trenches in the field. The additional references, not available at the time of writing the Paper, are most welcome.

Table 2A. Distance between pipe and trench over which short term risk of breakage or unserviceability is unacceptable

Soil type	Conditions as specified in Appendix 1 of the Paper[1]	x_2 increased to $(10+18)$ α mm^2	x_2 increased to $(10+18)$ α mm^2 x_3 reduced to $3\frac{3}{4}$ for surface restraint
Soft clay	Max short term strain at trench face=0·58 of breaking strain (the same)	2·3 m (1·9)	1·0 m (1·0)
Firm clay	2·5 m (1·9)	5·5 m (3·5)	4·3 m (2·9)
Stiff clay	6·5 m (2·7)	10 m (4·4)	8·5 m (3·9)
Very stiff clay	8·5 m (3·1)	12·5 m (4·7)	11 m (4·3)
Very stiff clay, x_2 $x\frac{1}{2}$	8·5 m (2·1)	12·5 m (3·2)	11 m (2·6)

[1] From worked example of Appendix 1 of the Paper existing pipe external diameter $D=0{\cdot}286$ m spun iron. Depth of existing pipe 2 m. Trench in light construction country road. Depth of trench $H=5$ m. Length of trench $L=20$ m. Excavation deeper than pipe. Movement x_2 allow 3 mm^2 to fill voids.
[2] Value halved for very stiff clay.

The calculus of variations applied to stability of slopes

REVILLA, J. & CASTILLO, E. (1977). *Géotechnique* **27**, No. 1, 1–11.

S. Barussaud, Laboratoire de l'Est Parisien, 93350 Le Bourget, France

The variational method presented by the Authors appears to be a powerful tool seldom applied to soil mechanics. The writer had used it for the determination of slip surfaces in cohesive soils, in a somewhat different way.

Considering horizontal forces F acting between vertical slices of soil, and noting σh the vertical stress on the slip surface $y\ (x)$, the following equation is obtained:

$$F = \int_{x_0}^{x_1} \sigma h y' \, \mathrm{d}x - c \int_{x_0}^{x_1} y'^2 - c(x_o - x_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Among all the possible slip surfaces $y\ (x)$ the writer considered the one which gives the maximum value of F at the point x, this value being equal to zero for a safety factor of 1.

Using Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that gives

$$\frac{\partial \sigma h}{\partial x} - 2cy'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we suppose no shear stress between vertical slices,

$$\sigma h = \gamma(f - y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

And the equation (3) becomes the equation (13) given by the Authors.

The writer applied this method to various problems: for instance behind a retaining wall the value of F represents the horizontal force needed by the stability of the wall and classical values of Rankine's earth pressure theory are easily obtained.

The writer applied it also to the slope stability analysis and obtained results identical to those of the Authors. He also noted that N tends to the value of 0·25 for an infinitely deep slip surface since the influence of the parabolic portion of the slip surface becomes negligible compared to the influence of the straight parts of it. This result comes from the assumption $\sigma_h = \gamma(f-y)$ at a certain depth, the distribution of σh is more uniform than that given by the equation (4). The slip surface so obtained is too 'sharp' and a small displacement will produce shear stresses among the sliding mass, increasing the safety factor.

It could be said that the Taylor's slip circle method gives an upper limit of S, since the slip surface fits the requirements of kinematics but does not necessarily represent the most critical surface, while the variational method gives a lower limit since it does not fit the requirements of kinematics. Applied to shallow foundations, the variational method gives results concerning the safety factor as well as considerations on the shape of the slip surface.

However, most interesting results are obtained in cases where the traditional methods are inadequate. For instance it can be shown that at the boundary of two different cohesive soils, the curvature of the slip surface changes in the ratio of the cohesions. Moreover, equation (2) can be solved (by a finite differences method for instance) accounting for any distribution of $\sigma y(x, y)$.

