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Valuation of step-down knock-in in one stock linked security using numerical and Monte Carlo integration

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Abstract

This paper shows a new methodology for evaluating the value and sensitivity of autocall knock-in type equity-linked securities. While the existing evaluation methods, Monte Carlo simulation and finite difference method, have limitations in underestimating the knock-in effect, which is one of the important characteristics of this type, this paper presents a precise joint probability formula for multiple autocall chances and knock-in events. Based on this, the calculation results obtained by utilizing numerical and Monte Carlo integration are presented and compared with those of existing models. The results of the proposed model show notable improvements in terms of accuracy and calculation time.

Keywords Numerical integration, Hitting time distribution, Knock-in, Monte Carlo Integration, Stepdown ELS

Paper type Research paper

1. Introduction

Equity-linked security (ELS) is a product of which interest or redemption amount is linked to specific stock performance and is characterized by providing somewhat higher returns than plain bonds as compensation for taking a risk of stock price fluctuations. At the end of 2021, ELS market trading volume was worth KRW 34 trillion and the number of securities was 11,008 [1]. In 2021, the total number of ELS issuances was 12,672 and 3,960 securities among them have a structure of step-down autocall knock-in (SDKI) [2]. In addition, 289 of them are linked to one stock price. The ELS market, which had shown solid growth since its launch, peaked at KRW 54.5 trillion at the end of 2018, and now appears faltering due to regulatory pressures from the financial authority for the investor protection and the instability in risk hedging [3], but it still has a relatively high return compared to the bond market. The issuance of ELS has continuously been taking place and attracted the interest of investors.

Along this background, a rigorous analysis about the fair price and the inherent risk of SDKI at this time may be overdue but still a necessary task. Currently, a valuation methods applied to this type by asset pricing firms and financial companies is Monte Carlo simulation (MCS) and finite difference method (FDM). These two methods, however, have limitations in

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underestimating the knock-in (KI) characteristics of the SDKI. In general, KI refers to a characteristic in which a loss of some or whole principal amount occurs depending on the performance of underlying stock at maturity if the stock price has been set below KI barrier during the contract period, including intraday trading prices. In practice, the determination whether KI or not in MCS and FDM is made on the evaluation dates of autocall (usually, every six month). The longer the judgment period, the shorter the calculation time, since the possibility of KI in the interim period between two consecutive evaluation dates is not taken into account. Therefore, pricing firms or financial companies will prepare and apply their own compromising criteria between the incompatible goals of improving the accuracy of the pricing and of reducing the calculation time. Since these limitations are specific to the methodology, this paper proposes a new evaluation model to overcome them and analyzes its performance with those of existing methodologies. The proposed model is a method based on Monte Carlo integration (MCI) and numerical integration (NI). It has significantly better advantages over the performance of existing models in terms of price accuracy, Greek stability and calculation time. Although MCS has the advantage of being relatively easy to implement algorithms for the securities linked on multi underlying assets, it also has the disadvantage of inherent instability in price and sensitivity and is relatively time consuming. In case of FDM, the sensitivity is stable, but if it has three or more underlying assets, the implementation algorithm is complicated and also takes a plenty of time. In order to overcome these limitations, there are papers that have studied using a semi-closed-form solution approach Lee and Hong (2021), but the effect of reducing the calculation time appears not to be large because time consuming simulation is inevitably used to reflect the KI effect. Also, in Europe, this type is distributed under the name of "Express Certificates", and there is a case of studying its valuation Wilhelm (2009) [4]. The proposed model of this study also uses some simulation techniques to calculate the probabilities of multivariate normal distribution, but the method uses uniform random numbers rather than normal ones, so the time-saving effect appears not to be small. This kind of method is called Monte Carlo Integration. Deák (1980) applied this method to calculate the probability value of the multivariate normal distribution and calculated the exact value to the third decimal place. Since then, Schervish (1984), Genz (1992), Deák (1986) and Szántai (2001) have made significant progress in calculating multivariate normal distribution probability values. Genz (1992) proposed a concise formula for easy calculation with MCI. Comparing the probability values of the multivariate normal distribution using MCI and direct NI, the same digits are produced up to 13 decimal places in the case of bivariate normal distribution, and in the case of trivariate, there is a slight difference at the level of 0.1 basis point (bp). In terms of calculation time, direct NI takes 1.7 s for three variables and more than 1 min for four variables, so it turned out to be unreasonable to apply this method. Therefore, we adopt Genz (1992) with higher implementation extensibility. This methodology provides the marginal errors implied by the model according to the number of variables. The errors up to six variables show up to the level of 0.1 bp. In addition to the multivariate normal distribution, the method for evaluating the KI and no-KI probabilities uses NI because it appears bivariate integration when there is only one underlying asset.

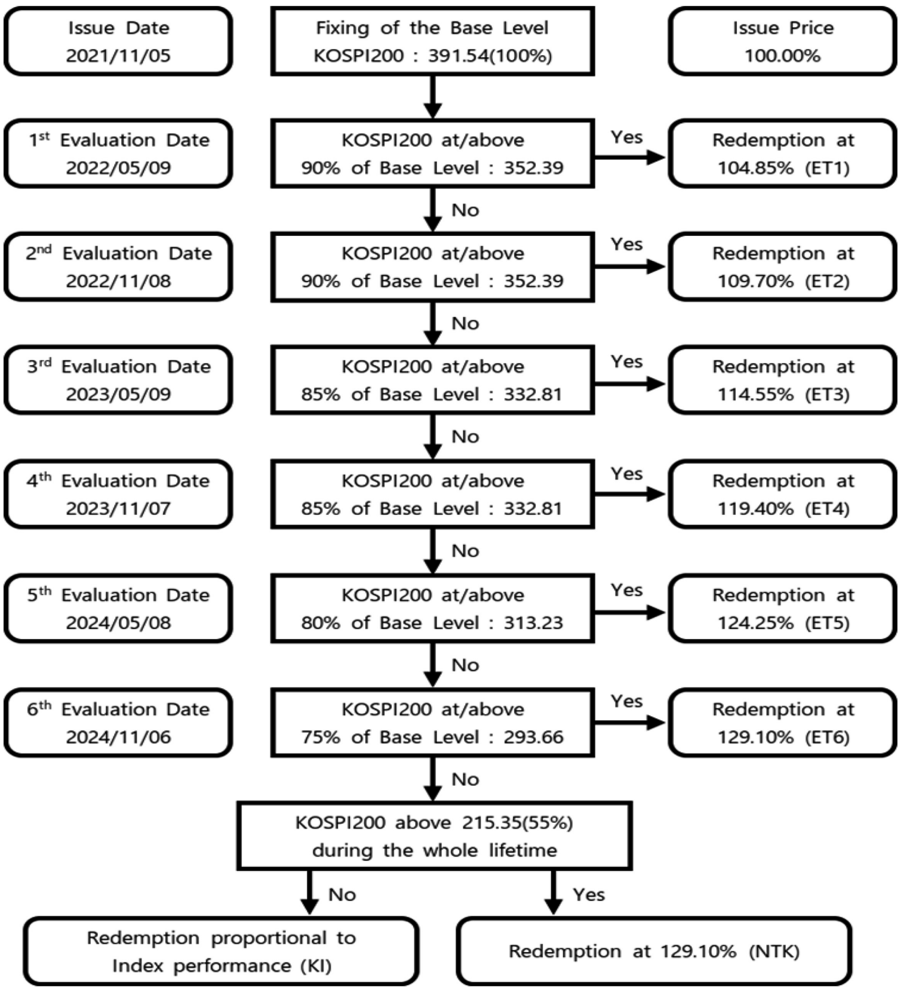
The paper is structured as follows. The proposed model is explained in section 2 and the analysis results are given in section 3. Finally, the conclusion is presented in section 4.

