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Form finding of non-minimal tensile membrane structures based on the general theory of curved interfaces using physics-informed neural networks

Sounak Kabasi PhD

Assistant Professor, Department of Civil Engineering, National Institute of Technology Jamshedpur, Jamshedpur, India (corresponding author: sounak.kabasi001@gmail.com; sounak.ce@nitjsr.ac.in)

Allan L. Marbaniang PhD

Assistant Professor, Department of Civil Engineering, Indian Institute of Technology Delhi, Delhi, India

Siddhartha Ghosh PhD

Professor, Structural Safety, Risk and Reliability (SSRR) Lab, Department of Civil Engineering, Indian Institute of Technology Bombay, Mumbai, India

Tensile membrane structures (TMS) are growing in popularity for their ability to span vast regions, lightweight properties, and material utilisation efficiency. Given that the shape of the structure is unknown in advance and is dependent on the boundary and stress conditions, the designer must use an approach known as ‘form-finding’ which seeks to discover a static equilibrium configuration of TMS. Unfortunately, standard methods suffer from several convergence issues. Therefore, the authors propose an innovative approach for identifying the TMS form based on the theory of general curved interfaces by solving the generalised form of the Laplace equation, which stems from the mechanical and thermodynamic principles for describing the phenomena of curved interfaces. This theory is implemented to develop a novel form-finding equation that can incorporate non-minimal shapes and therefore is specifically tailored to fit a TMS’s structural requirements. The suggested methodology entails using a mesh-less framework called a physics-informed neural network to solve this newly obtained equation. Several numerical examples highlight the advantages of the proposed formulation.

Keywords: computational mechanics/curved interface/form finding, tensile membrane structures/Laplace equation/physics-informed neural networks

Notation

a	determinant of the surface metric tensor
a_α, a_β	tangential basis vectors
$a^{\alpha\beta}$	surface metric tensor
B	interfacial bending moment
$b_{\alpha\beta}$	surface curvature tensor components
c_1, c_2	principal curvatures at a point on the surface
D	deviatoric curvature at a point on the surface
ds	scalar element of the surface
e_{11}, e_{22}	alternating components
e_{12}, e_{21}	alternating components
f	force exerted per unit mass
$f(x, t)$	known source term in the PDE
f_s	surface density of excess free energy
H	surface mean curvature
K	Gaussian curvature at a point on the surface
M_I, M_{II}	eigenvalues of the surface moment tensor
$M^{\alpha\beta}$	surface moment components
$M^{\alpha\beta}_{,\alpha\beta}$	covariant derivatives of $M^{\alpha\beta}$
m	analogous counterpart of body forces
N	surface moment tensor
N_f	number of collocation points for enforcing PDE

N_u	number of boundary points
$N^{\alpha(n)}$	transverse component of the surface moment tensor
$N^{\alpha\beta}$	tangential component of the surface moment tensor
n	outward unit normal vector to the surface S
P_I, P_{II}	pressures in fluid bulk phases I and II
$q^{\alpha\beta}$	deviatoric curvature tensor
S	surface enclosing the volume V
s_s	surface density of entropy
T	temperature
T	stress tensor
t	time variable
U_s	surface unit tensor
$u_\theta(x, t)$	neural network approximation of $u(x, t)$
V	volume of material
v	velocity field
w	angular velocity
w_s	surface density of grand thermodynamic potential
$X_b = \{x_b, y_b\}$	boundary collocation points
$X_b = \{x_b, y_b\}$	neural network approximation of TMS surface

$X_I = (x_i, y_i)$	membrane collocation points
α	coefficient accounting for moment of inertia
Γ	mass density per unit surface area
Γ_k	number of molecules of component k
γ	thermodynamic surface tension
$\delta\alpha$	local dilation of an elemental surface area ΔA
ε	surface alternator
ζ	shearing tension
Θ	interfacial torsion moment
θ^*	optimal PINN parameters
μ_k	chemical potential
ρ	mass density
σ	surface stress tensor
σ_x, σ_y	eigenvalues of the surface stress tensor
σ_x, σ_y	orthogonal warp and weft prestresses (during form-finding)
$\sigma^{\alpha\beta}$	tangential shear stress components
$\mathcal{B}$	boundary condition operator
$\mathcal{L}$	differential operator defining the PDE
$\mathcal{L}_{\text{tot}}(\theta)$	total loss function for PINN training
∇_s	two-dimensional gradient operator on the surface

1. Introduction

In recent years, tensile membrane structures (TMSs) have become extremely popular for long-span structures. They are readily utilised to cover huge areas because they are made with the aid of lightweight textiles. As a result, they are mostly utilised as the roofing structures for airports, stadiums, shopping centres, parking lots, and so on. The membrane fabric used for TMS has negligible flexural stiffness and low shear strength. The membrane action of in-plane prestresses and the anticlastic curvature are the main contributors to TMS's stability. The fact that TMS may be prefabricated gives them a significant advantage over conventional structures such as reinforced cement concrete, steel, or masonry structures in terms of quality control. The flexibility in architectural design that TMS can provide is another benefit of such systems, as they have been employed to generate a large number of aesthetic structures.

The main challenging aspect of TMS design stems from a procedure known as 'form finding'. The form of TMS is not known beforehand and is dependent on the prescribed boundary and prestress condition. During this stage, the TMS is expected to be in static equilibrium, subjected to the prescribed boundary conditions and the prescribed prestress by forming a double-curved anticlastic surface. It is expected that the form-finding framework will produce optimal structural forms that meet the functional and durability requirements while also being affordable and aesthetically acceptable. The TMS's structural behaviour is determined by orthogonal fibres that make up the membrane material. The prestress in the membrane is applied along the orthogonal 'warp' and 'fill' ('weft') material fibre directions, respectively. When the TMS has a minimal shape with the smallest surface area, the warp and fill prestress values become equal, resulting in an isotropic prestress shape. However, in practical situations, minimal shapes may not

always meet design requirements, such as clearance height or aesthetics. Therefore, anisotropic prestress (unequal warp and weft prestress) is needed, resulting in a non-minimal shape. The final shape of a non-minimal TMS is determined by the ratio of warp to fill prestress. By increasing the prestress along one direction, the corresponding anticlastic curvature is decreased (Marbaniang *et al.*, 2022).

Nowadays, several mesh-based numerical methods are employed for the form-finding process. Some of them include the force density method, dynamic relaxation (DR) method, stiffness method, and so on (Lewis, 2003; Topping and Ivanyi, 2007). Traditional mesh-based form-finding methods typically rely on mesh discretisation and subsequent application of numerical methods to achieve equilibrium within the system. Despite their popularity, these methods often encounter convergence issues, especially when dealing with anisotropic prestressed membranes (Marbaniang *et al.*, 2022). An assessment of various standard TMS form-finding software and tools revealed significant discrepancies in results, particularly for non-minimal shapes with anisotropic prestress (Gosling *et al.*, 2013). Customary form-finding frameworks suffer from several shortcomings, including computationally intensive stiffness matrix calculations, ambiguous force-density parameters in geometric stiffness methods, and challenges in determining optimal parameters for dynamic equilibrium-based methods (Veenendaal and Block, 2012). Incorrect parameter selection can lead to numerical anomalies, such as mesh distortion and non-convergent solutions (Veenendaal and Block, 2012; Wakefield, 1999). Consequently, the computational performance of traditional form-finding techniques heavily relies on initial geometry and hyperparameter selection (Nouri-Baranger, 2004; Veenendaal and Block, 2012). In addition, the computational complexity of mesh-based methods escalates in high-dimensional problems, further exacerbated by the computational demands of meshing algorithms (Dutta and Ghosh, 2021). Moreover, tangential fluctuations on the discretised surface pose additional challenges that need separate consideration (Bletzinger and Ramm, 1999; Veenendaal and Block, 2012; Wüchner and Bletzinger, 2005). Detailed numerical experiments conducted by Kabasi *et al.* (2023) highlighted drawbacks and convergence issues of traditional mesh-based form-finding frameworks, such as DR (Topping and Ivanyi, 2007) and conventional finite-element method (FEM). DR exhibited sensitivity to initial shape and mesh density, leading to convergence and mesh distortion problems. Similarly, FEM-based form finding showed sensitivity to pseudo-material properties, resulting in non-convergence or significant deviation of Cauchy stress from the target prestress. The study also emphasised the significant impact of the chosen initial form on the convergence of the FEM-based approach. Given the challenges associated with mesh-based techniques, Kabasi *et al.* (2023) proposed a mesh-less form-finding technique based on the Euler-Lagrange equation, utilising physics-informed neural networks (PINNs) (Raissi *et al.*, 2019). While this method

offers potential solutions to conventional mesh-based limitations, it is currently limited to minimal TMS subjected to isotropic prestress along the warp and weft directions of the fabric.

This research proposes a new mesh-free PINN-based algorithm for non-minimal TMS form finding. This unique form-finding strategy makes use of the theory of a general curved interfaces, that is mostly used in the scientific domains of biophysics and colloidal sciences. In general, there are two approaches for analysing the contours of curved membranes—the mechanical approach characterised by surface stress and moment tensors, which are subsequently used for balancing linear and angular momentum, and the thermodynamical approach characterised by shear tensions, scalar dilation, bending moments, and torsion. Upon comparison of these two approaches, Kralchevsky *et al.* (1994) found out that these two seemingly diametrically opposite approaches of describing the same phenomena are ultimately complementary to each other and subsequently derived an elegant form of a generalised Laplace equation. Although this theory is applied for describing the surfaces of lipid vessels, microemulsions, cell membranes, capillary waves at interfaces, thin liquid films, and so on, the basic physics of these membranes is conjectured to be similar to that of a TMS. Hence, the generalised Laplace equation mentioned above is investigated for the form finding of TMS. A more suitable equation specialised towards TMS is derived, and this equation is solved using PINN. Most importantly, the proposed formulation is also applicable for the form finding of non-minimal shapes having anisotropic prestress. Extensive numerical case studies are undertaken to demonstrate that the proposed framework can be used to get an accurate estimation of the form-found shape of TMS while alleviating the convergence issues faced by traditional approaches.