Although the application of the variational method to soils in which $\phi \neq 0$ appears to be difficult, it is doubtless that general criteria ($\partial\sigma h/\partial x - 2cy'' = 0$ for cohesive soils) will lead to a simplification of all problems involving slip surfaces such as slope stability, retaining walls or footing foundations problems.

J. L. Justo, University of Seville, Spain

The Authors should be congratulated, because their method looks really promising. I should like to comment upon the differences between Taylor's and the Authors' results. Slides in natural deposits of homogeneous and isotropic clay occur through quasi circular surfaces, and the same happens in centrifuge tests (Salas, Justo and Serrano, 1976).

The exact formula for Janbu's method, using the authors' terminology is

$$S = \frac{\int_{x_0}^{x_1} c(1+y'^2)\,dx}{\int_{x_0}^{x_1} \gamma[y-f(x)]y'\,dx + \int_{x_0}^{x_1} T'(x)y'\,dx} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

It is well known that

$$\int_{x_0}^{x_1} T'(x)\,dx = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Where $T(x)$ = shearing force between slices.

But

$$\int_{x_0}^{x_1} T'(x)y'\,dx \neq 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

because y' is not a constant.

The Authors employ Janbu's simplified method, which assumes that:

$$\int_{x_0}^{x_1} T'(x)y'\,dx = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

This method is acceptable when the slide is shallow, because then, the variation of y' is small. On the other hand, when the sliding surface is deep the term

$$\int_{x_0}^{x_1} T'(x)y'\,\mathrm{d}x$$

becomes important and is not negligible at all. Under these circumstances, Janbu's method gives values of the factor of safety lower than more exact methods (Salas, Justo and Serrano, 1976; Escario, 1966).

Now, if we look at the Authors' Fig. 7, we see that the resulting sliding surfaces are very deep for low values of α. Under these circumstances Janbu's simplified method is rather inexact.

We may add that Taylor's results are in good agreement with the results of centrifugal tests (Mikasa and Takada, 1973).

These comments are not related with the variational method which looks, as stated earlier, promising. What we have tried to show is that the difference with Taylor's method is due to the errors of Janbu's simplified method, and not to errors related to the use of circular surfaces in Taylor's approach.

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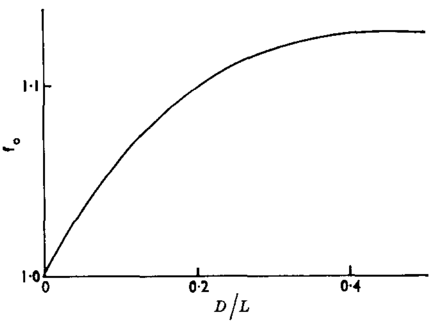

Fig. 1. f_0 factors for $\phi = 0$ soils

P. Friedli and M. Giger, Geotechnik, Zurich

The Authors apply the calculus of variations to the problem of locating the critical slip line in a cohesive soil. The stability analysis is performed according to Janbu's method of slices.

The ground surface is defined by a set of functions. The expression for the factor of safety according to Janbu's method contains the functions of the slip line. By solving Euler's equation the minimum factor of safety and the critical slip line are found. The method presented by the Authors applies to cohesive soils only. It would be interesting to have the method include c/ϕ soils.

The comparison with Taylor's 1948 solution shows a rather limited agreement in the range

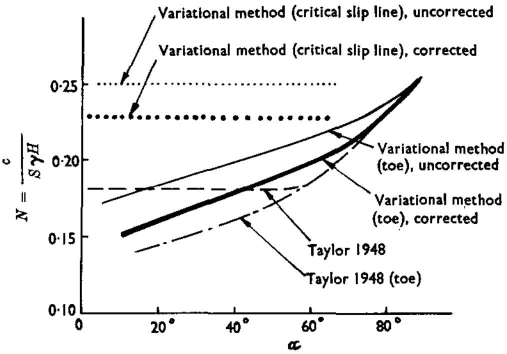

Fig. 2. Variations of stability number N with slope angle α

of flat slopes. The agreement is better for steep slopes. We suggest to apply Janbu's factor f_0 expressing the influence of the T-forces which is especially important with long slip lines (Fig. 1). Figure 2 is the Author's Fig. 9 corrected according to the above procedure. The agreement with Taylor's solution is at least 50% better.