2. SDKI model

2.1 SDKI ELS

A general structure of SDKI is as follows. There are three states in ELS which comprise not terminated and knocked-in (NTK) state, early termination (ET) state and knock-in (KI) state. ET state occurs when the underlying stock price appears greater than or equal to the ET barrier at the ET evaluation date and then ELS repays principal (1) and predetermined interest(c_i) at the ET payment date [5] and terminated. If stock price is determined to be below ET barrier on all the past ET evaluation dates and stock price has an experience of being below the KI barrier after contract effective date, it is called KI state. If the ET condition at the last evaluation date is not satisfied in the KI state, the principal reduced by the ratio of the stock price on the maturity date to the base price is repaid, resulting in a principal loss. If the NTK state continues until maturity, it is a common market practice for the principal and interest to be repaid equal to those of the last ET state. Explaining this process in terms of state transition, it starts with NTK on the contract effective date and may stay in that state over time, or reach ET on the evaluation date and the contract terminates or enter KI state as the stock price declines. Even if it was in KI state, it is possible to move on to ET if the stock price rises sufficiently. In summary, NTK state may stay in that state or transit to ET or KI. KI state can stay in KI or transit to ET. There exist N chances to determine whether ET or not, and KI is judged by whether the stock price hits the KI barrier from the inception of the contract to the expiration date. The KI barrier is set at a level sufficiently lower than the last ET barrier and remains at the same level throughout the contract period. The following Figure 1 describes the actual issuance case (KR6KW0000KT0). The contract period for this product is three years, and there are total of six automatic call opportunities every six months including maturity. In case of ET, interest is accrued for the elapsed period at an annual rate of 9.7%. In other words, the early repayment amount is 104.85% for the first period and 129.1% for the sixth period. The ET barriers are set at 90%, 90%, 85%, 85%, 80% and 75% of the base price. If the ET conditions are not satisfied until maturity and the stock price has not hit the KI barrier during the contract period, the same amount as the last ET redemption amount is paid, but if the KI barrier has been touched, the principal is paid in the ratio of the final stock price to the base price which means principal loss. In general, the ET barriers for each period tend to remain at or below the previous level over time, and for this reason, step-down is used as an alias for automatic callable ELS. To explain automatic early repayment as an example, in the case of early repayment in the 3rd round in Figure 1, the KOSPI 200 index was below 352.39 on May 9, 2022 and November 8, 2022, and is greater than or equals to 332.81 in May 9, 2023.

Let's look at Figure 2 to understand the possible paths of state transition over time from the viewpoint of SDKI valuation. Figure 2 shows the state transition of ELS contract starting from NTK on the contract effective date (t_0) and moving to the right over time. There are total N opportunities for ET, and each early payment evaluation date is expressed as t_i. The i^{th} ET occurs, while the stock price falls below the ET barrier at past $i-1$ evaluation dates, the stock price is determined above the ET barrier at the i^{th} evaluation date. Of course, if the ET condition is satisfied on each evaluation date, the interest and principal are paid and the contract is terminated. It should be noted that the assessment of the ET condition is made on a specific day. In other words, even if the stock price at the business day just before the ET evaluation date exceeds the barrier, if it falls below the barrier on the evaluation date, ET is not made. In the figure, there are two arrows reaching ET state for each evaluation date. This indicates that ET can occur from KI state as well as NTK state. KI occurs when the stock price falls below the KI barrier during the contract period (i.e. during the entire period including the intraday trading quotes). Even in this state, if the ET condition is satisfied on the one of the subsequent evaluation dates, the ET is made, otherwise, the principal loss occurs at maturity. NTK state occurs when the stock price exceeds the KI barrier over the entire period and falls below the ET barrier on each evaluation date. In normal SDKI, the end points of state transition paths, that is, the point at which the contract is terminated after the payment of principal and interest cash flows, comprise N ET points($ET_1, ET_2, \cdots, ET_N$), one NTK_N and one KI_N at maturity, for a total of N+2 points. If it is an automatic callable ELS without KI, the reachable states can be divided into two states – early termination and no early termination. The early termination state is the same as ET which is mentioned above, but if the ELS continues until maturity with no-ET, the principal reduced by the ratio of the price on the last



Note(s): This SDKI is issued on November 5, 2021 and there are 6 ET chances on every 6 months with 3-year maturity. The price of 391.54 on January 4, 2021 is set as the base price 100%. If the 6 ET evaluation date prices are more than 90, 90, 85, 85, 80, and 75%, respectively, the contract will be terminated. If the conditions for early repayment are not satisfied and the price does not hit 55% during the contract period, normal principal and interest will be paid, if hits, loss occurs

Source(s): Figure1 of Wilhelm(2009), modified for a domestic case

Figure 1.
Stepdown autocall ELS: an actual case

evaluation date to the base price is paid. In other words, it can be seen as a product with a relatively high risk from a probabilistic point of view because it has a structure in which losses occur even in the NTK state as in KI state of SDKI. Comparative analysis is performed by producing the no KI price for model verification in the latter part.

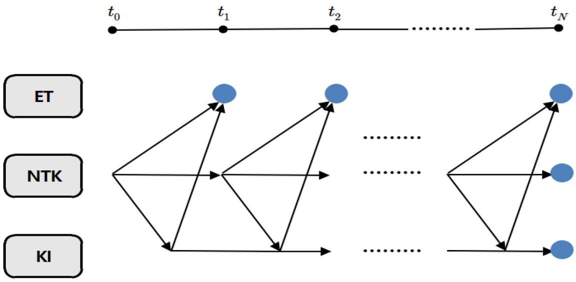


Figure 2. Autocall ELS: state transition diagram

Note(s): SDKI starts as NTK on the contract effective date and transits to ET when the ET condition is satisfied on each evaluation date, and either stays in NTK if it is not satisfied or transits to KI when the KI condition is satisfied. At maturity, NTK occurs when both the ET and KI conditions are not satisfied. KI can either transit to ET or continue

2.2 Valuation model

If the price of SDKI at time t is V_t and there are N remaining ET evaluation dates, the pricing formula is expressed as follows.