The rest of the manuscript is organised in the following order. Section 2 introduces the mechanical and thermodynamic theories of a general curved interface and their relation to each other. The derivation of a novel form-found surface equation for TMS based on these theories is given in Section 3. A brief discussion on the framework of PINN is given in Section 4. The proposed methodology of the form finding of TMS is described in Section 5, followed by the form-finding case studies in Section 6. Section 7 provides a brief summary and some important concluding points.

2. Theory of curved interfaces and membranes

2.1 Overview

According to different reviews in biophysics and colloid science, the classical Young–Laplace theory (Kralchevsky *et al.*, 1994), which describes the contour of a membrane or curved interface as a two-dimensional continuum subjected to isotropic tension and position independence, proves ineffective in many cases. Gibbs

(1961) introduced a simplified framework for situations where the effects of curvature are significant. To account for the effects of bending alongside vanishing tensions in biomembranes and lipid membranes, Helfrich (1974) considered different mechanical parameters such as bending and torsion moduli and spontaneous curvature, which subsequently became an important part of the literature on microemulsions (Kralchevsky *et al.*, 1994). Kralchevsky and Ivanov (1990) assumed biomembranes as a two-dimensional continuum having shearing membrane tension, bending, and torsional moments and found satisfying results. Examples of such cases include erythrocytes (Markin, 1981), microemulsions (Ljunggren and Eriksson, 1992), and surfactant micelles (Eriksson and Ljunggren, 1985).

There are two main ways to approach the theoretical and mathematical description of a general curved interface. The first approach comes from a mechanical standpoint and is based on the Kirchhoff–Love theory of plates and shells (Ciarlet, 1997). This theory is often used to describe the surface of biomembranes and lipid bilayers due to its strong mathematical foundations based on two-dimensional continuum mechanics, differential geometry, and tensor analysis. The second approach originates from Gibbs’ theory of capillarity (Gibbs, 1961), which is based on thermodynamics and is commonly cited in the fields of physical chemistry and surface sciences. In their literature review, Kralchevsky *et al.* (1994) examined these two approaches and explored the relationship between them. They concluded that the two approaches were equivalent and complemented each other, leading to the discovery of an elegant form of the generalised Laplace equation. They suggested that this equation could be used to describe the surface of any curved surface, thin film, or membrane.

2.2 Mechanical approach

A brief discussion on the main equations of the mechanical approach is presented here. The detailed derivations can be found in Kralchevsky *et al.* (1994). Consider a volume of material, denoted as V , that is surrounded by a surface represented as S , with an outward unit normal vector, n . As per Newton’s second law, it is hypothesised that (Serrin, 1959):

$$1. \quad \frac{d}{dt} \int_V dV \rho v = \int_S ds n \cdot T + \int_V dV \rho f$$

In this context, the variable t represents time, ρ represents the mass density, v represents the velocity field, T represents the stress tensor, ds denotes a scalar element of the surface, and f represents the force exerted on the body per unit of mass. By balancing the surface linear momentum, the Young–Laplace equation of capillarity (Laplace, 1805) can be derived for the case of a quasi-static

deformation of a fluid interface between two fluid bulk phases having pressures P_I and P_{II} , respectively.

$$2. \quad 2H\sigma = P_{II} - P_I$$

where H is the surface mean curvature. The theory underlying this equation is limited by the assumption that the fluid interface has uniform surface tension. However, this assumption does not hold true for strongly curved surfaces, such as those present in microemulsions and biomembranes, where surface bending moments play a significant role. Moreover, the surface tension is often neither isotropic nor constant. To address these complexities, it is necessary to postulate the surface angular momentum balance equation (Naghdi, 1972).

$$3. \quad \alpha \Gamma \frac{dw}{dt} = \nabla_s \cdot N + \Gamma m + n[(\sigma : \epsilon)U_s + \sigma \cdot \epsilon]$$

The surface moment tensor N is further given as a summation of the tangential and transverse components defined by:

$$4. \quad N = a_\alpha a_\beta N^{\alpha\beta} + a_\alpha n N^{\alpha(n)}$$

Here, α is a coefficient accounting for the moment of inertia, and the angular velocity is represented by w . ∇_s is the two-dimensional gradient operator on the surface, and σ is the surface stress tensor. U_s is the surface unit tensor. Γ is the mass density per unit surface area. m is the analogous counterpart of body forces, and the surface alternator is given by:

$$5. \quad \begin{aligned} \epsilon &= a^\alpha a^\beta \epsilon_{\alpha\beta} = a^\alpha a^\beta \sqrt{a} e_{\alpha\beta} \\ e_{12} &= -e_{21} = 1, e_{11} = e_{22} = 0 \end{aligned}$$

where a is the determinant of the surface metric tensor. By applying various mathematical substitutions, it is possible to derive the generalised Laplace equation for situations where surface moments ($M^{\alpha\beta}$) are involved. This equation can be expressed as (Naghdi, 1972):

$$6. \quad b_{\alpha\beta} \sigma^{\alpha\beta} - M^{\alpha\beta}_{,\alpha\beta} = P_{II} - P_I$$

Here, $b_{\alpha\beta}$ and $M^{\alpha\beta}_{,\alpha\beta}$ are the components of the surface curvature tensor and the covariant derivatives of M , respectively. $\sigma^{\alpha\beta}$ are the tangential shear stress components.

2.3 Thermodynamic approach

The thermodynamic method developed by Gibbs (1961), which was subsequently expanded by Boruvka and Neumann (1977), considers a multiphase system as consisting of bulk, surface, and linear phases. Each phase is defined by a fundamental equation. The local form of the fundamental equation of the surface is given by Kralchevsky and Ivanov (1990):

$$7. \quad \delta f_s = -s_s \delta T + \sum_{k=1}^N \mu_k \delta \Gamma_k - w_s \delta \alpha + \delta w_s$$

where the surface densities of the excess free energy, surface densities of the entropy, surface densities of the grand thermodynamic potential, the number of molecules of components, temperature, and chemical potential are given by f_s , s_s , w_s , Γ_k , k , T , and μ_k , respectively. The local dilation of an elemental surface area ΔA is given by:

$$8. \quad \delta \alpha = \frac{\delta(\Delta A)}{\Delta A}$$

The work done due to surface deformation per unit area for dilation ($\delta \alpha$), shear ($\delta \beta$), bending (δH), and torsion (δD) is given by:

$$9. \quad \delta w_s = \gamma \delta \alpha + \zeta \delta \beta + B \delta H + \Theta \delta D$$

γ , ζ are the thermodynamic surface tension and shearing tension, respectively, and B and Θ are the interfacial bending and torsion moments, respectively. By thermodynamic definitions, it can be said that:

$$10. \quad w_s = f_s - \sum_{k=1}^N \mu_k \Gamma_k$$

By combining all these equations, Equation 7 can be written as:

$$11. \quad \delta w_s = -s_s \delta T - \sum_{k=1}^N \Gamma_k \delta \mu_k + (\gamma - w_s) \delta \alpha + \zeta \delta \beta + B \delta H + \Theta \delta D$$

After imposing the conditions of thermodynamic and mechanical equilibrium, the surface equilibrium equation is obtained as (Kralchevsky *et al.*, 1994):

$$12. \quad \begin{aligned} 2H\gamma + 2D\zeta - (H^2 + D^2)B - 2HD\Theta - \frac{(a^{\alpha\beta}B + q^{\alpha\beta}\Theta)_{,\alpha\beta}}{2} \\ = P_{II} - P_I \end{aligned}$$

H and D are mean curvatures and deviatoric curvatures at a point on the surface, respectively. $a^{\alpha\beta}$ is the surface metric tensor, and $q^{\alpha\beta}$ is the deviatoric curvature tensor:

$$13. \quad q^{\alpha\beta} = \frac{(b^{\alpha\beta} - Ha^{\alpha\beta})}{D}$$

2.4 Relating the two approaches

The thermodynamic dilation and shearing tension can be related to the mechanical surface tension and shear using the following relations (Gurkov and Kralchevsky, 1990).

$$14. \quad \begin{aligned} \gamma &= \sigma + \frac{BH}{2} + \frac{\Theta D}{2} \\ \zeta &= \eta + \frac{BD}{2} + \frac{\Theta H}{2} \end{aligned}$$

By substituting these equations into Equation 12, the generalised Laplace equation as derived can be written elegantly as:

$$15. \quad 2H\sigma + 2D\eta - \frac{(a^{\alpha\beta}B + q^{\alpha\beta}\Theta)_{,\alpha\beta}}{2} = P_{II} - P_I$$

where

$$16. \quad \sigma = \frac{(\sigma_x + \sigma_y)}{2}; \quad \eta = \frac{(\sigma_x - \sigma_y)}{2}$$

$$17. \quad B = M_1 + M_2; \quad \Theta = M_1 - M_2$$

σ_x, σ_y are the eigenvalues of the surface stress tensor. M_I, M_{II} are the eigenvalues of the surface moment tensor.

