CORRESPONDENCE

The Secretary,
The Institution of Civil Engineers.

DEAR SIR,

THE YIELDING OF CLAY

Regular readers of *Géotechnique* will be familiar with the Paper "On the yielding of soils" by Roscoe, Schofield and Wroth (1958)[1] and the recent Paper "A theoretical and experimental study of strains in triaxial compression tests on normally consolidated clays" by Roscoe and Poorooshasb (1963), the typescript of which was made available to the Writer by Mr Roscoe before publication.

In the first Paper the Authors point out (p. 36) that "in all cases when a triaxial test sample contracts . . . the sample increases in strength throughout the test and . . . work harden(s)". The Writer aims at exploring, in a preliminary fashion, the hypothesis that "*normally consolidated clay is a strain-hardening plastic material*" within the established mathematical theory of plasticity (see, for example, Goodier and Hodge (1958), §§ 1 and 2).

As will be shown, the hypothesis appears to be justified by some, but by no means all, of the available experimental evidence. In particular, there is no close agreement in detail with the strain data of Roscoe and Poorooshasb (1963), which is the first available consistent general strain data. This lack of agreement may be a consequence of the patent non-uniformities of deformation in the triaxial test, which have been ignored in the present work. However, in spite of this obvious shortcoming the present note may shed a little light on the subject.

As in the Paper by Roscoe and Poorooshasb, attention is restricted to the "wet" side (as defined, Roscoe *et al* (1958)) of the C.V.R. line.

STRESS AND STRAIN VARIABLES

Following Ref. 1 we use as stress variables p and q, defined by:

$$\left.\begin{aligned} p &= \tfrac{1}{3}\sigma_1' + \tfrac{2}{3}\sigma_3' \\ q &= \sigma_1' - \sigma_3' \end{aligned}\right\} \qquad (1)$$

where σ_1' and $\sigma_3'(=\sigma_2')$ are the principal effective stresses in the conventional triaxial test. Compressive stress is reckoned positive. For incremental strain variables we use:

$$\left.\begin{aligned} \delta\epsilon_p &= -\delta V/V \\ \delta\epsilon_q &= -\delta L/L + \tfrac{1}{3}\delta V/V \end{aligned}\right\} \qquad (2)$$

where V and L represent the current volume and length of the sample (which is assumed to be cylindrical throughout the test), and the operator δ represents a positive increment. Thus a small decrease in volume represents a positive change in ϵ_p, etc. The subscripts p and q suggest the association of the stress and incremental strain vectors in pairs. Superscripts p and e in $\delta\epsilon_p^p$, $\delta\epsilon_p^e$, $\delta\epsilon_q^p$, $\delta\epsilon_q^e$ denote the plastic (irreversible) and elastic (fully reversible) parts of the incremental strains which are assumed, as usual, to be separable. Relations (2) are chosen (uniquely) in such a way that over a plastic strain increment $\delta\epsilon_p^p$, $\delta\epsilon_q^p$ the plastic work done per unit (current) volume is:

$$p\delta\epsilon_p^p + q\delta\epsilon_q^p$$

[1] The references are given on p. 255.

This is a prerequisite for application of the "normality" condition (see Goodier and Hodge (1958), p. 55).

The crucial idea of the present work may be introduced most easily by means of an analogy between the triaxial test and the experiments of Taylor and Quinney (1931) on thin metal tubes under combined axial pull and torque.

First we compare normal consolidation of a clay in equilibrium and its associated elastic recovery (Fig. 1, (a) and (b)) with the extension of the metal tube under a pull P, and elastic

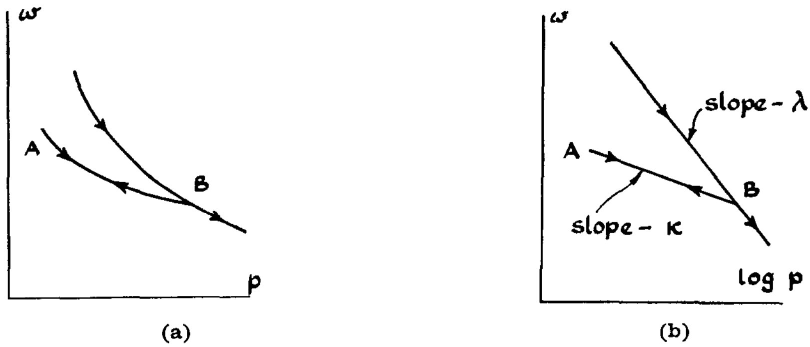

Fig. 1.

unloading (Fig. 2) : x represents the elongation of the tube. Figs 1 and 2 are both schematic in that, as usual, hysteresis effects, etc., have been ignored, and that only one typical unloading has been shown. Figs 1 and 2 represent closely analogous behaviour.