$$V_t = \sum_{i=1}^{N} e^{-r(t_i-t)} E^Q\left[1_{\{ET_i\}}\right](1+c_i) + e^{-r(T-t)}\left(E^Q\left[1_{\{NTK_N\}}\right](1+c_N) + E^Q\left[1_{\{KI_N\}}\frac{S_T}{S_B}\right]\right) \quad (1)$$

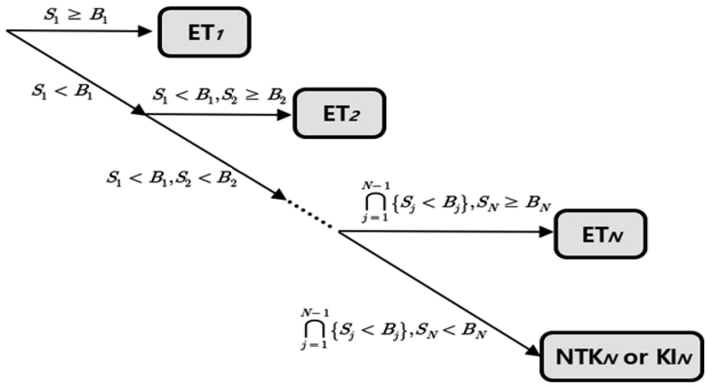
In Equation (1), r is a risk-free interest rate, E^Q is an expectation operator on risk neutral measure which has risk-free money market account as Numeraire, and $1_{\{\bullet\}}$ is an index function which returns 1 if the condition in the brace is satisfied, otherwise 0, $T = t_N$, S_T means the stock price at maturity, and S_B means the base price. Hereafter, without loss of generality, the stock price on the i^{th} early termination evaluation date (t_i) is expressed as S_i. The only random variable, the movement of the stock price, follows the following equation in the Q measure.

$$S_{t+\Delta} = S_t e^{\left(r-\sigma^2/2\right)\Delta + \sigma\sqrt{\Delta}\,W_{t+\Delta}} \quad (2)$$

In the above equation, Δ is an elapsed time, σ is a constant volatility of stock return process and W is a Wiener process. The first expectation operator on the right hand side of Equation (1) represents the expected value of the ET state and is expressed as follows.

$$E^Q\left[1_{\{ET_i\}}\right] = P^Q[ET_i] = P^Q\left[\bigcap_{j=1}^{i-1}\{S_j < B_j\}, S_i \geq B_i\right] \quad (3)$$

In the above formula, B_i denotes the ET barrier applied to the i^{th} ET evaluation date. The probabilities of reaching ET states are developed in a similar way to those of geometric distribution, except that the success probability of the geometric distribution, that is, the ET probability, is correlated with each other and the period is finite. The following Figure 3 shows the results of state transitions related to ET states in more detail. Whether or not to early terminate is judged only by the closing stock price on the ET evaluation date. In the figure, the first ET is determined by comparing the stock price (S_1) on the first ET date and the first ET barrier (B_1). If the ET condition is not satisfied, the judgment is passed on to the next evaluation date. The probabilities of ET for the second or higher evaluation dates should be calculated by simultaneously considering the no early termination condition of the past



Note(s): The ET probability structure is similar to that of a geometric distribution. The *i*th ET probability, however, is expressed as *i*-variate normal distribution, and the probabilities are correlated with each other. If the ET condition is not satisfied at maturity, the probability is divided into whether the KI condition is satisfied or not

Figure 3.
ET state arrival probability

evaluation dates. Since the stock price process is assumed to follow a log-normal one, the log-transformed stock price follows a normal distribution and the i^{th} ET probability can be calculated using the *i*-variate normal distribution. In the figure, there are N+1 terminal points of the binomial tree. The sum of path probabilities for the terminal points is guaranteed to be 1.

For Equation (3), the numerical integration method adopted in this paper for calculating the probabilities of multivariate normal distribution will be described in the next section. The second expectation term on the right hand side of Equation (1) can be expressed as follows.

$$E^Q\left[1_{\{NTK_N\}}\right] = P^Q[NTK_N] = P^Q\left[\bigcap_{j=1}^{N}\{S_j < B_j\}, min_{\tau\in(t_0,T)}S_\tau \geq LB\right] \tag{4}$$

In the above equation, LB means the KI barrier applied during the whole period including intraday. The right-hand side of Equation (4) means the probability that the stock price is less than the ET barrier at each ET evaluation date and higher than the KI barrier over the entire period. For ease of convenience and comprehension, we express as $\{min_{\tau\in(t_0,T)}S_\tau \geq LB\} \leftrightarrow \{\tau_{LB} \geq T\}$, then, Equation (4) is converted as follows. In equation below, $S_{i|i-1}$ means the stock price at time t_i given the stock price at time t_{i-1}.

$$P^Q\left[\bigcap_{j=1}^{N}\{S_j < B_j\}, \tau_{LB} > T\right] = P^Q[S_1 < B_1, \tau_{LB} > t_1] \times \prod_{i=2}^{N} P^Q\left[S_{i|i-1} < B_i, \tau_{LB} > t_i - t_{i-1} \middle| LB < S_{i-1} < B_{i-1}\right] \tag{5}$$

Equation (5) shows that the whole period joint probability of non-ET and non-KI events is expressed as the product of one period joint conditional probabilities of those. This equation can be derived relatively easily by using the Strong Markov and hitting time properties of Brownian motion. The derivation process for three periods is as follows.

$$P^Q\left[\bigcap_{j=1}^{3}\{S_j < B_j\}, \tau_{LB} > T\right] = P^Q[S_1 < B_1, \tau_{LB} > t_1]P^Q[S_2 < B_2, S_3 < B_3, \tau_{LB} > t_3 | S_1 < B_1, \tau_{LB} > t_1] \tag{5.1}$$

$$= P^Q[S_1 < B_1, \tau_{LB} > t_1]P^Q[S_2 < B_2, \tau_{LB} > t_2 | S_1 < B_1, \tau_{LB} > t_1] \times P^Q[S_3 < B_3, \tau_{LB} > t_3 | S_1 < B_1, \tau_{LB} > t_1, S_2 < B_2, \tau_{LB} > t_2] \tag{5.2}$$

$$= P^Q[S_1 < B_1, \tau_{LB} > t_1]P^Q\left[S_{2|1} < B_2, \tau_{LB} > t_2 - t_1 \middle| LB < S_1 < B_1\right] \times P^Q\left[S_{3|2} < B_3, \tau_{LB} > t_3 - t_2 \middle| LB < S_2 < B_2\right] \tag{5.3}$$

