2k-1},\\&\phi_{2k}\sum_{j=0}^{N-1}a_j\lambda_{2k}^j+\psi_{2k}\sum_{j=0}^{N-1}b_j\lambda_{2k}^j=-\lambda_{2k}^N\phi_{2k},\\&\phi_{2k-1}\sum_{j=0}^{N-1}c_j\lambda_{2k-1}^j+\psi_{2k-1}\sum_{j=0}^{N-1}d_j\lambda_{2k-1}^j=-\lambda_{2k-1}^N\psi_{2k-1},\\&\phi_{2k}\sum_{j=0}^{N-1}c_j\lambda_{2k}^j+\psi_{2k}\sum_{j=0}^{N-1}d_j\lambda_{2k}^j=-\lambda_{2k}^N\psi_{2k}.\end{aligned}\tag{9}$$

Nonetheless, the $4n$ equations presented in equations (9) are inadequate for uniquely determining the $4N$ unknown functions A_j's, B_j's, C_j's and D_j's. We consider the Taylor expansions $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k-1}+\epsilon}=0$ and $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k}+\epsilon}=0$ at $\epsilon=0$:

$$\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k-1}+\varepsilon,\lambda_{2k}+\varepsilon}=\sum_{\zeta=0}^{m_k}\sum_{\iota=0}^{\zeta}\Delta^{[N](\iota)}|_{\lambda=\lambda_{2k-1},\lambda_{2k}}\Phi^{(\zeta-\iota)}|_{\lambda=\lambda_{2k-1},\lambda_{2k}}\varepsilon^{\zeta}+O(\varepsilon^{m_k}),$$

$$\Delta^{[N](\zeta)}|_{\lambda=\lambda_{2k-1}}=\begin{pmatrix} C_N^{\zeta}\lambda_{2k-1}^{N-\zeta}+\sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}a_j\lambda_{2k-1}^{j-\zeta} & \sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}b_j\lambda_{2k-1}^{j-\zeta} \\ \sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}c_j\lambda_{2k-1}^{j-\zeta} & C_N^{\zeta}\lambda_{2k-1}^{N-\zeta}+\sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}d_j\lambda_{2k-1}^{j-\zeta} \end{pmatrix},$$

$$\Delta^{[N](\zeta)}|_{\lambda=\lambda_{2k}}=\begin{pmatrix} C_N^{\zeta}\lambda_{2k}^{N-\zeta}+\sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}a_j\lambda_{2k}^{j-\zeta} & \sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}b_j\lambda_{2k}^{j-\zeta} \\ \sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}c_j\lambda_{2k}^{j-\zeta} & C_N^{\zeta}\lambda_{2k}^{N-\zeta}+\sum\limits_{j=\zeta}^{N-1}C_j^{\zeta}d_j\lambda_{2k-1}^{j-\zeta} \end{pmatrix},$$

$$\Phi^{(\zeta)}|_{\lambda=\lambda_{2k-1}}=\begin{pmatrix}\varphi_{2k-1}^{(\zeta)}\\ \psi_{2k-1}^{(\zeta)}\end{pmatrix}=\frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k-1}^{\zeta}}\Phi|_{\lambda=\lambda_{2k-1}}=\begin{pmatrix}\frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k-1}^{\zeta}}\varphi_{2k-1}\\ \frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k-1}^{\zeta}}\psi_{2k-1}\end{pmatrix}, \tag{10}$$

$$\Phi^{(\zeta)}|_{\lambda=\lambda_{2k}}=\begin{pmatrix}\varphi_{2k}^{(\zeta)}\\ \psi_{2k}^{(\zeta)}\end{pmatrix}=\frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k}^{\zeta}}\Phi|_{\lambda=\lambda_{2k}}=\begin{pmatrix}\frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k}^{\zeta}}\varphi_{2k}\\ \frac{1}{\zeta!}\frac{\partial^{\zeta}}{\partial\lambda_{2k}^{\zeta}}\psi_{2k}\end{pmatrix},$$

where ϵ is a small parameter, $\iota=0,1,\ldots,\zeta$, $\zeta=0,1,\ldots,m_k$ and $C_N^{\zeta}=\frac{N!}{\zeta!(N-\zeta)!}$.

We truncate $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k-1}+\epsilon}=0$ and $\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k}+\epsilon}=0$ to the m_k-th order and obtain:

$$\lim_{\epsilon\to 0}\left(\Delta^{[N]}\Phi\right)|_{\lambda=\lambda_{2k-1}+\epsilon,\lambda_{2k}+\epsilon}\epsilon^{-\zeta}=0,$$

i.e.:

$$\sum_{\iota=0}^{\zeta}\Delta^{[N](\iota)}|_{\lambda=\lambda_{2k-1}}\Phi^{(\zeta-\iota)}|_{\lambda=\lambda_{2k-1}}=0,\ \sum_{\iota=0}^{\zeta}\Delta^{[N](\iota)}|_{\lambda=\lambda_{2k}}\Phi^{(\zeta-\iota)}|_{\lambda=\lambda_{2k}}=0. \tag{11}$$