In their celebrated experiments Taylor and Quinney (1931) found that if a tube was loaded by an axial pull P_B to point B, Fig. 2, unloaded to point C and then loaded by a steadily increasing torque Q (the axial pull being held steady at P_c) a "kink" in the Torque/Twist curve was observed at a particular value, say Q_y, of Q. The tests were repeated for different positions of point C, Fig. 2, and it was found that the locus of points P_c, Q_y in P,Q space for which yield occurred for a tube initially loaded by a pull P_B was an elliptical curve, shown as $P_B Q_B$, in Fig. 3. The corresponding locus for a tube originally loaded to a point such as D, Fig. 2, was similar, as indicated by the broken curve in Fig. 3.

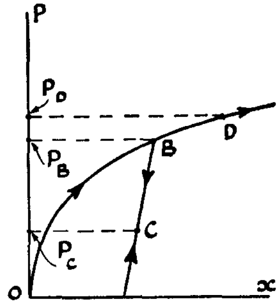

Fig. 2.

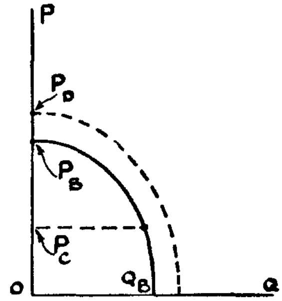

Fig. 3.

Roscoe *et al* (1958) showed that for a saturated clay normally consolidated to a pressure corresponding to a point such as B in Fig. 4 and subsequently loaded in such a way that the material "hardens", the water content w is a unique function of p and q and that the loading path in p,q,w space lies on a curved surface of the type shown in Fig. 4. In accordance with the terminology of Roscoe and Poorooshasb (1963), this surface will be called the "state boundary surface" (and not the "yield surface", as it is called in Roscoe *et al* (1958)).

Let us now consider a set of triaxial tests analogous to the tube tests of Taylor and Quinney. We perform a normal consolidation test to a pressure corresponding to point B, Fig. 4, and unload elastically to a pressure corresponding to point H. Then, keeping $p(=\frac{1}{3}\sigma_1' + \frac{2}{3}\sigma_3')$ constant, we increase the value of q steadily. It is reasonable to suppose that the load-point follows a path normal to the Owp plane until it reaches the state boundary surface, after which it follows a path on the surface, the direction of this part of the path depending on the conditions of the test—i.e. whether the test is undrained, fully drained, etc. In other words, we are making the hypothesis that *the projection of BF on to the Opq plane (Fig. 4) is the yield curve in p, q space for the non-linear elastic material formed by consolidation to B and subsequent unloading.*

Tests similar to the one just described have in fact been made (undrained tests of overconsolidated clay): the result relevant to the present work is shown in Fig. 4 of Roscoe *et al.* (1959) for overconsolidation ratio 1·7, and it does not contradict the above hypothesis if the word "kink" does not imply an abrupt change in direction. (Fig. 1 (b) is, of course, an idealization of the actual experimental data.) Now in Fig. 4 curve BE represents the intersection of the state boundary surface with a plane w = constant; that is, it is the p, q trajectory for an undrained test. (It is known, of course, that curves of intersection of the surface with all planes w = constant are similar (see Fig. 9 of Roscoe *et al* (1958) and Fig. 9a of Roscoe and Poorooshasb (1963)). Thus, in view of the facts (*a*) that experimentally determined curves BE are available and (*b*) that the curves in Fig. 1 (b) are straight, and that all elastic recovery curves are parallel, it is possible to construct geometrically the projection of BF on to plane Opq from a typical curve BE. A construction is shown in Fig. 5. G is any point on BE and H is the foot of the perpendicular from G to Op. J, the point on BF corresponding to G, lies on OG in such a position that:

$$\frac{OG}{OJ} = \left(\frac{OB}{OH}\right)^{\alpha}$$

where $\alpha = \kappa/(\lambda - \kappa)$

λ and κ being respectively the slopes of the lines in Fig. 1(b). In Fig. 5 curve BE is taken from Fig. 9(a), Roscoe and Poorooshasb (1963).

THE NORMALITY CONDITION

If curve BF in Fig. 5 is a yield curve within the theory of plasticity, the *normality* condition for plastic strain increments will hold. In terms of the present variables the normality condition states that if a plastic strain increment takes place at a load combination represented by a point such as R (Fig. 5) on the (current) yield curve, the incremental strain vector $\delta\epsilon_p^p$, $\delta\epsilon_q^p$, when plotted on axes which are parallel to the p,q axes respectively and have their origin at R, is normal to the yield curve at R. Thus, for example, in terms of the experiments of Taylor and Quinney, the normality condition would require the curves in Fig. 3 to intersect both axes normally because (*a*) in plastic extension when $Q = 0$ there is no plastic twist and (*b*) in plastic twisting when $P = 0$ there is no plastic extension of the tube.