Equations (5.1) and (5.2) are the results of transformations for period 1 and 2, respectively, using the fact that the long-term hitting time event is a proper subset of the short-term hitting time event. In Equation (5.3), the period range in the condition part is replaced with the end point according to the Strong Markov theorem, and the joint probability part is converted into a single period condition, respectively. The joint probabilities after the second period have similar functional form as the one period joint probability. As a well-known probability, this represents the joint probability of an event in which the stock price has not been touched the KI barrier (LB) during the consideration period (t_0, t_1) and an event in which the stock price (S_1) is determined to be less than the first ET barrier at the end point of the period(t_1). This probability is derived using the reflection principle of Brownian motion [6]. S_1 appears in the form of a conditional transition probability given S_0. In general, since S_0 is a constant, the expression seems to be an unconditional probability for S_1. In case of transition from S_{i-1} to S_i, since S_{i-1} is also a variable, the second probability is expressed as the product of the transition probability and the probability density function of S_{i-1}. The formula for calculating this probability is as follows.

$$P^Q[S_i < B_i, \tau_{LB} > t_i | LB < S_{i-1} < B_{i-1}] = \frac{\iint_{LB}^{B} \varphi(S_i|S_{i-1})\varnothing(S_{i-1})dS_i dS_{i-1}}{\int_{LB}^{B_{i-1}} \varnothing(S_{i-1})dS_{i-1}} \tag{6}$$

In the above equation, $\varphi(\bullet)$ is same as the function that calculates the first term on the right side of Equation (5), $\varnothing(\bullet)$ is the lognormal probability density function, and each is defined as follows [7].

$$\varphi(S_i|S_{i-1}) = \frac{1}{\sigma\sqrt{\Delta_i}\,S_i}\left[n\left(\frac{ln\left(\frac{S_i}{S_{i-1}}\right) - (r - \sigma^2/2)\Delta_i}{\sigma\sqrt{\Delta_i}}\right) - \left(\frac{LB}{S_i}\right)^{\frac{2r}{\sigma^2}-1} n\left(\frac{ln\left(\frac{S_i}{S_{i-1}}\right) - 2ln\left(\frac{LB}{S_{i-1}}\right) - (r - \sigma^2/2)\Delta_i}{\sigma\sqrt{\Delta_i}}\right)\right] \tag{7}$$

$$\varnothing(S_{i-1}) = \frac{1}{S_{i-1}\sigma\sqrt{2\pi}} exp\left(-\frac{\left(\ln S_{i-1} - \ln S_t - \left(r + \frac{\sigma^2}{2}\right)(t_{i-1} - t)\right)^2}{2\sigma^2}\right) \tag{8}$$

In Equation (7), $\Delta_i = t_i - t_{i-1}, n(\bullet)$ means a standard normal probability density function. It is necessary to understand the background of KI probability. This is an interesting subject studied as a hitting time probability in the field of traditional probability theory. So far, for univariate and bivariate, Black and Cox (1976) and He *et al.* (1998) have published calculation formulas, respectively. For more than three variables, exact formulas are not published yet. One of the reasons why financial institutions or asset pricing firms cannot help adopting the MCM or the FDM when evaluating this type of products is the multiple hitting time probability problem. When deciding KI, one should consider whether the KI barrier has been touched for all prices quoted during the contract period. MCM or the FDM calculates the probability approximately by selecting daily, monthly, or early termination evaluation days due to the excessive calculation time. In other words, the days during the interim periods not selected as the KI evaluation dates are ignored. In this context, it is meaningful to study calculation formulas and methods that enable more accurate and rapid calculation. The left-hand side of Equation (5), that is, the problem of calculating the joint probability of the KI barrier non-touching event and no-early termination event during the whole period is a frequently raised question when evaluating financial derivatives. It is known that the exact calculation formula has never been published so far. Among the existing studies, Wilhelm (2009) attempted an analysis borrowing the results of Chuang (1996) on the same problem. As for the result, it is possible to calculate the joint probability of the KI and one past ET, but only the results are presented without explanation for the joint probability calculation process for two or more past ET events. It is hard to say that the joint probability calculation process for this problem was solved. Therefore, to our knowledge, equations (5) and (6) are the first for the joint distribution of multi-time point's random variables and a hitting time for the entire period. This joint distribution is expected to be applicable not only to SDKI, but also to the valuation of nth to default type credit derivatives based on multiple reference assets. Finally, the third expectation in Equation (1) is expressed in the form of a combination of two stochastic events and it is calculated by applying the traditional numeraire transform as follows.

$$
\begin{aligned}
E^Q\left[1_{\{KI_N\}}\frac{S_T}{S_B}\right] &= E^Q\left[1_{\left\{\bigcap_{j=1}^{N}\{S_j<B_j\},m_\tau\le LB\right\}}\frac{S_T}{S_B}\right] \\
&= \frac{S_t}{S_B}P^{Q^S}\left[\bigcap_{j=1}^{N}\{S_j<B_j\},m_\tau\le LB\right] \\
&= \frac{S_t}{S_B}\left(P^{Q^S}\left[\bigcap_{j=1}^{N}\{S_j<B_j\}\right]-P^{Q^S}\left[\bigcap_{j=1}^{N}\{S_j<B_j\},m_\tau>LB\right]\right)
\end{aligned}
\tag{9}
$$

In the above equation, m_τ is the minimum value of the underlying asset price during the contract period, and Q^S is the risk neutral measure when the stock price is numeraire. The first term in the third line on the right side of Equation (9) is calculated by applying multivariate normal distribution under the Q^S probability measure, and the second term is calculated by decomposing it as in Equation (5) under the Q^S probability measure as well.

In summary, the value of SDKI ELS is calculated through Equations (3), (5), and (9) and substituting them into Equation (1). Equation (5), however, must be preceded by Equation (6), which is an expression for calculating the hitting time probability. All the other probabilities are calculated using the multivariate normal distribution. If there is no KI condition, it can be calculated using a multivariate normal distribution in the form of excluding the hitting time part in Equation (9). In this case, the same logic applies even if there are two or more underlying asset prices. An additional point to mention is that there is a case in which the early termination evaluation price is determined as the average of the three-day closing prices. In this case, the probability can be calculated by applying the first order Taylor approximation.

2.3 Monte Carlo integration for multivariate normal distribution

Equation (1) mentioned above shows that the price of SDKI ELS consists of expected values for three states. From a technical point of view, in the ET state, if there are N ET opportunities remaining, the ET probability is expressed in the form of multiple integral of N-variate normal probability density function. The NTK and the KI states, respectively, are expressed as double integral because they are in the form of product of the transition probability function of the stock price and the probability density function of the previous stock price when the number of underlying stock is one. In this paper, the method adopted for calculating the pricing formula is to apply numerical integration for double integral and MCI for triple or more integral. Though NI produces an accurate integral value, it has a disadvantage of being very slow when there are three or more variables. For ET state probabilities, MCI is applied if there are more than three variables. The existing Monte Carlo simulation (MC) for ELS price calculation is a method focusing on a stock price scenario with normal random numbers, generating redemption cash flows for each scenario, discounting and averaging them to generate the final price. MCI is basically a method to quickly calculate the probability value of a multivariate probability distribution through uniform random number generation. In this paper, this method is applied to the probability calculation of the price formula.