Combining Transformations (2), equations (6), N-Fold DT Matrix (7) and equations (11), we calculate out the Nth-order solutions for System (1) as:

$$\begin{aligned}&p^{[N]}=2\sqrt{\beta^2+\alpha}P^{[N]},\ q^{[N]}=4\left(\beta^2+\alpha\right)Q^{[N]}-2\sqrt{\beta^2+\alpha}\left(\sqrt{\beta^2+\alpha}+\beta\right)\left(P^{[N]}\right)_x,\\ &P^{[N]}=P^{[0]}-\left[\ln(1+c_{N-1})\right]_x,\ Q^{[N]}=(1+c_{N-1})\left(Q^{[0]}+b_{N-1}\right),\\ &b_{N-1}=\frac{\Upsilon_b}{\Upsilon},\ c_{N-1}=\frac{\Upsilon_c}{\Upsilon},\\ &\Upsilon=\mathrm{Det}\left(\Upsilon_1^T,\quad\Upsilon_2^T,\quad\cdots,\quad\Upsilon_n^T\right)^T,\end{aligned}$$

$$
\begin{aligned}
&(\Upsilon_k)_{\varsigma,\tau} = \begin{cases}
\displaystyle\sum_{\rho=0}^{\varsigma-1} C_{N-\tau}^{\rho}\lambda_{2k-1}^{N-\tau-\rho}\varphi_{2k-1}^{(\varsigma-1-\rho)}, \; 1 \le \varsigma \le m_k+1, \; 1 \le \tau \le N,\\
\displaystyle\sum_{\rho=0}^{\varsigma-1} C_{2N-\tau}^{\rho}\lambda_{2k-1}^{2N-\tau-\rho}\psi_{2k-1}^{(\varsigma-1-\rho)}, \; 1 \le \varsigma \le m_k+1, \; N+1 \le \tau \le 2N,\\
\displaystyle\sum_{\rho=0}^{\varsigma-m_k-2} C_{N-\tau}^{\rho}\lambda_{2k}^{N-\tau-\rho}\varphi_{2k}^{(\varsigma-2-m_k-\rho)}, \; m_k+2 \le \varsigma \le 2(m_k+1), \; 1 \le \tau \le N,\\
\displaystyle\sum_{\rho=0}^{\varsigma-m_k-2} C_{2N-\tau}^{\rho}\lambda_{2k}^{2N-\tau-\rho}\psi_{2k}^{(\varsigma-2-m_k-\rho)}, \; m_k+2 \le \varsigma \le 2(m_k+1), \; N+1 \le \tau \le 2N,
\end{cases}\\
&(\upsilon)_{2N\times 1} = (\upsilon_1, \upsilon_2, \cdots, \upsilon_n)^T, \; (\mu)_{2N\times 1} = (\mu_1, \mu_2, \cdots, \mu_n)^T,\\
&(\upsilon_k)_{1,\varsigma} = \begin{cases}
-\displaystyle\sum_{\rho=0}^{\varsigma-1} C_N^{\rho}\lambda_{2k-1}^{N-\rho}\varphi_{2k-1}^{(\varsigma-1-\rho)}, \; 1 \le \varsigma \le m_k+1,\\
-\displaystyle\sum_{\rho=0}^{\varsigma-m_k-2} C_N^{\rho}\lambda_{2k}^{N-\rho}\varphi_{2k}^{(\varsigma-m_k-2-\rho)}, \; m_k+2 \le \varsigma \le 2(m_k+1),
\end{cases}\\
&(\mu_k)_{1,\varsigma} = \begin{cases}
-\displaystyle\sum_{\rho=0}^{\varsigma-1} C_N^{\rho}\lambda_{2k-1}^{N-\rho}\psi_{2k-1}^{(\varsigma-1-\rho)}, \; 1 \le \varsigma \le m_k+1,\\
-\displaystyle\sum_{\rho=0}^{\varsigma-m_k-2} C_N^{\rho}\lambda_{2k}^{N-\rho}\psi_{2k}^{(\varsigma-m_k-2-\rho)}, \; m_k+2 \le \varsigma \le 2(m_k+1),
\end{cases}
\end{aligned}
\tag{12}
$$

where Υ_k is a $2(m_k + 1) \times 2N$ matrix, $1 \le \varsigma \le 2(m_k + 1)$, $1 \le \tau \le 2N$, $(\Upsilon_k)_{\varsigma,\tau}$ denotes the element in the ςth row and τth column of Υ_k, Υ_b can be obtained via the modified Υ with its $(N + 1)$ th column replaced by the vector $(\upsilon)_{2N\times1}$, Υ_c can be obtained via the modified Υ with its 1st column replaced by the vector $(\mu)_{2N\times1}$.

In conclusion, N-Fold DT Matrix (7) and The Nth-Order Solutions (12) together establish an N-fold GDT for System (1). In The Nth-Order Solutions (12), the variable n indicates the number of sets of spectral parameters utilized, the variable m_k represents that the kth set of spectral parameters is iterated m_k times (a total of $m_k + 1$ times) and the value of N represents the total order of The Nth-Order Solutions (12). Notably, when $n = N$, i.e. $m_k = 0$, each of the n spectral parameters is iterated only once. In this case, the resulting GDT simplifies to the N-fold DT proposed by Shan *et al.* (2012) and Wang *et al.* (2010).

In relation to the horizontal velocity field and height of the deviation from the equilibrium position, the previously obtained GDT allows for multiple iterations of the spectral parameters, leading to the generation of the multi-pole solitons, multi-pole breathers, higher-order rogue waves and mixed waves. This study might assist in exploring the complex and dynamic natural mechanisms underlying real-world shallow water waves.

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