3. Derivation of a novel form-found surface equation for TMS based on the generalised Laplace equation

The generalised Laplace equation was derived for finding the stable configuration of biomembranes and other general curved surfaces in the fields of biophysics and colloids. Nonetheless, there is no fundamental impediment to the application of the same physical principles to TMS. The equations that govern these structures may require some modifications to account for their unique characteristics. Therefore, in this study, a novel equation based on the generalised Laplace equation is proposed to find the form-found surface of TMS. It is well established that TMS are incapable of withstanding bending stresses (Lewis, 2003). Therefore,

$M_I, 2H\sigma = P_{II} - P_I$, and consequently B and Θ are equal to zero. Hence, in Equation 15,

$$18. \quad a^{\alpha\beta}B + q^{\alpha\beta}\Theta = 0$$

Furthermore, in the form-finding exercise, all external loads are ignored. Hence,

$$19. \quad P_{II} = P_I = 0$$

Therefore, Equation 15 is reduced to the equation of form-found shape of a TMS as:

$$20. \quad H\sigma + D\eta = 0$$

Now, we know from differential geometry (Lane, 1940):

$$21. \quad H = \frac{c_1 - c_2}{2}$$

$$22. \quad D = \frac{c_1 - c_2}{2}$$

$$23. \quad K = c_1c_2$$

where K is the Gaussian curvature at a point on the surface, and c_1, c_2 are the principal curvatures at a point on the surface. Again, the differential equations of H, K are (Lane, 1940):

$$24. \quad H = \frac{r(1 + q^2) - 2pqs + t(1 + p^2)}{2(1 + p^2 + q^2)^{1.5}}$$

$$25. \quad K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}$$

where

$$26. \quad p = \frac{\partial z}{\partial x}; \quad q = \frac{\partial z}{\partial y}; \quad r = \frac{\partial^2 z}{\partial x^2}; \quad s = \frac{\partial^2 z}{\partial x \partial y}; \quad t = \frac{\partial^2 z}{\partial y^2}$$

Furthermore, from Equations 21–23, we have:

$$27. \quad D = \pm \sqrt{H^2 - K}$$

Due to the $\pm$ sign in Equation 27, it becomes numerically difficult to solve Equation 20 directly due to the possibility of non-unique solutions. Furthermore, the square root term also causes numerical instabilities and convergence issues. In order to avoid this conundrum, a weight function is utilised as is commonly used in solving partial differential equations (PDEs) using finite-element analysis (Reddy, 2019). Instead of solving Equation 20 directly, $H\sigma - D\eta$ is multiplied to both sides of the equation, and the integral is solved. The derived equation proposed for the form finding of TMS with non-minimal prestress is given by the following simplified integral:

$$28. \quad \int_{\Omega} (H\sigma + D\eta)(H\sigma - D\eta) = 0$$

The integral is solved using a Monte Carlo integration (Morokoff and Caflisch, 1995) scheme, leading to the following formulation:

$$29. \quad \int_{\Omega} (H\sigma + D\eta)(H\sigma - D\eta) = 0$$

where B is the area of the x - y domain, and N_f is the number of randomly sampled points on the domain. $[\cdot]_i$ denotes the quantities inside the parentheses are calculated at the i th collocation point. Thus, putting the value of D from Equation 27, we get:

$$30. \quad \sum_{i=1}^{N_f} \left[(H\sigma)^2 - (H^2 - K)\eta^2 \right]_i = 0$$

$$\Rightarrow \sum_{i=1}^{N_f} \left[\left(\frac{H}{2}(\sigma_x + \sigma_y) \right)^2 - (H^2 - K) \frac{(\sigma_x - \sigma_y)^2}{4} \right]_i = 0$$

Putting the curvature formulas from Equations 24 and 25, the final equation of the form-found shape is given by:

$$31. \quad \sum_{i=1}^{N_f} \left[\left(\frac{r(1+q^2) - 2pqs + t(1+p^2)}{4(1+p^2+q^2)^{1.5}} (\sigma_x + \sigma_y) \right)^2 \right]_i$$

$$- \sum_{i=1}^{N_f} \left[\left(\left\{ \frac{r(1+q^2) - 2pqs + t(1+p^2)}{2(1+p^2+q^2)^{1.5}} \right\}^2 - \frac{rt-s^2}{(1+p^2+q^2)^2} \right) \times \frac{(\sigma_x - \sigma_y)^2}{4} \right]_i = 0$$

We consider the principal components of the prestress to be in the same direction of the principal curvatures, which is the case in form finding of TMS. Here, σ_x, σ_y are analogous to the prestresses acting along the ‘warp’ and ‘weft’ fibres of the TMS fabric, making the formulation robust enough to accommodate anisotropic prestress. The global x - y axes are oriented along these material fibre directions.

3.1 The connection between the generalised Laplace equation and the Euler–Lagrange equation

When a TMS is under the action of isotropic prestress, $\sigma_x, \sigma_y, \eta = 0$, Equation 20 becomes:

$$32. \quad H = 0$$

This equation is popularly known as the Euler–Lagrange equation of minimal surfaces (Lewis, 2003). One limitation of the Euler–Lagrange equation for TMS form finding is that this equation is only valid for minimal TMS subjected to isotropic prestress. However, as was discussed before, most practical TMS requires anisotropic prestressing, leading to non-minimal design. This is the main flaw in the form-finding methodology proposed by Kabasi *et al.* (2023), as the authors tackled the form finding of TMS by solving Equation 32 using PINN. Although the mesh-less method has several advantages with respect to conventional mesh-based form-finding techniques, the method falls short in the sense that it is not applicable for non-minimal TMS subjected to anisotropic prestress. Equation 31 is formulated in this study to precisely alleviate this particular flaw of the PINN-based method formulated in Kabasi *et al.* (2023), as the proposed formulation can easily accommodate the different ‘warp’ and ‘weft’ prestresses along the global x - y axes for form finding of practical non-minimal TMS.

4. Physics-informed neural network

PINNs are a class of machine learning models that incorporate physical laws and constraints into their architecture to improve their performance in solving PDEs and related problems. The idea of combining neural networks with physics-based modelling dates back to the 1990s when researchers started using neural networks to approximate PDEs. However, the approach was limited by the need for large amounts of training data and the inability of neural networks to enforce physical constraints.

The modern concept of PINNs was introduced by Raissi *et al.* (2019), who proposed using neural networks to learn the solution to PDEs while enforcing the physical laws that govern the system by leveraging automatic differentiation (Baydin *et al.*, 2017). PINNs provide a framework for combining the strengths of deep learning with the power of physics-based modelling, enabling efficient and accurate solutions to PDEs with limited data.

PINNs work by approximating the solution to a given PDE using a neural network. The neural network takes the spatial and temporal coordinates of the PDE as input and outputs the predicted solution. In addition to the usual loss function that measures the error between the predicted and actual solutions (for boundary/initial constraints), PINNs also include a physics-based loss function that enforces the governing equations of the system. The physics-based loss function is derived from the PDE itself and is used to constrain the predictions of the neural network to be consistent with the underlying physics. Given a PDE of the form:

$$33. \quad \mathcal{L}\left(u(x, t), \partial_t u(x, t), \partial_x u(x, t), \dots\right) = f(x, t)$$

where $\mathcal{L}$ is a differential operator, u is the unknown function, and f is a known source term.

We seek to find the solution u that satisfies the PDE and any associated boundary conditions. To do this, a neural network $u_\theta(x, t)$ is constructed with learnable parameters θ that approximates the solution $u(x, t)$, such that:

$$34. \quad \mathcal{L}\left(u_\theta(x, t), \partial_t u_\theta(x, t), \partial_x u_\theta(x, t), \dots\right) = f(x, t)$$

This equation is referred to as the physics-informed loss and is used to train the neural network.

In addition, any available boundary conditions can be incorporated into the loss function by adding a boundary loss term of the form:

$$35. \quad \mathcal{B}\left(u_\theta(x_b, t_b), g(x_b, t_b)\right) = 0$$

where x_b and t_b are the coordinates of the boundary points, $g(x_b, t_b)$ is the known boundary condition, and $\mathcal{B}$ is a boundary operator that enforces the boundary condition. The total loss function for training the PINN is then given by:

$$36. \quad \mathcal{L}_{\text{tot}}(\theta) = \sum_{i=1}^{N_f} \left| \mathcal{L}\left(u_\theta(x_i, t_i), \partial_t u_\theta(x_i, t_i), \partial_x u_\theta(x_i, t_i), \dots\right) - f(x_i, t_i) \right|^2 + \alpha \sum_{b=1}^{N_u} \left| \mathcal{B}\left(u_\theta(x_b, t_b), g(x_b, t_b)\right) \right|^2$$

where N_f is the number of data points used for training to enforce the PDE, N_u is the number of boundary points, and α is a weighting parameter that balances the importance of the physics-informed loss and the boundary loss. By minimising the total loss

function $\mathcal{L}_{\text{tot}}$ with respect to the learnable parameters θ , an approximation of the solution $u(x, t)$ can be obtained that satisfies both the PDE and the boundary conditions.

Overall, PINNs provide a powerful tool for solving complex physical problems that involve PDEs and related constraints and have been successfully applied in a wide range of fields, including fluid dynamics, solid mechanics, and so on (Raissi *et al.*, 2019).