The method suggested by Genz (1992) to calculate the probability value of the multivariate normal distribution is as follows. Firstly, the objective function to be calculated, that is, the N-variate cumulative normal distribution function, is as shown in Equation (10) below.

$$F(a,b) = \frac{1}{\sqrt{|\Sigma|(2\pi)^N}} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_N}^{b_N} exp\left(-\frac{1}{2}\theta^t\Sigma^{-1}\theta\right)d\theta \tag{10}$$

In the above equation, $a = (a_1, a_2, \cdots, a_N)^t$, $b = (b_1, b, \cdots, b_N)^t$, $\theta = (\theta_1, \theta_2, \cdots, \theta_N)^t$, Σ is a $N\times N$ covariance matrices and t means transpose of a matrix. Applying numerical integration directly to the above equation is very cumbersome and takes a lot of computation time. To solve this problem, Genz introduced three-step variable transformation. The first transform is a method of separating variables by Cholesky decomposition of the covariance matrix. Let the lower triangular Cholesky matrix of the covariance matrix be C. If we consider a transformation $\theta = Cy$, then y becomes independent random vector ($\Sigma = CC^t$; $\theta^t\Sigma^{-1}\theta = y^tC^t(C^t)^{-1}C^{-1}Cy = y^ty$). Expression (10) is expressed as y as follows.

$$F(a,b) = \frac{1}{\sqrt{(2\pi)^N}} \int_{a'_1}^{b'_1} e^{-\frac{y_1^2}{2}} \int_{a'_2(y_1)}^{b'_2(y_1)} e^{-\frac{y_1^2}{2}} \cdots \int_{a'_N(y_1,y_2,\cdots y_{N-1})}^{b'_N(y_1,y_2,\cdots y_{N-1})} e^{-\frac{y_N^2}{2}} dy \tag{11}$$

where $a'_i(y_1, y_2, \cdots, y_{i-1}) = (a_i - \sum_{j=1}^{i-1} c_{ij}y_j)/c_{ii}$, $b'_i(y_1, y_2, \cdots, y_{i-1}) = (b_i - \sum_{j=1}^{i-1} c_{ij}y_j)/c_{ii}$

Each integral function in Equation (11) can be expressed as follows using the univariate cumulative normal distribution.

$$F(a,b) = \int_{d_1}^{e_1} \int_{d_2(z_1)}^{e_2(z_1)} \cdots \int_{d'_N(z_1,z_2,\cdots,z_{N-1})}^{e_N(z_1,z_2,\cdots,z_{N-1})} dz \tag{12}$$

where $d_i(z_1 : z_{i-1}) = N((a_i - \sum_{j=1}^{i-1} c_{ij}N^{-1}(z_j))/c_{ii})$, $e_i(z_1 : z_{i-1}) = N((b_i - \sum_{j=1}^{i-1} c_{ij}N^{-1}(z_j))/c_{ii})$,
$(z_1 : z_{i-1}) = (z_1, z_2, \cdots, z_{i-1})$, $z_i = N(y_i)$, $N(\bullet)$: cumulative standard normal distribution

Equation (12) seems to be simple, but it is still tedious to apply the numerical integration algorithm because the integration boundaries become complicated. In order to solve this problem, converting to $z_i = d_i + w_i(e_i - d_i)$ and rearranging, it is expressed as the following Equation (13).

$$F(a,b) = (e_1 - d_1)\int_0^1 (e_2 - d_2)\cdots\int_0^1 (e_N - d_N)\int_0^1 dw \tag{13}$$

where $d_i(w_1 : w_{i-1}) = N((a_i - \sum_{j=1}^{i-1} c_{ij}N^{-1}(d_j + w_j(e_j - d_j)))/c_{ii})$,

$e_i(w_1 : w_{i-1}) = N((b_i - \sum_{j=1}^{i-1} c_{ij}N^{-1}(d_j + w_j(e_j - d_j)))/c_{ii})$

Equation (13) shows that given w_i, the desired multivariate cumulative normal distribution probability value can be calculated sequentially. Genz (1992) presents the procedures of generating a uniform random number for w_i and calculating Equation (13) through MCI. A more detailed simulation approach can be found by referring to Genz (1992). This function is provided as a library in *R*, a programming language. The price calculation algorithm in this study was written based on *R*, and pmvnorm() provided by *R* was used as a function of the multivariate normal distribution probability value. Computation time is a very important practical issue. Comparing calculation time for major programming languages, the general simulation calculation time is empirically identified in the order FORTRAN $\leq$ C, C++ < JAVA << R < Python. In particular, when calculating the price of SDKI ELS, it is shown that the time is reduced by about 1/10 compared to *R* for JAVA.

3. Analysis result

3.1 Monte Carlo simulation

This section describes MC method for evaluating the value of SDKI ELS set as a comparison target in this study. The detailed procedure of MC method is as follows.

(1) Obtaining product and market information

(2) Creating a date vector for a stock price scenario

(3) Creating an early termination evaluation date index in the stock price date vector

(4) Creating a cash flow date index and discount function vector

(5) Generating a stock price scenario for the stock price date vector

(6) Determining whether ET on the ET evaluation dates for each stock price scenario

(7) Deciding whether or not to KI for the non-ET scenario

(8) Matching redemption amounts according to ET/KI/NTK for each scenario

(9) Discounting the redemption amount and averaging

3.2 Description on the target product and related parameters

For the target product for verification of calculation results, a simple SDKI structure is assumed and the results of MC and the proposed model are compared under flat interest rate curve. After that, the results of a pricing firm and the proposed model are compared with the SDKI case (KR6KW0000KT0) and the Step Down No Knock-In (SDNoKI) case (KR6KW00010L8) released on the market. Product and market information of SDKI ELS for performance comparison of the proposed model is set as follows.

(1) Maturity(T): 3 year

(2) Principal Amount: 1

(3) Stock Initial and Base Price ($S_0 = S_B$): 100

(4) Riskfree Interest Rate (r): 3%

(5) Stock Volatility (σ): 20%

(6) ET Evaluation: Every six month

(7) ET Barriers (B_i): 90%, 90%, 85%, 85%, 80% and 80%

(8) KI Barrier (LB): 55%

(9) ET Returns (c_i): 3.5%, 7%, 10.5%, 14%, 17.5%, 21% and 21%

The 7th rate of ET return above means that the rate of return in the case of not satisfying the ET and the KI condition (NTK) is matched with the final ET return rate. All calculations were made using a laptop computer with CPU Intel I5-8265U 1.6 GHz. If MC is performed under these computer specifications, the maximum possible number of scenarios of stock prices per day excluding weekends for three years is 400,000, which means a total of 312 million (=3*260*400,000) random number generation. Since MC is already widely known, here, the implementation method for the above example will be described to help the understanding of the proposed model. Calculation of Equation (3) is done through Equation (8), and it is calculated by inputting the lower end (a) and upper end (b) of the individual integral for each cycle. The threshold for ET for each period is calculated in the same way as d_2 of Black and Scholes (1973). The *ith* boundary value is as follows.

