A METHOD OF ANALYSIS OF THE STABILITY OF EMBANKMENTS ASSUMING PARALLEL INTER-SLICE FORCES

(Spencer, E., *Géotechnique* **17**, No. 1, 11–26)

I accept most of the first paragraph of Newland's letter (1968), except the reference to the fulfilment of the moment equilibrium condition in Bishop's (1955) rigorous method. In my opinion the moment condition of equilibrium is satisfied in both rigorous and simplified versions of Bishop's method. The evidence on which this opinion is based is described below.

Bishop's equation (4) equates the moments about the centre of rotation of the weight of the soil within the slip circle and the moment of the external forces which act on the sliding surface

$$\sum Wx = \sum slR$$

Clearly this is an equation of moment equilibrium.

By substituting in this equation an expression for s which includes P (the total force normal to the base of a typical slice) and then substituting for P, Bishop derives equation (13). Since this equation follows directly from (4), equation (13) must also be an equation of moment equilibrium.

Equation (13) must be satisfied in both of Bishop's methods though in the simplified version the assumption is made that the sum of the vertical components of the pair of inter-slice forces which act on each slice is zero, i.e. the resultant Q of each pair of inter-slice forces is horizontal. With this restriction, which applies only in the simplified method, it seems reasonable to claim that the moment equilibrium conditions are satisfied in both of Bishop's methods.

Further evidence as to the validity of this claim seems to have been misunderstood by Newland. My Paper included an expression (5) for the resultant Q for a typical slice and also expressions (7) and (8) which were respectively equations of force and moment equilibrium for the earth mass within the slip circle:

$$\sum Q = 0$$

$$\sum [Q \cos(\alpha - \theta)] = 0$$

In these expressions α and θ are respectively the slopes relative to the horizontal of the base of the slice and of Q. If the value of θ is taken as zero in these two equations then it can be shown that equation (8S)[1] becomes identical to Bishop's equation (13). A proof of this is given in the Appendix.

A statement regarding the identity of these two equations was made in my Paper but it was not made as clear as it might have been to which of Bishop's equations the statement referred. Unfortunately, Newland has presumed that the reference was to equation (18B) instead of (13B). However, the identity of the moment equilibrium equation (8S) with equation (13B) when $\theta = 0$ seems to be clear evidence that Bishop's equation (13) is also a moment equilibrium equation.

Results were given in my Paper of the solutions of a large number of embankment stability problems. Each problem was solved, first by finding values of F and θ which satisfied equations (7S) and (8S) and second by taking θ as zero and finding the value of F which satisfied equation (8S). The latter value, as Newland has remarked, was taken to be the value which would be obtained using Bishop's simplified method. The reason why this assumption was made was not simply, as Newland has stated, that the equilibrium condi-

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[1] Equation numbers with suffixes B, N and S refer to corresponding equations in Bishop's paper, Newland's letter and my Paper, respectively.

tions were identical, but that (as shown in the Appendix) the two equations (8S) and (13B) become identical when $\theta=0$. It is therefore rather surprising that Newland should have found a discrepancy, amounting to about $1\frac{1}{2}\%$, between the values of F given for a particular problem, first by Bishop's simplified method and second by mine (with $\theta=0$).

It must, however, be pointed out that one of the two equations used by Newland appears to be incorrect. His equation (2) was apparently derived from equations (5S) and (8S) taking $\theta=0$. If this was in fact the source of his equation (2), then the term $WF \sin \alpha$ has been omitted from the numerator of the left-hand side of his equation (2) (see Appendix).

Further, Newland used only five slices in his analyses. If this was the same in both solutions and all the other values that he used were equal, then it is unlikely that any discrepancy would have arisen on this account. However, he has compared his results with the value of 1·039 I obtained for this problem but, as was stated in the Paper, this value was obtained using 32 slices. It is therefore a little surprising to find that he obtained exactly the same answer; one would expect to find a discrepancy of at least $1\frac{1}{2}\%$ when comparing a solution obtained with five slices with one obtained using 32.

Newland has also compared a value of F of 1·018 (for $\theta=0$) quoted in the Paper with a value of 1·09 obtained by reference to charts published by Bishop and Morgenstern (1960). He takes the discrepancy between these two values as '. . . further numerical proof that Bishop's simplified method does not satisfy the equilibrium condition with respect to moments'. The justification for this statement seems doubtful to me. In comparing results obtained from these charts with those obtained using my method (with $\theta=0$), excellent agreement has been obtained in many cases. A more likely reason for the discrepancy found by Newland would seem to lie in the fact that, in order to apply the charts to the solution of this problem (in which the slope was $1\frac{1}{2}$:1), he had to extend the curves in the charts into a range beyond the maximum slope of 2:1 for which they were intended.