5. The proposed form-finding algorithm

The approximate PDE that governs the shape of the form-found shape of TMS subjected to a general initial prestress condition is given by Equation 31. This section introduces a framework for finding the solution of this equation with the help of PINN, thus finding the form-found shape while respecting the given boundary conditions within a low-tolerance value. The premise of the proposed framework lies in finding the unknown z coordinates of the TMS (having x - y domain, Ω , and boundary $\partial\Omega$) subjected to the prescribed orthogonal warp (σ_x) and weft (σ_y) prestresses acting along the x - y axes. This is done using a neural network $z_\theta(x, y)$ for satisfying Equation 31 and the boundary constraints, $B(x, y) = 0$, simultaneously by employing the concepts of a PINN. The neural network parameters θ are initialised using a randomised Xavier initialisation (Glorot and Bengio, 2010).

Consider a set of collocation points $X_I = (x_i, y_i) \in \Omega$; $i = \{1, 2, \dots, N_f\}$ generated using Latin hypercube sampling (LHS; McKay *et al.*, 1979). Then the loss function for enforcing the proposed governing equation for form-found TMS configuration (Equation 31) can be written as:

$$37. \quad \mathcal{L}_{ff} = \left[\sum_{i=1}^{N_f} \left(\frac{r(1+q^2) - 2pqst + t(1+p^2)}{4(1+p^2+q^2)^{1.5}} (\sigma_x + \sigma_y) \right)^2 - \sum_{i=1}^{N_f} \left(\left\{ \frac{r(1+q^2) - 2pqst + t(1+p^2)}{2(1+p^2+q^2)^{1.5}} \right\}^2 - \frac{rt - s^2}{(1+p^2+q^2)^2} \right) \frac{(\sigma_x - \sigma_y)^2}{4} \right]$$

where

$$38. \quad p = \frac{\partial z_\theta(x_i, y_i)}{\partial x}; \quad q = \frac{\partial z_\theta(x_i, y_i)}{\partial y}; \quad r = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial x^2};$$

$$s = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial x \partial y}; \quad t = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial y^2}; \quad \forall (x_i, y_i) \in \Omega$$

are found by leveraging the automatic differentiation modules.

The boundary constraints need to be satisfied for N_u number of equidistant points $X_b = (x_b, y_b) \in \partial\Omega$; $b = \{1, 2, \dots, N_u\}$ using the following loss function:

$$B = \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u}$$

$$39. \quad + \sqrt{\frac{\sum_{i=1}^{N_u} \left\{ |z_\theta(x_b, y_b) - z(x_b, y_b)|^2 - \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u} \right\}^2}{N_u - 1}}$$

In the above equation, the term $\frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u}$ represents the mean of square of the errors between the z coordinates at the boundary collocation points obtained by using the neural network, $z_\theta(x_b, y_b)$, and the prescribed boundary coordinates $z(x_b, y_b)$. The other term is the standard deviation of the squared errors. The standard deviation is also added to increase the precision of the PINN solution (Kabasi *et al.*, 2023). Thus, the total loss function for enforcing both the governing equation and the boundary constraints concurrently to find the form-found shape of the TMS is given by:

$$40. \quad \mathcal{L}_{tot}(\theta) = \mathcal{L}_{ff} + B$$

$$\mathcal{L}_{tot}(\theta) = \left[\sum_{i=1}^{N_f} \left(\frac{r(1+q^2) - 2pqs + t(1+p^2)}{4(1+p^2+q^2)^{1.5}} (\sigma_x + \sigma_y) \right)^2 - \sum_{i=1}^{N_f} \left(\left\{ \frac{r(1+q^2) - 2pqs + t(1+p^2)}{2(1+p^2+q^2)^{1.5}} \right\}^2 - \frac{rt - s^2}{(1+p^2+q^2)^2} \right) \times \frac{(\sigma_x - \sigma_y)^2}{4} \right]^2$$

$$41. \quad + \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u}$$

$$+ \sqrt{\frac{\sum_{i=1}^{N_u} \left\{ |z_\theta(x_b, y_b) - z(x_b, y_b)|^2 - \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u} \right\}^2}{N_u - 1}}$$

The optimum PINN parameters, θ^* is obtained from the following optimisation problem.

$$42. \quad \theta^* = \underset{\theta}{\operatorname{argmin}} \mathcal{L}_{tot}(\theta)$$

In the proposed framework, the L-BFGS-B optimiser (Virtanen *et al.*, 2020) is employed to train the neural network. The optimiser options used in the proposed algorithm are chosen to be the same as those provided in Kabasi *et al.* (2023). Hence, the ordered triplet $[x, y, z_\theta(x, y)]$ describes the form-found configuration of the TMS, where $x, y \in \Omega \cup \partial\Omega$.

Algorithm 1: The proposed form-finding algorithm

1. Initialise $\theta \rightarrow z_\theta(x, y)$
2. Generate $X_I = \{x_i, y_i\} \in \Omega$; $i = \{1, 2, \dots, N_f\}$
3. Generate $X_b = \{x_b, y_b\} \in \partial\Omega$; $b = \{1, 2, \dots, N_u\}$

Automatic differentiation

$$p = \frac{\partial z_\theta(x_i, y_i)}{\partial x}; \quad q = \frac{\partial z_\theta(x_i, y_i)}{\partial y}; \quad r = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial x^2};$$

$$s = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial x \partial y}; \quad t = \frac{\partial^2 z_\theta(x_i, y_i)}{\partial y^2}; \quad \forall \{x_i, y_i\} \in X_I$$

Loss function for governing equation

$$\mathcal{L}_{ff} = \left[\sum_{i=1}^{N_f} \left(\frac{r(1+q^2) - 2pqs + t(1+p^2)}{4(1+p^2+q^2)^{1.5}} (\sigma_x + \sigma_y) \right)^2 - \sum_{i=1}^{N_f} \left(\left\{ \frac{r(1+q^2) - 2pqs + t(1+p^2)}{2(1+p^2+q^2)^{1.5}} \right\}^2 - \frac{rt - s^2}{(1+p^2+q^2)^2} \right) \times \frac{(\sigma_x - \sigma_y)^2}{4} \right]^2$$

Loss function for boundary constraints

$$B = \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u}$$

$$+ \sqrt{\frac{\sum_{i=1}^{N_u} \left\{ |z_\theta(x_b, y_b) - z(x_b, y_b)|^2 - \frac{\sum_{b=1}^{N_u} (z_\theta(x_b, y_b) - z(x_b, y_b))^2}{N_u} \right\}^2}{N_u - 1}};$$

$$\forall \{x_b, y_b\} \in X_b$$

Total loss function $\mathcal{L}_{tot}(\theta) = \mathcal{L}_{ff} + B$

Optimisation for finding optimum PINN parameters $\theta^* = \underset{\theta}{\operatorname{argmin}} \mathcal{L}_{tot}(\theta)$

Final form-found coordinates $[x, y, z_{\theta^*}(x, y)]; \forall x, y \in \Omega \cup \partial\Omega$.

6. Numerical study

6.1 Parametric studies

The proposed algorithm is explored for four frame-supported non-minimal (minimal is not performed in this study since the formulation reduces to Kabasi *et al.* (2023)) TMS case studies: (a) plane (Kabasi *et al.*, 2023), (b) tent (Kabasi *et al.*, 2023), (c) hypar (Lewis, 2003; Marbaniang *et al.*, 2022), and (d) twin hypar (Marbaniang *et al.*, 2022). The details of these case studies are given in Table 1. The corresponding boundary collocation points are shown in Figures 1(a), 2(a), 3(a), and 5(a). An extensive parametric study is conducted by varying (N_u, N_f) for two combinations of target (anisotropic) prestress, (σ_x, σ_y) —(3, 4) and (2, 4) kN/m. The chosen PINN architecture of this case study is taken as 2 hidden layers and 20 neurons per hidden layer (Kabasi *et al.*, 2023). The results of this parametric study are tabulated in Tables 2–4.

The trained PINN is used to predict the z coordinates of another 490 000 uniformly distributed sample points in the domain, and the obtained form-found coordinates are validated by calculating the unbalanced residual forces as shown in Table 2. These nodes are discretised using constant-strain triangles for the calculation of the unbalanced residual forces at the unconstrained nodes (Topping and Ivanyi, 2007). This is because the usual convention for checking the validity of a form-found solution is by observing the maximum value of the unbalanced residual force R , which must be less than some tolerance value (Kabasi *et al.*, 2023; Topping and Ivanyi, 2007). The 490 000 nodal coordinates obtained by the proposed approach are plotted for each form-finding case study for all the stress cases. It is to be noted that this is just a post-processing step for demonstrating the validity of the solution. The original form-finding framework is still mesh-less.

The maximum absolute prediction error ($\delta_1\%$) at the boundary, considering different numbers of training collocation points for different stress conditions as a percentage of the maximum dimension, is also noted down in Table 3 to check whether the proposed PINN-based method is able to satisfy the boundary conditions.

The corresponding form-found shapes obtained by a mesh-based method called the updated weight method (UWM) (Marbaniang *et al.*, 2022) are also plotted. The proposed method is further validated by comparing the coordinates of the form-found shape using the PINN-based approach with those obtained using UWM. The maximum absolute difference between the z coordinates of the PINN and the z coordinates obtained by UWM at the free nodes is also calculated as a percentage of the maximum dimension of the structure and is denoted by δ_2 . The values of δ_2 are shown in Table 4.

The PINN algorithm is implemented in Google Colab using Python, while UWM is performed in MATLAB. The results are plotted using MATLAB.