$$d_{2i} = \frac{-\log\left(\frac{S_0}{B_i \times SB}\right) - \left(r - \frac{\sigma^2}{2}\right)t_i}{\sigma\sqrt{t_i}}$$

In the case of the *ith* ET probability, from the first to the $(i-1)th$, the lower end is $-\infty$ and the upper end is $d_{2j}, 1 \leq j \leq i-1$, the *ith* lower end is d_{2i} and the upper end is ∞. Genz (1992) targets the multivariate standard normal distribution with correlation. Since the stock price dynamics is defined as Wiener process, the covariance matrix is defined as follows.

$$\Sigma_{jk} = \begin{cases} 1, & j = k \\ \dfrac{min(t_j - t_0, t_k - t_0)}{\sqrt{(t_j - t_0)(t_k - t_0)}}, & j \neq k \end{cases}, \quad 1 \leq j, k \leq i.$$

In Equation (5), since the probability function is defined as stock level, not stock return rate, the lower end is entered as LBxSB and the upper end as BixSB. Equation (9) is expressed in the form of deducting the probability of non-ET and non-KI from the probability of non-ET until maturity. For the probability of non-ET, the lower end is entered as $-\infty$ and the upper end is entered as d_{1j}, $1 \leq j \leq N$ of Black and Scholes (1973) in Equation (10). In the case of non-ET and non-KI probability, equations (7) and (8) are modified to reflect drift term transformation of numeraire change, and the integral boundary is the same as before.

3.3 Comparison analysis of the results

The following Table 1 shows the results of price, state probabilities and calculation time in seconds with the MC and the model proposed in this study (Proposed) for the target product. The price of the proposed model is 20bp lower than the MC price, which is thought to be because the KI opportunities not considered in the simulation is reflected. For example, if the stock price during a day hits the KI barrier and is determined to be above the barrier at the closing price, the Monte Carlo evaluation shows no-KI effect, and in the proposed model, it is likely to appear. Due to this difference, it can be seen that the KI probability is relatively lower by 50bp in the case of the proposed model. In the case of the ET probabilities, the first ET probability differs by about 10bp and thereafter is less than 3bp, similar to the proposed model and simulation results. This phenomenon is believed to be because ET is determined preferentially regardless of KI. In the case of calculation time, the simulation is shown from 2.68 s to 831.57 s, and in the case of the proposed model, it takes significantly less time at 0.21 s. Of course, if JAVA, C or C++ instead of R is used, the time decreases by about 1/10, but the case of the proposed model also decreases, so the orders are maintained.

Here, we look at the comparison results of price evaluations for actual market products. Table 2 shows the prices evaluated by Pricing Firm A for the SDKI and SDNoKI products in circulation and the prices evaluated by the proposed model. The no-KI type has a structure in which NTK status does not exist and only ET or loss occurs. Both products have three-year maturity and offer opportunities to repay every six months. The valuation date is a day when six ET opportunities remain in the future. In the case of SDKI type, the prices according to the three models are shown in the following order: Pricing Firm A > MC > Proposed. The first inequality indicates that the possibility of KI is relatively low because financial institutions run the KI check simulation only on each ET evaluation date for the purpose of reducing calculation time. Of course, the 5bp price difference is considered to be in the tolerance level when doing interest rate interpolation or simulation, and the 37bp difference from the proposed model indicates that there is a relatively large difference between discrete and continuous KI evaluations. Since SDNoKI does not have a KI option, the price difference shows 4bp and 6bp, respectively. These are in the tolerance level of the simulation, so it is difficult to see a meaningful difference between the three models.

Next, looking at the sensitivity calculation results for the example products in Table 1, from Tables 3–5 are the results of comparing price sensitivities to interest rates, stock prices and volatility shocks, respectively. In the case of the proposed model, it can be seen that the result of the simulation scenario 400,000 is relatively similar, but the sensitivity is slightly fluctuated according to the number of scenarios. In the practical field, the sensitivity of ELS must be calculated for hedging or a risk calculation in line with the FRTB BASEL III standard. It is, however, common to calculate the number of simulations by setting below 50,000 due to computation time and resource allocation issues. In this case, the generated sensitivity result seems to be more or less unstable, so there is a possibility that a problem may arise in risk or hedging.

In order to understand the change in the sensitivity of the simulation, ELS price and delta calculation results for each underlying asset starting price are presented in Figure 4.

The top two figures in Figure 4 show prices and deltas of the simulation and the proposed model. Although the prices of the two models look similar, it can be seen that the simulation delta shows somewhat unstable shape near the KI barrier, but in the case of the proposed model, it appears in the form of a smooth curve. The characteristics of the delta appearing in the two models can be interpreted as follows. Firstly, if the stock price is very low, the probability of occurrence of

Table 1.
SDKI ELS valuation results

			Probability								
Normal	Scenario	Value	ET1	ET2	ET3	ET4	ET5	ET6	NTK	KI	Calc. Time
MC	10,000	1.011165	78.21%	6.49%	4.48%	1.74%	1.97%	1.07%	2.89%	3.150%	2.68
	30,000	1.010716	78.05%	6.72%	4.55%	1.89%	1.96%	0.98%	2.60%	3.247%	7.80
	50,000	1.010572	78.32%	6.73%	4.56%	1.85%	1.98%	0.88%	2.44%	3.242%	11.78
	1,00,000	1.010025	78.28%	6.88%	4.48%	1.85%	1.86%	0.90%	2.42%	3.333%	33.78
	2,50,000	1.010162	78.19%	6.81%	4.58%	1.83%	1.82%	0.91%	2.53%	3.328%	244.88
	4,00,000	1.010094	78.26%	6.90%	4.61%	1.82%	1.80%	0.86%	2.44%	3.321%	831.57
Proposed	–	1.008047	78.16%	6.90%	4.61%	1.82%	1.83%	0.88%	1.92%	3.886%	0.21

KI is high and the redemption amount is linked to the ratio of the maturity stock price to the base stock price. Therefore, it can be seen that the delta of the low stock level appears horizontally around 0.6. If the stock price is high, the probability of early redemption is high, so the promised fixed interest is paid. In this case, the delta appears as 0 because the redemption amount does not increase no matter how much the stock price increases above a certain price level. These characteristics appear in common in the results of both models. On the other hand, the simulation results are unstable at the mid-stock price level, especially near the KI barrier. Simulation shows excellent performance in distinguishing above and below based on a specific boundary. As in step-down ELS, however, when it is necessary to distinguish whether KI is present and reflect the return of the last stock price in case of KI, the degree of freedom of stock prices at maturity time inevitably decreases, so it is understood that they appear unstable. The lower two figures of Figure 4 show the price and delta difference between the two models, respectively. In the figure, it can be seen that both ends of the initial stock price appear similar to each other.