M. Garber, Technion–Israel Institute of Technology, Haifa

THE CRITICAL NOTES

The Paper deals with the application of variational methods for evaluation of the slope stability. The Authors do not refer to any work dealing with this subject. However, several works exist dealing with it. Moreover, these works are more general as well as more fundamental than the one under discussion. Some of these works (Dorfman, 1965, 1970; Garber, 1972, 1973a) are published in Russian journals, but the remaining (Biernatowski, 1976; Chen and Shitbham, 1975; Garber, 1973b; Kopacsy, 1957, 1961) are published in Western journals.

The Authors indicate that the safety factor usually can be expressed as a functional, which is a quotient of two integrals, their equation (1). The first part of the article deals with the investigation of the functional of this type. As a result, the Authors obtain the system of equations which are necessary conditions for an extremum. In this connection, it must be noted, that this system is a partial case of the system of equations derived by Garber (1973b) for a functional of more general type, than that which had been studied by the authors.

In the second part of the Paper, several problems are solved. The Authors use Janbu's stability presentation for frictionless soil. The functional to be minimized is given by their equation (12). This functional, actually, is a partial case of the functional K (Garber, 1972, 1973a, b), which follows from the stability presentation by Solov'ev (1962).

$$K = \frac{\int_{x_0}^{x_n} [\gamma h(x)\psi + c(1+y'^2)]\,\mathrm{d}x}{\int_{x_0}^{x_n} \gamma h(x) y'\,\mathrm{d}x} \quad \text{(D1)}$$

where

x_0, x_n end points of the critical line,
ϕ angle of internal friction,
ψ $\tan\phi$,
$h(x)$ $f(x)-y(x)$.

On the basis of the functional K the problem of the plane slope was solved by Garber (1972, 1973, a, b) for all the cases ($c \neq 0$, $\phi \neq 0$), ($c \neq 0$, $\phi = 0$), ($c = 0$, $\phi \neq 0$). Thus, the solution for ($c \neq 0$, $\phi = 0$) obtained by the Authors had already been obtained earlier.

To acquaint the Western reader with the full variational solution of the plane slope the summary of the results is given below.

VARIATIONAL SOLUTION OF THE PLANE SLOPE STABILITY (BASED ON SOLOV'EV'S PRESENTATION)

Solov'ev (1962) in his presentation starts from the following assumptions:

(a) At the first instant of failure, the horizontal component of displacement is identical at all points of the failing section;

(b) The energy dissipated for displacements along the vertical planes can be ignored in comparison with the energy dissipated for displacements along with the failure line.

The principle of virtual displacements is applied to the failure section. As a result, the functional K, is defined, equation (D1). The fulfilment of the virtual displacement principle is equivalent to the fulfilment of all equilibrium equations. Therefore the expression, equation (D1), completely describes the limit equilibrium.

K is the safety functional; $\hat{t} = \min K$ is the safety factor. The variational solution of the plane slope leads to the following results.

In the case ($c \neq 0$, $\phi \neq 0$, $\alpha < 90°$) the critical line always passes below the toe (Fig. 1). The stability equation, connecting all the problem parameters, is written as

$$\bar{N}^3(1-m\nu)^2 - 48\bar{N}(1+\nu^2) + 128 = 0 \qquad \text{(D2)}$$

where

$\bar{N} = 1/N = H\gamma\hat{t}/c$ stability coefficient,
$\nu = (\tan\phi)/\hat{t}$ friction coefficient,
$\hat{t} = \min K$ factor of safety,
α angle of the slope inclination,
$m = \cot\alpha$.