6.2 Discussions

In order to test the proposed form-finding framework, numerous form-finding exercises are conducted under different stress conditions. The form-finding metrics obtained for various stress conditions are tabulated in Tables 2–4. The loss convergence curve, including the boundary loss and the PDE loss for Hypar for $(\sigma_x, \sigma_y) = (4, 3)$ kN/m, is shown in Figure 4.

Typically, increasing the number of boundary collocation points reduces the prediction error at the boundary of the form-found shape obtained by the PINN. However, Table 3 shows that such a linear trend is not observed. A concrete reasoning for this observation is not straightforward, as was noted by Raissi *et al.* (2019), who observed that the optimum number of collocation points to reach a global optimum is quite difficult to obtain. Similar indiscernible patterns were also observed in the PINN-based approach proposed in Kabasi *et al.* (2023). The non-linear trend in δ_2 can

Table 1. Details of the frame-supported TMS case studies

Example	Domain: m	Boundary equation	(σ_x, σ_y) : kN/m	Proposed	UWM
Plane	$x, y \in (-5, 5)$	$z = x + y$	(3, 4)	Figure 1(b)	Figure 1(c)
Plane	$z = x + y$	$z = x + y$	(2, 4)	Figure 1(d)	Figure 1(e)
Tent	$x \in (-2.5, 2.5), y \in (-5, 5)$	$z = 1.0 - 0.2 y $	(3, 4)	Figure 2(b)	Figure 2(c)
Tent	$x \in (-2.5, 2.5), y \in (-5, 5)$	$z = 1.0 - 0.2 y $	(2, 4)	Figure 2(d)	Figure 2(e)
Hypar	$x, y \in (-25, 25)$	$z = -\frac{xy}{125} + 5$	(3, 4)	Figure 3(b)	Figure 3(c)
Hypar	$x, y \in (-25, 25)$	$z = -\frac{xy}{125} + 5$	(2, 4)	Figure 3(d)	Figure 3(e)
Twin hypar	$x \in (-30, 30), y \in (-15, 15)$	$z = -0.011 y x + 0.167 y + 2.5$	(3, 4)	Figure 5(b)	Figure 5(c)
Twin hypar	$x \in (-30, 30), y \in (-15, 15)$	$z = -0.011 y x + 0.167 y + 2.5$	(2, 4)	Figure 5(d)	Figure 5(e)

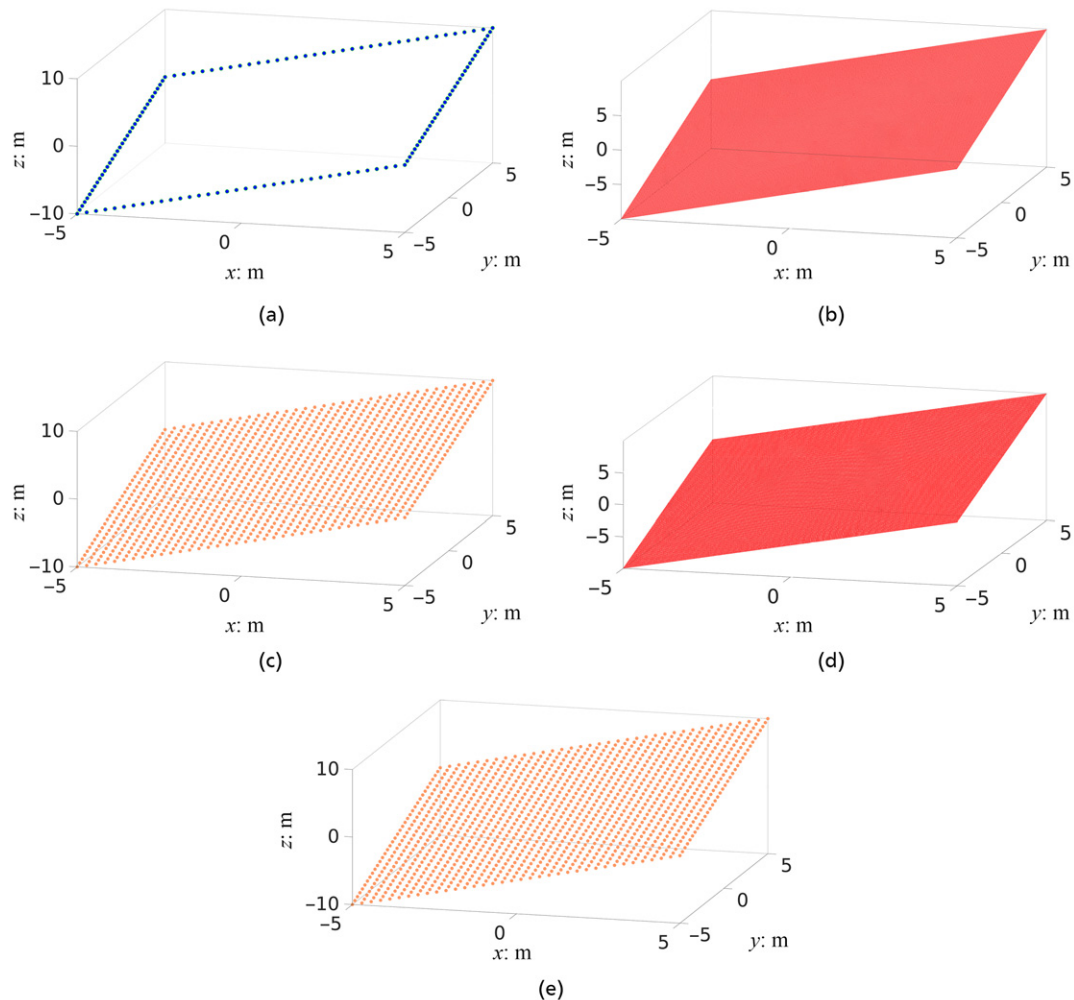


Figure 1. Example 1: plane

also be attributed to numerical approximations and errors of the linearly meshed UWM. However, the δ_2 are relatively low for all the cases, specifically for $(N_u, N_f) = (80, 20\,000)$.

Despite the incoherent patterns obtained in the parametric study, all the values of the unbalanced residual R are within the acceptable tolerance of traditional form-finding approaches, which is usually taken in the range of 10^{-2} and 10^{-3} kN (Kabasi *et al.*, 2023; Lewis, 2003; Topping and Ivanyi, 2007). Thus, the final shapes obtained by the proposed framework are all in static equilibrium and therefore represent the correct form-found surfaces. Furthermore, the low values of δ_1 suggest that the proposed method is able to successfully satisfy the boundary constraints. In addition, the low values of δ_2 as seen in Table 4 suggest that the form-found shape obtained by the proposed framework and the traditional UWM approach closely

correspond to each other, adding a second layer of validation to the proposed methodology.

6.3 Computational efficiency for high-dimensional cases

The proposed method is tested against DR and UWM with respect to computational efficiency for form-finding of a twin hyper for three sets of collocation points (5000, 10 000, and 20 000) on an Intel 657 Core i7 3.00 GHz 8-core CPU with 16 GB RAM workstation (no GPU), and the results are noted in Table 5. The traditional methods outperform the proposed method for the smallest set of collocation points. However, for higher node counts, such as 10 000 and 20 000 nodes, DR fails to converge. This outcome is expected considering the previously discussed convergence issues of DR attributed to the high dimensionality of the problem. In