The final result to be mentioned is a comparison of the hedging performance according to the two models. This analysis does not seek the optimal hedging strategy, but aims to identify the applicability of the model by comparing the tracking errors of simple delta hedging for each model for the same stock price scenario. The delta hedging algorithm applied to this analysis is as follows.

At time 0:

$$\Delta_0 = \frac{V_0(1.01 \times S_0) - V_0(S_0)}{0.01 \times S_0}$$

$$BS_0 = \Delta_0 S_0$$

$$BA_0 = V_0 - BS_0$$

At time t:

$$\Delta_t = \frac{V_t(1.01 \times S_t) - V_t(S_t)}{0.01 \times S_t}$$

$$BS_t = \Delta_t S_t$$

$$BA_t = BA_{t-1} \times (1 + rdt) + (\Delta_{t-1} - \Delta_t) S_t$$

$$TE_t = \Delta_{t-1} S_t + BA_{t-1}(1 + rdt) - V_t$$

where,

Δ_t: Delta at t, V_t: ELS Price at t, BS_t: Stock Amount at t,

BA_t: Amount to short(borrow) or long(saving) bonds at t,

r: Risk-free rate, : 1/365, TE_t: Tracking Error at t

Figure 5 is a scatterplot of accumulated tracking error after daily delta hedging according to the above equations for each scenario after generating 1,000 stock price scenarios for 10 days after

Type	Product	Pricing firm A(a)	MC(b)	Proposed(c)	(b)-(a)	(c)-(a)	Val. Date
SDKI	KR6KW0000KT0	1.048356	1.047879	1.044686	−0.05%	*−0.37%*	03/01/2022
SDNoKI	KR6KW00010L8	1.028927	1.028546	1.028334	−0.04%	−0.06%	11/08/2022

Note(s): As a result of the price calculation for the market product, the three prices showed no significant difference in the absence of KI, but the results of the proposed model for the KI type showed a significant difference

Table 2. Market ELS valuation results

r+1BP	Scenario	Value	Probability ET1	ET2	ET3	ET4	ET5	ET6	NTK	KI	PV01
MC	10,000	1.011185	78.22%	6.52%	4.47%	1.74%	1.96%	1.06%	2.90%	3.130%	0.20
	30,000	1.010648	78.05%	6.73%	4.56%	1.90%	1.96%	0.98%	2.59%	3.243%	−0.68
	50,000	1.010490	78.33%	6.73%	4.56%	1.85%	1.98%	0.88%	2.43%	3.242%	−0.82
	1,00,000	1.009995	78.29%	6.89%	4.48%	1.85%	1.86%	0.90%	2.42%	3.323%	−0.30
	2,50,000	1.010107	78.20%	6.81%	4.58%	1.82%	1.81%	0.91%	2.53%	3.323%	−0.55
	4,00,000	1.010036	78.27%	6.89%	4.61%	1.83%	1.80%	0.85%	2.43%	3.317%	−0.58
Proposed	–	1.007994	78.17%	6.90%	4.61%	1.82%	1.83%	0.87%	1.92%	3.880%	−0.53

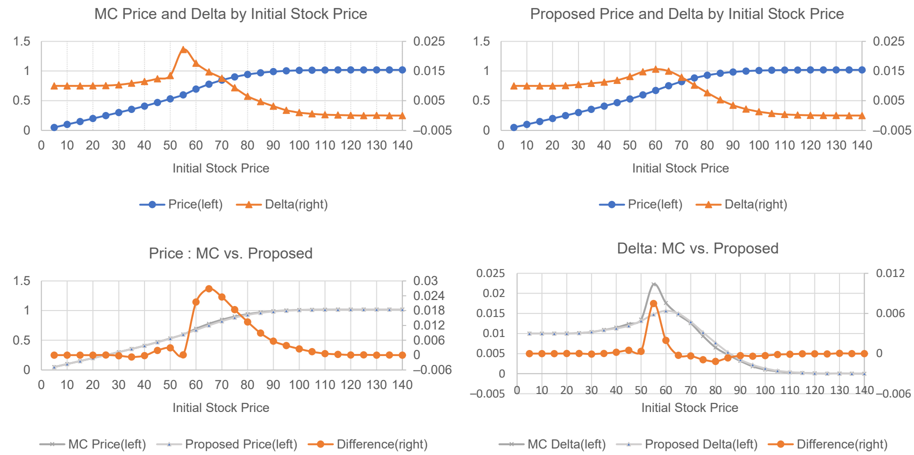
Table 3.
Valuation results for interest rate 1bp shock

Sx101%	Scenario	Value	Probability								Delta
			ET1	ET2	ET3	ET4	ET5	ET6	NTK	KI	
MC	10,000	1.011797	80.48%	5.98%	3.98%	1.58%	1.74%	0.95%	2.45%	2.840%	0.06
	30,000	1.011767	80.00%	6.18%	4.22%	1.76%	1.77%	0.85%	2.33%	2.893%	0.11
	50,000	1.011572	80.43%	6.17%	4.12%	1.60%	1.82%	0.80%	2.17%	2.892%	0.10
	1,00,000	1.011187	80.25%	6.31%	4.08%	1.67%	1.68%	0.83%	2.23%	2.970%	0.12
	2,50,000	1.011382	80.17%	6.28%	4.19%	1.66%	1.64%	0.82%	2.28%	2.946%	0.12
	4,00,000	1.011188	80.21%	6.36%	4.21%	1.67%	1.64%	0.75%	2.20%	2.967%	0.11
Proposed	–	1.009353	80.17%	6.35%	4.19%	1.65%	1.65%	0.79%	1.73%	3.478%	0.13

Table 4.
Valuation results for stock price 1% shock

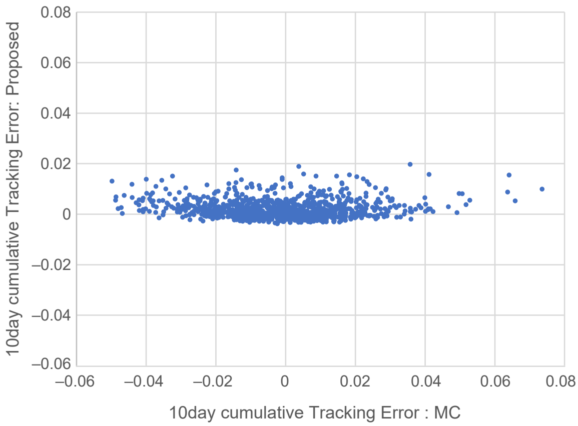
Table 5.
Valuation results for stock volatility 1% shock