APPENDIX

Taking $\theta=0$ in equation (5S)

$$Q \cos \alpha = \frac{\frac{c'b}{F} \sec \alpha + \frac{\tan \phi'}{F} (W \cos \alpha - ub \sec \alpha) - W \sin \alpha}{1 + \frac{\tan \phi'}{F} \tan \alpha}$$

From equation (8S), if $\theta=0$, $\sum[Q \cos \alpha]=0$. Hence

$$\sum \left[\frac{c'b \sec \alpha + \tan \phi'(W \cos \alpha - ub \sec \alpha) - WF \sin \alpha}{F + \tan \phi' \tan \alpha}\right] = 0$$

If $u=r_u \gamma h$ then $ub=r_u(\gamma hb)=r_u W$. Therefore

$$\sum \left[\frac{c'b \sec \alpha + W \tan \phi'(\cos \alpha - r_u \sec \alpha) - WF \sin \alpha}{F + \tan \phi' \tan \alpha}\right] = 0$$

If this equation is compared with Newland's equation (2) the omission of the factor $WF \sin \alpha$ from the latter is apparent. Continuing

$$\sum \left\{\left[\frac{c'b}{F} + W \frac{\tan \phi'}{F} (\cos^2 \alpha - r_u) - W \sin \alpha \cos \alpha\right] \frac{\sec \alpha}{1 + \frac{\tan \phi'}{F} \tan \alpha}\right\} = 0$$

$$\sum \left\{\left[\frac{c'b}{F} + W \frac{\tan \phi'}{F} (1 - \sin^2 \alpha - r_u) - W \sin \alpha \cos \alpha\right] \frac{\sec \alpha}{1 + \frac{\tan \phi'}{F} \tan \alpha}\right\} = 0$$

$$\sum \left\{\left[\frac{c'b}{F} + W \frac{\tan \phi'}{F} (1 - r_u) - W \left(\sin \alpha \cos \alpha + \frac{\tan \phi'}{F} \sin^2 \alpha\right)\right] \frac{\sec \alpha}{1 + \frac{\tan \phi'}{F} \tan \alpha}\right\} = 0$$

$$\sum \left\{\left[\frac{c'b}{F} + W \frac{\tan \phi'}{F} (1 - r_u) - W \sin \alpha \cos \alpha \left(1 + \frac{\tan \phi'}{F} \tan \alpha\right)\right] \frac{\sec \alpha}{1 + \frac{\tan \phi'}{F} \tan \alpha}\right\} = 0$$

$$\sum\left\{\left[\frac{c'b}{F}+W\frac{\tan\phi'}{F}(1-r_u)\right]\frac{\sec\alpha}{1+\frac{\tan\phi'}{F}\tan\alpha}\right\}-\sum[W\sin\alpha]=0$$

$$\frac{1}{F}\sum\left\{[c'b+W\tan\phi'(1-r_u)]\frac{\sec\alpha}{1+\frac{\tan\phi'}{F}\tan\alpha}\right\}=\sum[W\sin\alpha]$$

$$F=\frac{1}{\sum[W\sin\alpha]}\sum\left\{\left[\frac{c'b}{F}+W\tan\phi'(1-r_u)\right]\frac{\sec\alpha}{1+\frac{\tan\phi'}{F}\tan\alpha}\right\}$$

This equation is identical to Bishop's equation (13) if in the latter $\bar{B}=r_u$ and $(X_n-X_{n+1})=0$ (i.e. $\theta=0$).

REFERENCES

Bishop, A. W. (1955). The use of the slip circle in the stability analysis of slopes. *Géotechnique* **5**, No. 1, 7–17.

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The Author proposes a two-dimensional limiting equilibrium method of slope stability analysis employing a circular failure surface and parallel inter-slice forces. Then he reports (Spencer, 1968) development of a method of analysis similar to that of Morgenstern and Price (1965) in which a distribution is assumed for the variation in slope of the resultants of the inter-slice forces. He suggests that the most promising type of distribution for the inclination of inter-slice forces is parabolic. I believe that parallel inter-slice forces give better results in certain classes of practical problems.

The inclination of inter-slice forces depends on a number of factors including slope geometry and soil zones, failure surface geometry, soil properties and pore water pressures within the slope. The most important of these factors seems to be pore water pressures within the slope. My recent work with the Morgenstern–Price method of analysis indicates the following:

(*a*) Parallel inter-slice forces generally give the most reasonable results when pore pressures on the failure surface are quite low, i.e. average $r_u < 0{\cdot}15$, and a phreatic surface corresponding to steady state seepage, or the set of r_u corresponding to this condition, is used in the stability analysis rather than a constant value of r_u.

(*b*) A parabolic or half sine wave distribution of the inclination of inter-slice forces generally gives the most reasonable results when pore pressures on the failure surface are constant and quite high, i.e. $r_u > 0{\cdot}35$, and the slope, or at least a significant part of it, is submerged, e.g. the upstream slope of an earth dam under partial or full reservoir conditions.

(*c*) The most suitable inclination distribution for other pore pressure conditions must in general be found by trial and error; parallel inter-slice forces are recommended as a first trial.