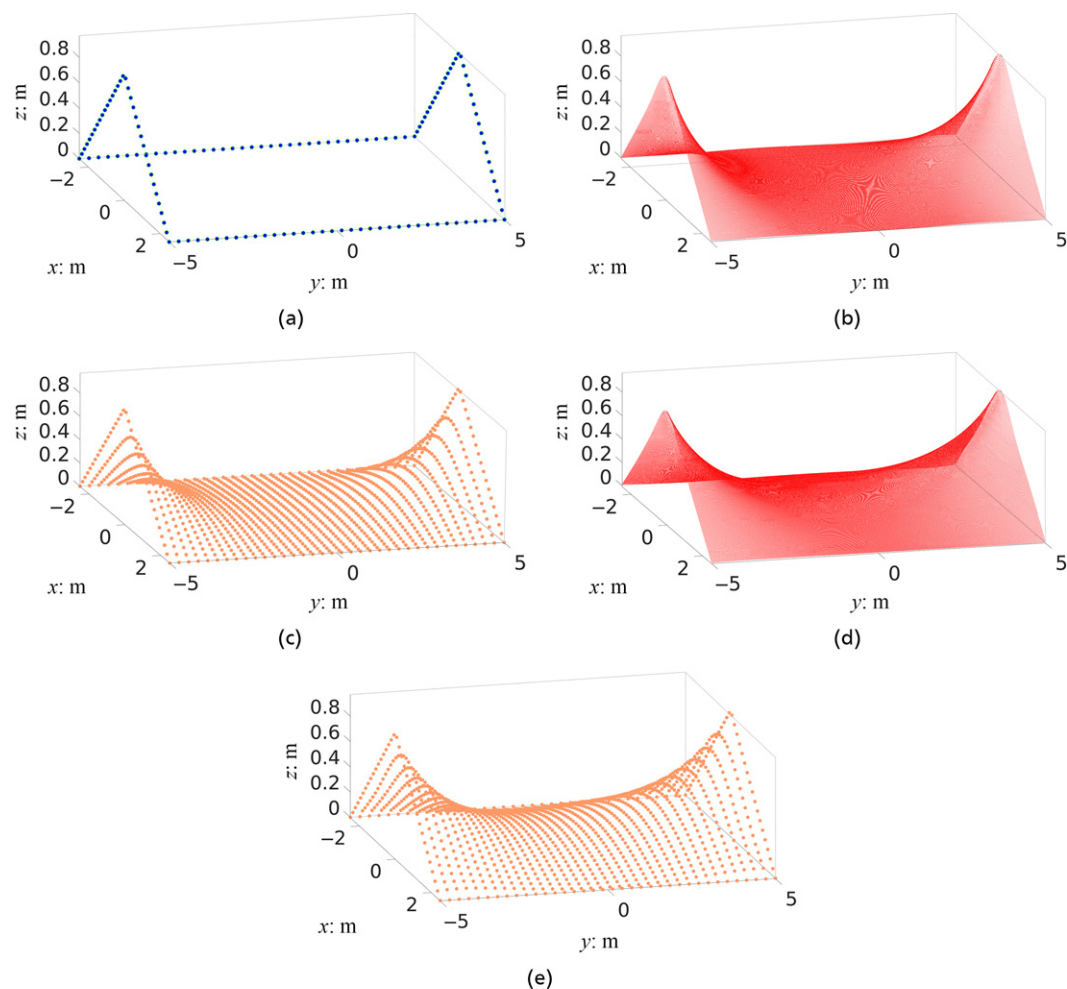


Figure 2. Example 2: tent

addition, PINN outperforms UWM for 10 000 nodes. This observation implies that UWM becomes increasingly computationally demanding as the dimensionality increases. Furthermore, for 20 000 nodes, UWM encounters a complete failure due to exceeding the memory capacity of the PC. This limitation arises from UWM's necessity to compute a Hessian matrix, which becomes a $60\,000 \times 60\,000$ matrix (a square matrix of size three times the number of nodes) in this case, resulting in an out-of-memory error and subsequent non-convergence. In contrast, the proposed approach achieves convergence within 62 s in this scenario, highlighting its superior performance in terms of dimensionality.

Conventional form-finding methods rely on free nodal coordinates as unknown variables, significantly increasing the computational challenge in achieving convergence. In contrast, for the selected gPINN architecture with 2 hidden layers and 20 neurons per hidden layer, the number of unknown variables to be optimised remains constant

at 501 (460 weight parameters and 41 bias parameters), regardless of the number of collocation points. Consequently, even when employing second-order classical optimisers for training, the Hessian size remains at 501×501 for the proposed approach, compared with $60\,000 \times 60\,000$ for UWM when the number of nodes/collocation points is 20 000. Thus, the proposed approach significantly surpasses UWM in terms of space complexity. As a result, regardless of the problem's complexity in terms of a large number of nodes/collocation points, the PINN computation remains significantly more efficient, establishing the proposed approach as a superior performer in terms of dimensionality compared with standard methods.

6.4 Advantages of the proposed method

The proposed framework takes advantage of the mesh-less nature of the PINN, resulting in a form-finding framework that has several advantages with respect to conventional mesh-based form-finding techniques:

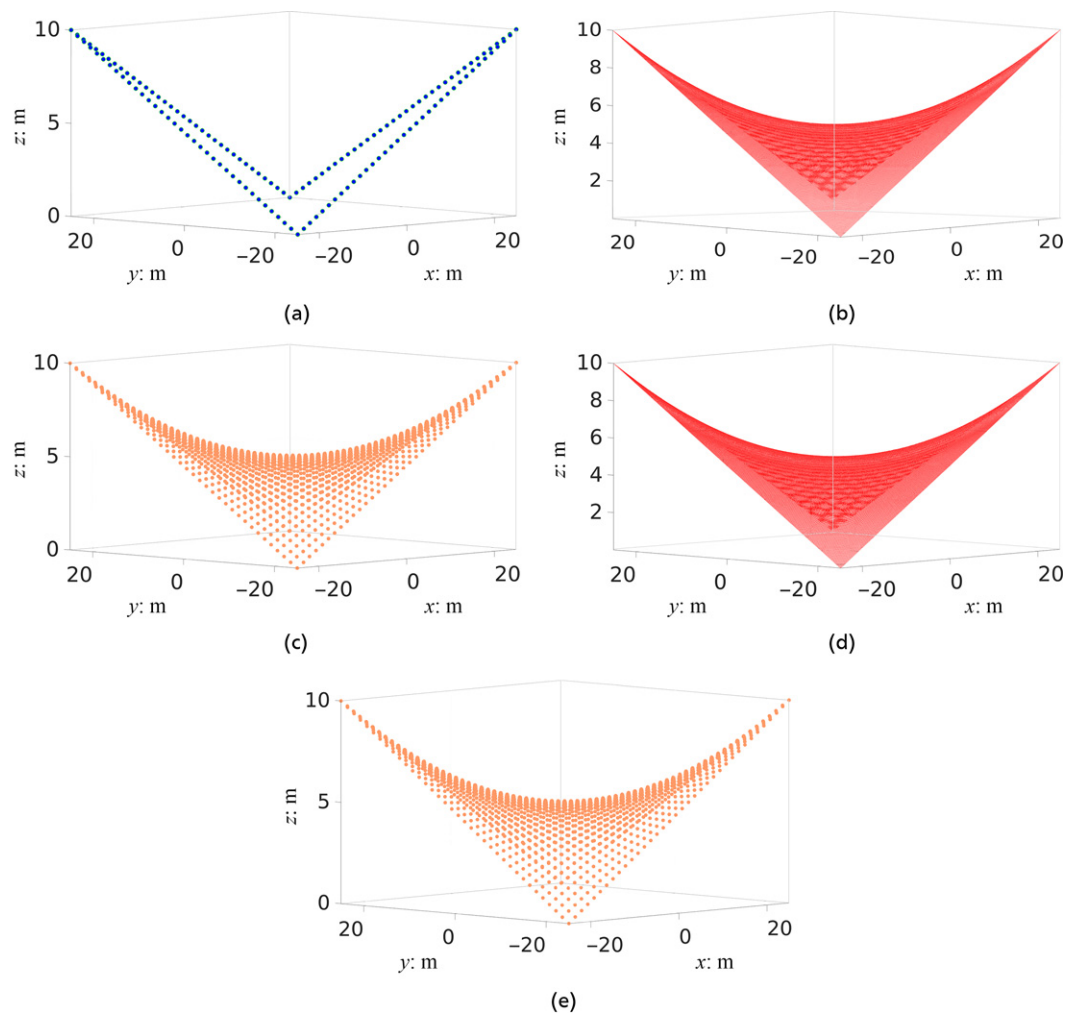


Figure 3. Example 3: hypar

Table 2. Maximum unbalanced forces R (kN) considering different numbers of training collocation points for different stress conditions

Example	(σ_x, σ_y) : kN/m	(N_u, N_r) (24, 1000)	(40, 10 000)	(80, 20 000)	(120, 30 000)
Plane	(3, 4)	1.93×10^{-5}	1.96×10^{-5}	2.13×10^{-5}	1.73×10^{-5}
	(2, 4)	3.15×10^{-5}	1.93×10^{-5}	1.56×10^{-5}	3.15×10^{-5}
Tent	(3, 4)	1.69×10^{-4}	8.03×10^{-5}	9.86×10^{-5}	1.27×10^{-4}
	(2, 4)	3.83×10^{-4}	3.70×10^{-4}	3.93×10^{-4}	3.49×10^{-4}
Hypar	(3, 4)	2.98×10^{-4}	2.51×10^{-4}	2.94×10^{-4}	2.28×10^{-4}
	(2, 4)	3.96×10^{-4}	3.38×10^{-4}	2.06×10^{-4}	2.24×10^{-4}
Twin hypar	(3, 4)	5.10×10^{-4}	5.23×10^{-4}	9.08×10^{-4}	1.10×10^{-3}
	(2, 4)	6.16×10^{-4}	1.13×10^{-3}	2.12×10^{-3}	2.45×10^{-3}

- Computationally more efficient than traditional approaches for high-dimensional cases.
- Demonstrates stable convergence in high-dimensional cases, unlike traditional methods that exhibit non-convergence for high-dimensional cases.
- No mesh distortion (as seen in DR) or ‘floating meshes’ (Linhard and Bletzinger, 2010) as the method is mesh-less.
- Independent of any pseudo-material parameters (as in FEM (Kabasi *et al.*, 2023)).