In the triaxial test we have two almost similar conditions:

(*a*) when $q = 0$, $\delta\epsilon_q^p = 0$, because by symmetry (assuming as usual that the material is isotropic) $\delta L/L = (1/3)(\delta V/V)$ under equal all-round pressure (see equations (2)).

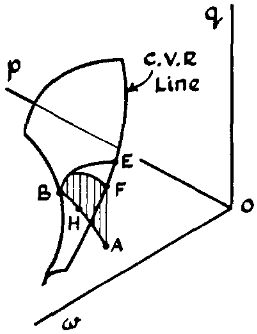

Fig. 4.

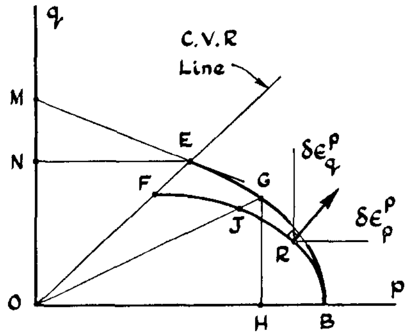

Fig. 5.

So, if the normality condition is to be satisfied in normal consolidation, BF must intersect Op normally at B. This is tantamount to requiring BE to be normal to Op at B. This is so for Thurairajah's data on Kaolin (Roscoe and Poorooshasb (1963), Fig. 9a) which is recent test data carefully obtained.

(*b*) At points on the C.V.R. line, the sample may undergo large axial shortening with no change in volume. Assuming that $\delta\epsilon_p^e = 0$ in such a change, $\delta\epsilon_p^p = 0$. Therefore, if the normality condition is to be satisfied at the C.V.R. line, yield curve BF must have a tangent parallel to Op at F. This appears to be so in Fig. 5. In terms of curve BE, the construction described above is tantamount to the requirement that $MN/NO = \alpha$, where EM is the tangent to curve BE at E, EN is parallel to Op and M and N lie on Oq. From Thurairajah's data on Kaolin, $MN/NO = 0{\cdot}43$ (averaged for the four curves) while $\kappa/(\lambda - \kappa) = 0{\cdot}4$ (the value of κ having been supplied kindly by Mr Roscoe). Thus at the two points on the typical yield curve for which it is well known that $\delta\epsilon_p^p = 0$ and $\delta\epsilon_q^p = 0$ respectively, the hypothesis appears to be confirmed.

It seems reasonable to suppose that at *any* point in p,q space on an arbitrary loading path and at which plastic deformation is taking place, the current yield curve is that curve through the point which is similar to BF. This is strongly suggested by the uniqueness of the surface in Fig. 4 for "hardening" loading paths, and is confirmed qualitatively (although not, it appears, quantitatively) by the experimental evidence in Roscoe and Poorooshasb (1963) that for loading paths with constant q/p ("anisotropic consolidation tests") the strain increment ratio $\delta\epsilon_q/\delta\epsilon_p$ does not change as the test proceeds.

During an undrained test (BE, Fig. 4) the load trajectory passes through a sequence of similar yield curves of increasing size as shown in Fig. 6. It seems then that the material hardens "isotropically", i.e. that the current yield curves are all similar.

It is pertinent to investigate the form of hardening; that is, the criterion by which the yield curve expands as the test proceeds. The "hardening criterion" is in fact a very simple one. Assuming our "yield curve" hypothesis to be correct, all points on the curved surface ABF represent states of material which can be reached from each other without incurring a plastic strain change $\delta\epsilon_p^p$. In other words, just as curves like BE represent paths on the state boundary surface for which $\delta\epsilon_p^p + \delta\epsilon_p^e = 0$, so curves like BF represent paths on the state boundary surface for which $\delta\epsilon_p^p = 0$. It follows that in passing from a point on curve BF to any point on a larger similar curve on the state boundary surface, a unique change in $\delta\epsilon_p^p$ takes place. Thus, for example, in passing from point A to point B, Fig. 6, the change $\delta\epsilon_p^p$ is the same as the change in $\delta\epsilon_p^p$ in passing (in a normal consolidation test) from point C

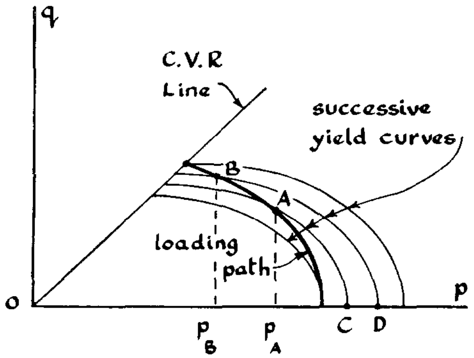

Fig. 6.