σ+1%	Scenario	Value	ET1	ET2	ET3	ET4 (Probability)	ET5	ET6	NTK	KI	Vega
MC	10,000	1.006649	76.96%	6.80%	4.48%	1.89%	2.02%	1.26%	2.56%	4.030%	−0.45
	30,000	1.006819	76.75%	7.05%	4.64%	2.02%	2.04%	1.00%	2.49%	4.013%	−0.39
	50,000	1.006849	76.98%	7.00%	4.71%	1.93%	2.10%	0.91%	2.40%	3.982%	−0.37
	1,00,000	1.006230	76.98%	7.14%	4.65%	1.98%	1.93%	0.94%	2.31%	4.075%	−0.38
	2,50,000	1.006396	76.92%	7.11%	4.73%	1.92%	1.88%	0.97%	2.42%	4.056%	−0.38
	4,00,000	1.006264	76.93%	7.17%	4.76%	1.92%	1.89%	0.92%	2.34%	4.071%	−0.38
Proposed	–	1.004346	76.84%	7.18%	4.76%	1.93%	1.91%	0.93%	1.86%	4.591%	−0.37



Note(s): When the stock price is located at a level slightly higher than the KI barrier, the price and delta of MC seem to be unstable compared to those of the proposed model

Figure 4. Comparison of price and delta between the two models



Note(s): As a result of delta hedging with MC and the proposed model for 1,000 daily stock price scenarios for 10 days, the cumulative tracking error of the proposed model is relatively stable than that of the MC

Figure 5. Comparison of delta hedging performance between two models

issuance. In the figure, the tracking error means the sum of daily tracking errors for 10 days. The delta hedging method is carried out by purchasing the underlying asset as much as the delta ratio with ELS sale amount on the effective date and saving the remaining amount. If the purchasing amount of the underlying asset is bigger than holding funds, borrowing occurs according to the shortfall. After constructing the portfolio, the stock weight is adjusted to the daily delta that fluctuates with the stock price, and the amount of savings (or borrowing) is adjusted according to the difference. For savings or borrowing rates, a risk-free rate is applied. The difference points between the delta hedging performance of the two models are the valuation price and the delta weight in daily rebalancing. The applied stock scenario is the same, and the horizontal axis is the profit/loss of the hedge portfolio according to the simulation and the vertical axis is the profit/loss of the proposed model. In the figure, it can be seen that the profit and loss volatility of delta hedging according to the simulation is relatively large, and in the case of the proposed model, the fluctuation range is relatively small.

Table 6 shows some descriptive statistics for the cumulative tracking errors by two models. A striking feature is the fact that the standard deviation of MCM tracking error is 4.8 times bigger than that of the proposed model tracking error, and the hedging according to the proposed model is relatively stable. Also, in the case of skewness, the result of the proposed model shows a long tail in the direction of profit (i.e. skew to the right) with a limited maximum loss size. As for the kurtosis [8], it can be seen that the probability of occurrence is higher than the normal distribution. The fundamental cause of this performance difference is believed to be due to the inherent instability of the simulation delta sensitivity. It is to be undesirable to rely solely on simulation results for rigorous risk analysis and hedging of ELS with KI characteristics. The following Figure 6 shows the daily tracking error calculated by hedging in two ways for 10 business days, that is, from January 4 to January 17, for products circulating in the market. The figure on the left is the result of applying normal delta ($\Delta V/\Delta S$), and the figure on the right is the result of hedging by applying the price difference (ΔV) to the delta part. In both cases, it can be seen that the tracking error of the proposed model shows relatively small variation. When delta was applied, the standard deviation of the tracking error was 6.33 bp for MC and 4.67 bp for proposed, and 26.41 bp and 10.05 bp, respectively, when the difference was applied. The reason for considering the difference other than the delta is that the delta hedging equation is defined as the derivative of the continuous function, but the calculation of this derivative is done through the difference. The problem of finding the optimal weight to be reflected in the price difference is judged to be a field that requires a lot of empirical research. In this paper, we finish to the extent that simple delta and simple difference among the alternatives are shown to be meaningful.

Table 6. Descriptive statistics for cumulative tracking error

	Mean	SD	Min	Max	Skew	Kurt
MC	−0.11%	1.79%	−4.98%	7.35%	0.16	0.69
Proposed	0.21%	0.37%	−0.37%	1.98%	1.39	2.43

Note(s): It can be confirmed that the standard deviation of the cumulative error of the proposed model appears smaller than that of the MC. Skewness of the proposed model means that the tail appears long in the direction of profit with limited maximum loss, and kurtosis implies the possibility that the tail event appears higher than that of the normal distribution

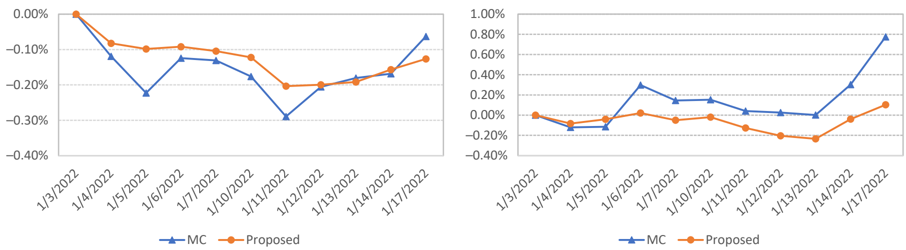


Figure 6. Comparison of delta hedging errors for KR6KW0000KT0 case

Note(s): As a result of delta hedging for market products, the daily tracking error results of the proposed model are showing more stable results using delta or difference by 1% shock as the delta proxy

4. Conclusion

This paper presents MCI model and its performance for evaluating the value of SDKI ELS with one underlying asset. The contributions of this paper compared to the existing literature are as follows. Firstly, by introducing a multivariate normal distribution for calculating the ET probability by period, a method that directly calculates the probability distribution rather than an approximate method such as simulation or FDM is adopted. Secondly, a method for calculating the probability of hitting time for the entire period is presented by sequentially applying the probability of hitting time for each period. The method is expected to be equally applicable even if the number of underlying assets increases. Thirdly, in relation to the KI effect, the limitations of existing simulations and FDMs were pointed out and a solution was provided through the proposed model. Fourthly, it provided a clue to solve the case of multiple underlying assets. The approach presented in this paper will be equally applied to the valuation of securities linked to multiple underlying assets. However, the problem of calculating the hitting time for multiple random variables and the problem of determining the rate of return linked to the worst performer among the underlying assets should be resolved. The latter problem of minimum return can be determined relatively easily, but the former multivariate hitting time distribution is an important issue of high interest in the relevant fields, and is only known for bivariate so far. Lastly, in terms of a practical contribution, the method presented in this paper is that the accuracy of the price and sensitivity of the products of a similar type was improved and the calculation time was shortened at the same time through the introduction of Monte Carlo numerical integration method.

Notes

1. www.seibro.or.kr
2. NICEP&I.
3. Jang (2020).
4. The author would like to thank an anonymous referee who informed us of this paper.
5. The ET payment date is determined two business days after the corresponding evaluation date.
6. Karatzas and Shreve (1998), Klebaner (2012).
7. Kwok (2008).
8. The kurtosis of the normal distribution is 3, but *R* gives the value of kurtosis minus 3.

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