Table 3. $\delta_1\%$ considering different numbers of training collocation points for different stress conditions

Example	(σ_x, σ_y) : kN/m	(N_u, N_f) (24, 1000)	(40, 10 000)	(80, 20 000)	(120, 30 000)
Plane	(3, 4)	0.042	0.058	0.022	0.061
	(2, 4)	0.083	0.022	0.020	0.0048
Tent	(3, 4)	0.20	0.32	0.36	0.89
	(2, 4)	0.22	0.32	0.47	0.47
Hypar	(3, 4)	0.059	0.0068	0.0048	0.0029
	(2, 4)	0.097	0.019	0.0019	0.0028
Twin hypar	(3, 4)	0.48	0.32	0.16	0.11
	(2, 4)	0.64	0.31	0.17	0.11

Table 4. $\delta_2\%$ for different numbers of training collocation points for different stress conditions

Example	(σ_x, σ_y) : kN/m	(N_u, N_f) (24, 1000)	(40, 10 000)	(80, 20 000)	(120, 30 000)
Plane	(3, 4)	0.79	0.20	0.34	0.38
	(2, 4)	0.61	0.12	0.31	0.67
Tent	(3, 4)	0.71	0.90	0.85	0.74
	(2, 4)	0.70	0.36	0.36	0.33
Hypar	(3, 4)	0.87	1.5	0.71	0.77
	(2, 4)	1.9	0.76	0.96	0.81
Twin hypar	(3, 4)	0.58	0.51	0.86	0.66
	(2, 4)	2.2	0.78	0.64	1.5

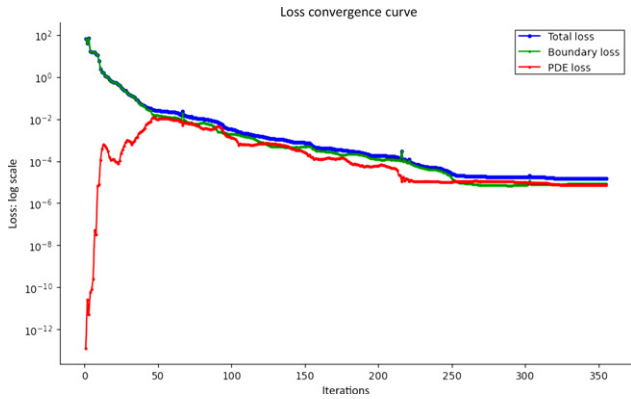


Figure 4. Example 3: loss convergence curve for Hypar TMS

- No computationally costly calculation of stiffness matrix (as in stiffness/geometric-based methods (Kabasi *et al.*, 2023)).
- Independent of choice of initial shape (unlike DR and FEM, which are highly sensitive (Kabasi *et al.*, 2023)), as the neural network parameters, θ , are initialised randomly.
- Implementation simplicity (using automatic differentiation to directly compute derivatives) compared with traditional methods.

The proposed method is also much more robust than the PINN-based form-finding method proposed in Kabasi *et al.* (2023), as the proposed method is applicable for both isotropic (in this case,

the equation reduces to the Euler–Lagrange equation) and anisotropic prestress. Table 6 summarises the computational advantages of the proposed method with respect to traditional methods. Therefore, the suggested method for form finding presents a strong and straightforward technique that is devoid of mesh-based issues and can be used to determine the stable configuration of non-minimal TMS that is typically common in real-world applications.

6.5 Guidelines for users

This section provides guidelines for future users who wish to implement the proposed algorithm based on the different case studies presented. First, it is suggested that the preferred choice of the architecture for the PINN should have two hidden layers, with each layer consisting of 20 neurons. This suggestion is based on metrics obtained from comprehensive form-finding exercises done in Kabasi *et al.* (2023). In this study, the authors studied the influence of the neural network architecture by varying the number of hidden layers and neurons per layer across nine different configurations. Specifically, networks with 2, 5, and 8 hidden layers are considered, each combined with 10, 15, and 20 neurons per layer. To ensure a fair comparison, the number of training collocation points was kept constant throughout this study. For each configuration, the convergence time (in seconds) and key performance metrics— R , $\delta_1\%$, and $\delta_2\%$ —were recorded upon convergence. The second parametric study focused on the effect of collocation point distribution by varying the number of collocation points at the boundary and within the domain across four different sets. A fixed network architecture of 2 hidden layers with 20 neurons

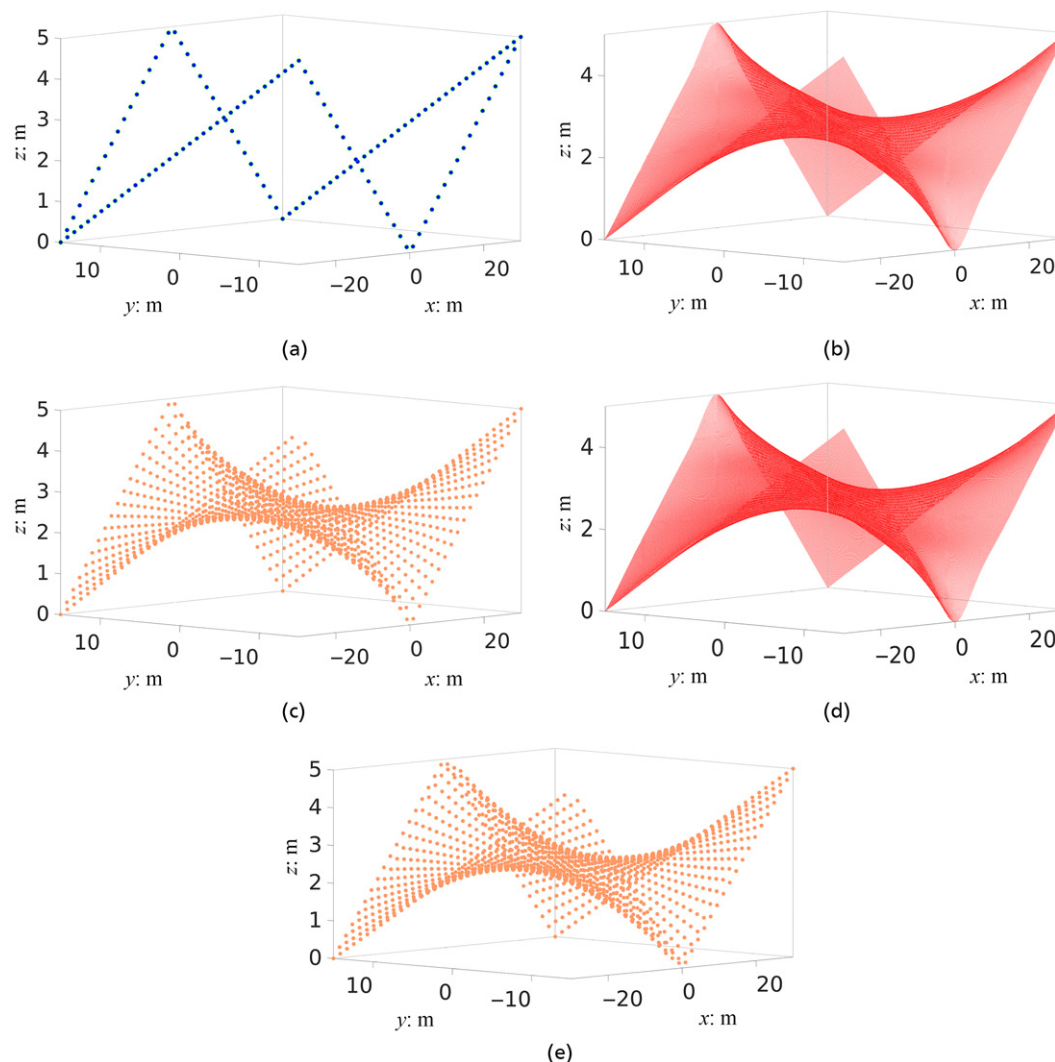


Figure 5. Example 4: twin hyperpar

Table 5. Comparison of convergence time in seconds for the twin hyperpar problem using the proposed method, DR, and UWM

Number of nodes/ collocation points	DR	UWM	PINN
5000	12	11	20
10 000	Does not converge	36	33
20 000	Does not converge	Does not converge	62

per layer was maintained to isolate the influence of collocation point selection. Similar to the first study, the key performance metrics R , $\delta_1\%$, and $\delta_2\%$ were tracked to evaluate the framework's response to changes in a number of collocation points. Based on

these observations made by Kabasi *et al.* (2023), it is recommended that the combination of collocation points should be $(N_u, N_f) = (80, 20\,000)$, with 20 000 collocation points being generated using LHS inside the domain (Kabasi *et al.*, 2023). The collocation points at the boundary should be distributed equidistantly. This combination is observed to yield a good representation of the form-found shape while satisfying the boundary within a low tolerance, as seen in the aforementioned form-finding tables (Tables 1–3).

Second, the activation function for all layers except the last one is suggested to be a hyperbolic tangent function (Kabasi *et al.*, 2023). This function has a range of $(-1, 1)$, which can accommodate both positive and negative activation values, unlike other

Table 6. Comparison of computational efficiency of different methods

Method	Scalability with node count	Memory efficiency	Computational complexity
Traditional (DR)	Fails for high node counts due to convergence issues in high dimensions	Moderate, but fails for large problems due to convergence limitations	Becomes infeasible for high-dimensional problems
Traditional (UWM)	Computational cost increases significantly due to dependence on node count	Highly inefficient as Hessian grows quadratically with node count	Very high due to second-order derivatives and Hessian computation
Proposed (PINN)	Scales well as the number of unknowns remains constant	Highly efficient, as Hessian remains small and fixed	Efficient, as the optimisation problem remains low dimensional

functions such as sigmoid or rectified linear unit. This allows for faster convergence and is therefore a desirable choice.

Finally, the parametric details of the LBFGS-B optimiser as given in Kabasi *et al.* (2023) can be followed. In the proposed form-finding methodology, the L-BFGS-B optimiser is utilised to train the PINN. Through preliminary experiments, the following convergence criteria have been identified as effective.

- A maximum of 500 variable metric corrections is maintained to construct the limited memory matrix, avoiding full Hessian computation.
- The line search procedure allows up to 500 steps per iteration.
- The total number of iterations is capped at 1500.
- The function evaluations are limited to a maximum of 1500.
- The optimisation process terminates if either the loss function tolerance or the gradient norm falls below 10^{-15} .
- Other parameters are retained at their default settings as specified in the optimiser's implementation.