In the domain where the solutions to equation (D1) are to be sought is restricted by inequalities

$$\begin{cases} \bar{N}(1-m\nu) - 4(1+\nu) > 0 \\ 1 - m\nu > 0 \end{cases} \qquad \text{(D3)}$$

This system of inequalities follows from physical and geometrical considerations, which characterize the present problem. It can be analytically proved that the solution of equation (D2) in the domain, equation (D3), exists and is unique for any stability problem (direct or reverse). We define a 'direct' problem as one of the determination of the safety factor for a given slope. We define a 'reverse' problem as one of the determination of the critical value of some physical or geometrical parameter for a given value of safety factor.

In the case ($c \neq 0$, $\phi \neq 0$, $\alpha = 90°$) the critical line passes through the toe (Fig. 2). The stability equation is written as

$$9\bar{N}^3\nu - 64\bar{N}^2\nu^2 + 48\bar{N}(1-4\nu) - 192 = 0 \qquad \text{(D4)}$$

In this case the only conditions which must be satisfied are $\bar{N} > 0$, $\nu > 0$. The solution of equation (D3) always exists and is unique.

In the case ($c \neq 0$, $\phi = 0$, $\alpha \leq 90°$) the critical line passes through the toe. The stability equation is written as

$$\bar{N} = 3(\sqrt{m^2 + 2m/3 + 1} + m - 1)/m \qquad \text{(D5)}$$

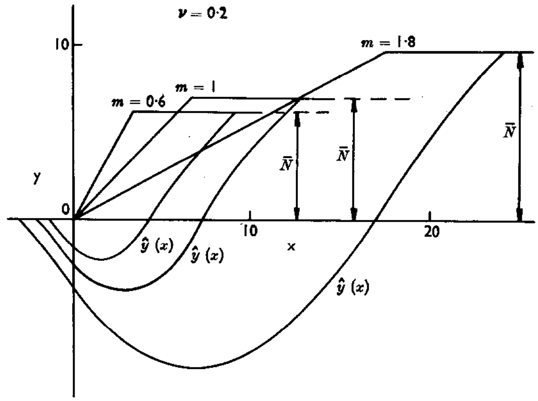

Fig. 1. Contours of the equally stable slopes and the critical lines corresponding to them

$X = x\gamma\hat{t}/c,\ Y = y\gamma\hat{t}/c$	**non-dimensional coordinates**
$Y(X)$	**critical line**
$\bar{N} = H\gamma\hat{t}/c,\ \nu = (\tan\phi)/\hat{t}$	$m = \cot\alpha$
$\hat{t}$	**safety factor**

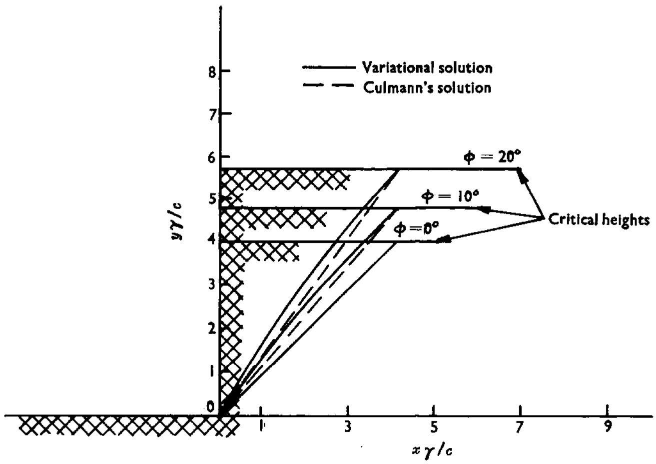

Fig. 2. Vertical slope. Comparison with the solution by Culmann (1966)

The stability coefficient $\bar{N}$ is a montonic function of m. The limits of $\bar{N}$ are

$$\lim_{\substack{m\to\infty \\ \alpha\to 0}} \bar{N} = 6 \qquad \lim_{\substack{m\to 0 \\ \alpha\to 90^\circ}} \bar{N} = 4 \qquad \text{(D6)}$$

The Authors of the Paper obtained the solution, equation (D5). However, from Fig. 9 of the Paper it follows that the Authors conclude that the solution of the problem is $N\equiv 1/4$. This solution corresponds to the deep line (a critical line, all points of which, are removed to infinity). However, an infinite volume of a homogeneous and isotropic soil does not exist in nature. Therefore a deep line is not real and we reject this solution.