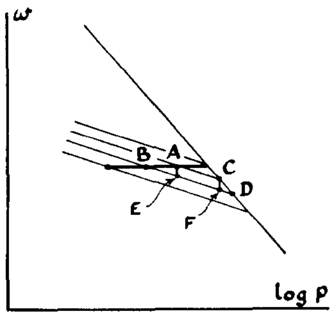

Fig. 7.

to point D. In this case there is a negative *elastic* change of volume ($\delta\epsilon_p^p + \delta\epsilon_p^e = 0$ in an undrained test) as the load point changes from A to B, which is reflected in a decrease in the value of p from p_A to p_B. It is instructive to follow the same path on the w, $\log p$ plane, as shown in Fig. 7. In passing from A to B (the letters correspond to those of Fig. 6) a plastic volume change represented by AE (which is equal to CF, see above) is cancelled by an elastic expansion represented by EB.

Thus it appears from the limited experimental evidence quoted that normally consolidated clay may be regarded as an "isotropic hardening" strain hardening plastic material for which the hardening depends on the plastic changes in volume.

COMMENTS

(i) In general the ideas presented in this Paper support those advanced by Roscoe and Poorooshasb (1963), who deduced from consistent incremental strain data a set of plastic equipotential curves in p,q space. The important experimental result in this Paper that strains in drained tests may be predicted accurately by superimposing strain increments corresponding to appropriate stress increments in undrained tests and anisotropic consolidation tests respectively, is strong evidence for the existence of expanding similar yield curves. The fact that the plastic equipotentials shown in the Paper differ significantly from the yield curves derived in this letter (although of course there are overall similarities) may not prove to be a major obstacle to future conceptual advances.

(ii) If the yield curves are nearly elliptical, as they appear to be, they may be expressed analytically by simple expressions in p and q. This suggests that a simple analytical approximation to the plastic stress/strain law may be established. As the material is clearly of the isotropic hardening sort, the stress/strain law will be a specialization of equation 6·3, p. 72 in Goodier and Hodge (1958).

Let f be a homogeneous function of degree 1 in p and q such that $f = k$ is the equation of the yield curve passing through point $p = k$ on the p axis. Using a superposed dot to denote differentiation with respect to a (dummy) time variable, and incorporating the hardening law described above:

$$\dot{\epsilon}_p^p = (\lambda - \kappa)\frac{\dot{f}}{f}$$

$$\dot{\epsilon}_q^p = (\lambda - \kappa)\frac{\dot{f}(\partial f/\partial q)}{f(\partial f/\partial p)}$$

where $\dot{f} \geqslant 0$

The Writer is particularly indebted to Dr A. N. Schofield for several discussions on the relevance of the theory of plasticity, and in particular the normality condition, to soil mechanics.

Yours faithfully,

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8 March, 1963.

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The Secretary,
The Institution of Civil Engineers.

DEAR SIR,

Mr Lo has presented a most interesting Paper on "Shear strength properties of a sample of volcanic material of the Valley of Mexico" (*Géotechnique*, 12 : 4 : 303–318).

To obtain the Hvorslev parameters for the undisturbed clay samples, Lo uses the $(\sigma_1 - \sigma_3)/2$ and $(\sigma_1' + \sigma_2')/2$ (at failure) curves. This procedure does not illustrate the effect of the scatter in test results, on the possible range in values of ϕ_e and c_e.

The data given in the Author's Tables 1 and 2 is shown plotted in Fig. 1. Mean values of ϕ_e and c_e may be obtained from this plot. The scatter of Lo's results is large; the following lines may be drawn for the undisturbed samples:

Table 1

Line	ϕ_e	k	Description
A	40°	0·057	Mean line for all samples
B	53°	0·023	Mean lines overconsolidated samples with $330 > w_i > 320\%$
C	28°	0·158	Mean line for normally consolidated samples

The equivalent preconsolidation pressure was taken as the vertical principal consolidation stress. Samples I-1, I-11 and I-8, consolidated in the laboratory along the recompression curve, were not plotted. It is evident from Fig. 1 that taking a mean value of 40° for ϕ_e with the available data is a rather bold step.

Comparison of lines *B* and *C* in Fig. 1 suggests that the structural breakdown that takes place in Mexico City clay, when the preconsolidation pressure is exceeded, may change the value of the true friction angle. It is well known that once the preconsolidation pressure is exceeded for this clay, a large variation in properties takes place. This effect is illustrated by