Further details regarding the convergence behaviour of the L-BFGS-B optimiser can be found in the SciPy documentation (Virtanen *et al.*, 2020). It is important to emphasise that the optimiser settings in the proposed approach differ from those in the foundational study by Raissi *et al.* (2019). These modifications were introduced to mitigate the risk of premature convergence. The selected optimiser parameters have been validated through detailed numerical investigations, demonstrating their suitability for the proposed form-finding strategy. It is also worth noting that stricter tolerance settings could be imposed to achieve greater accuracy; however, this may come at the expense of increased computational effort.

7. Conclusion

This study explores the applicability of an elegant form of the generalised Laplace equation typically used in the domains of biomembranes and microemulsions for the form-finding exercise. This equation originates from the theory of general curved interfaces, which is usually characterised by a combination of mechanical and thermodynamic approaches. Based on this theory, an equation for finding the form-found configuration of

TMS is derived. This PDE is solved using a recently introduced algorithm in the domain of supervised learning known as PINN. A parametric study is performed that involves finding the optimum number of training collocation points for the proposed PINN architecture with 2 hidden layers and 20 neurons per hidden layer.

The proposed framework takes advantage of the mesh-less nature of the PINN, resulting in a computationally efficient form-finding framework that has several advantages with respect to conventional mesh-based form-finding techniques, as summarised in Section 6.4. The proposed mesh-free form-finding approach does not use a mesh or nodal connectivity matrix; rather, the neural network is trained directly by utilising the physics of the problem as delineated by the form-found surface equation derived using the theoretical background of the mechanical and thermodynamic approaches for describing the curved interfaces. This is performed by employing collocation points and different loss functions for enforcing the governing differential equation and the boundary constraints. The inherent mesh-less nature of the proposed framework makes the method immune to mesh distortion and floating mesh issues seen in mesh-based form-finding approaches (Kabasi *et al.*, 2023; Wüchner and Bletzinger, 2005). In addition, the neural network parameters are initialised randomly, making the method independent of the choice of the initial shape, that is known to cause significant convergence issues in the case of traditional methods (Kabasi *et al.*, 2023). Furthermore, the method is independent of material properties (since form finding is a purely geometrical problem independent of material properties), unlike FEM-based methods (Kabasi *et al.*, 2023), and also avoids the requirement of a stiffness matrix (Veenendaal and Block, 2012). Overall, the proposed method can be thought of as a more robust and computationally efficient form-finding tool suited for practical TMS design (both minimal and non-minimal), as the formulation can accommodate different (anisotropic) prestress along the orthogonal 'warp' and 'weft' fibres of the TMS fabric, unlike the PINN-based framework proposed in Kabasi *et al.* (2023), which is only applicable for minimal TMS subjected to isotropic prestress. The suggested method for form-finding presents a strong and straightforward technique that is devoid of mesh-based issues and

can be used to determine the stable configuration of non-minimal TMS that is typically common in real-world applications.

One limitation of PINN-based approaches is the need to retrain the neural network from scratch for new boundaries or initial conditions. However, this challenge can potentially be addressed using advanced frameworks, such as operator learning (e.g. DeepONet (Lu *et al.*, 2019) and Fourier neural operators (Li *et al.*, 2020)), which enable the prediction of PDE solutions under varying boundary or initial conditions without requiring retraining. Future research should explore these methodologies to enhance the adaptability and efficiency of PINN-based approaches. Moreover, this study focuses exclusively on frame-supported TMS. This limitation arises from the requirement that boundary collocation points remain fixed, which is essential for defining the boundary loss function in the proposed framework. However, this assumption is not applicable to cable-supported TMS, as their boundary positions evolve during the form-finding process. As a result, further investigation is needed to extend the framework to accommodate cable-supported configurations.

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REFERENCES

- Baydin AG, Pearlmutter BA, Radul AA and Siskind JM (2017) Automatic differentiation in machine learning: a survey. *Journal of Machine Learning Research* **18**(1): 5595–5637.
- Bletzinger K and Ramm E (1999) A general finite element approach to the form finding of tensile structures by the updated reference strategy. *International Journal of Space Structures* **14**(2): 131–145.
- Boruvka L and Neumann A (1977) Generalization of the classical theory of capillarity. *The Journal of Chemical Physics* **66**(12): 5464–5476.
- Ciarlet PG (1997) *Mathematical Elasticity: volume II: theory of Plates*. Elsevier.
- Dutta S and Ghosh S (2021) Form-finding of frame-supported tensile membrane structures using stochastic optimisation. *Structures* **32**: 2211–2221.
- Eriksson JC and Ljunggren S (1985) The mechanics and thermodynamics of rod-shaped micelles. *Journal of the Chemical Society, Faraday Transactions 2* **81**(8): 1209–1242.
- Gibbs JW (1961) *The Scientific Papers of J. Willard Gibbs: Dynamics*. Dover Publication.
- Glorot X and Bengio Y (2010) Understanding the difficulty of training deep feedforward neural networks. In *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, pp. 249–256.
- Gosling PD, Bridgens BN, Albrecht A *et al.* (2013) Analysis and design of membrane structures: results of a round robin exercise. *Engineering Structures* **48**: 313–328, <https://doi.org/10.1016/j.engstruct.2012.10.008>.
- Gurkov T and Kralchevsky P (1990) Surface tension and surface energy of curved interfaces and membranes. *Colloids and Surfaces* **47**: 45–68.
- Helfrich W (1974) Blocked lipid exchange in bilayers and its possible influence on the shape of vesicles. *Zeitschrift Fur Naturforschung. Section C, Biosciences* **29C**(9–10): 510–515.
- Kabasi S, Marbaniang AL and Ghosh S (2023) Physics-informed neural networks for the form-finding of tensile membranes by solving the Euler–Lagrange equation of minimal surfaces. *Thin-Walled Structures* **182**: 110309, <https://doi.org/10.1016/j.tws.2022.110309>.
- Kralchevsky PA and Ivanov IB (1990) Micromechanical description of curved interfaces, thin films, and membranes: II. Film surface tensions, disjoining pressure and interfacial stress balances. *Journal of Colloid and Interface Science* **137**(1): 234–252.
- Kralchevsky P, Eriksson J and Ljunggren S (1994) Theory of curved interfaces and membranes: mechanical and thermodynamical approaches. *Advances in Colloid and Interface Science* **48**: 19–59.
- Lane EP (1940) *Metric Differential Geometry of Curves and Surfaces*. University of Chicago Press, Chicago, IL.
- Laplace PS (1805) *Traite de Mechanique Celestes; Supplements an Livre X, 1805-1806*. Chelsea Publishing Company, New York, English translation by Bowditch (1839); reprinted in 1966.
- Lewis W (2003) *Tension Structures -Form and Behaviour*. Thomas Telford Publishing, London.
- Li Z, Kovachki N, Azizzadenesheli K, *et al.* (2020) Fourier neural operator for parametric partial differential equations. *arXiv preprint arXiv:2010.08895*.
- Linhard J and Bletzinger KU (2010) Tracing the equilibrium—recent advances in numerical form finding. *International Journal of Space Structures* **25**(2): 107–116.
- Ljunggren S and Eriksson JC (1992) Minimal surfaces and Winsor III microemulsions. *Langmuir* **8**(5): 1300–1306.
- Lu L, Jin P and Karniadakis GE (2019) DeepONet: learning nonlinear operators for identifying differential equations based on the universal approximation theorem of operators. *arXiv preprint arXiv:1910.03193*.
- Marbaniang AL, Dutta S and Ghosh S (2022) Updated weight method: an optimisation-based form-finding method of tensile membrane structures. *Structural and Multidisciplinary Optimization* **65**(6): 169.
- Markin V (1981) Lateral organization of membranes and cell shapes. *Biophysical Journal* **36**(1): 1–19.
- McKay MD, Beckman RJ and Conover WJ (1979) A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics* **21**(2): 239–245.
- Morokoff WJ and Caflisch RE (1995) Quasi-Monte Carlo integration. *Journal of Computational Physics* **122**(2): 218–230.
- Naghdi PM (1972) The theory of shells and plates. In *Handbuch der Physik*, vol. VIa/2, Springer, Berlin, p. 425.
- Nouri-Baranger T (2004) Computational methods for tension-loaded structures. *Archives of Computational Methods in Engineering* **11**(2): 143–186.
- Raissi M, Perdikaris P and Karniadakis GE (2019) Physics-informed neural networks: a deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics* **378**: 686–707.
- Reddy JN (2019) *Introduction to the Finite Element Method*. McGraw-Hill Education.
- Serrin J (1959) *Mathematical Principles of Classical Fluid Mechanics*. Springer, Berlin.
- Topping B and Ivanyi P (2007) *Computer Aided Design of Cable Membrane Structures*. Saxe-Coburg Publications.

-
- Veenendaal D and Block P (2012) An overview and comparison of structural form finding methods for general networks. *International Journal of Solids and Structures* **49(26)**: 3741–3753, <https://doi.org/10.1016/j.ijsolstr.2012.08.008>.
- Virtanen P, Gommers R, Oliphant TE *et al.* (2020) SciPy 1.0: fundamental algorithms for scientific computing in Python. *Nature Methods* **17(3)**: 261–272, <https://doi.org/10.1038/s41592-019-0686-2>.
- Wakefield DS (1999) Engineering analysis of tension structures: theory and practice. *Engineering Structures* **21(8)**: 680–690, [https://doi.org/10.1016/S0141-0296\(98\)00023-6](https://doi.org/10.1016/S0141-0296(98)00023-6).
- Wüchner R and Bletzinger KU (2005) Stress-adapted numerical form finding of pre-stressed surfaces by the updated reference strategy. *International Journal for Numerical Methods in Engineering* **64(2)**: 143–166.

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