In the case ($c=0$, $\phi\neq 0$) the critical curve is the inclined part of the slope surface. The stability equation is written as

$$m\nu = 1 \qquad (H\text{—arbitrary}) \qquad \text{(D7)}$$

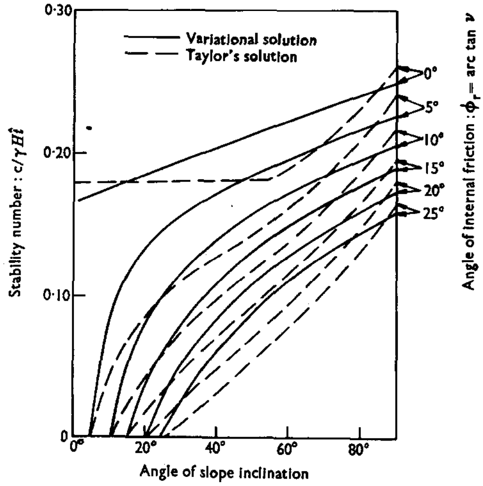

Fig. 3. Stability chart for a plane slope. Comparison with the solution by Taylor (1948). $\nu = (\tan\phi)/\hat{f}$ **is the friction coefficient**

From equation (D7) it follows that for the angles $\alpha \leq \phi$ a slope of any height is stable $-(\hat{f} \geq 1)$. The slope failure is expressed in the sliding of the inclined part of the surface. Figures 1 to 3 illustrate the presented variational solution.

EFFICIENCY OF THE VARIATIONAL METHODS

The preliminary condition for the application of the variational methods is the possibility to solve analytically Euler's equation, equation (2) of the Paper. This possibility depends on the character of functions entering in the safety functional. Therefore, the declaration of the Authors, that the variational methods hold for any slope surface and for anisotropic soil, is an exaggeration. However, in the cases when the variational methods can be used these methods have the obvious priority. The undoubted advantage of the variational methods is their efficiency for solving the reverse problems (problems of design). It must be noted that the variational methods permit to obtain not only the critical line, but also the critical stress distribution along the critical line without any preliminary assumptions (Baker and Garber, 1977).

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Authors' reply

Mr Barussaud indicates that at the boundary of two different cohesive soils, the curvature of the slip surface changes in the ratio of the cohesions. This, in our opinion, is not correct, at least with the model we have selected. On the boundary of two different cohesive soils it is the first derivative of the slip surface which changes in relation with the ratio of the cohesions.

We agree completely with Dr Justo regarding the fact that the differences with Taylor's method are due to the errors in Janbu's simplified method. In fact we are working at the present time at the University of Santander in applying the calculus of variations technique to the well known methods of Morgenstern and Price and Spencer, which satisfy all the equilibrium equations.

We want to indicate to Dr M. Garber that the Paper was submitted for possible publication in *Géotechnique* in May 1975, so some of the papers he refers to could not be consulted. On the other hand the rest of the papers related with the calculus of variations are not easily available to Western people.

Through Dr Garber's discussion we see that part of the work we have presented was similar to his work, with the only difference being the model. With our model, in the case ($c \neq 0$, $\phi \neq 0$, $\alpha < 90°$), the critical line does not always pass below the toe of the slope.

Dr Garber indicates that the preliminary condition for the application of the variational method is one possibility to solve analytically Euler's equation and expresses his doubts about the possibility of application of the variational methods for any slope surface and for anisotropic soils. In our opinion this is not correct because a numerical technique can be used to solve Euler's equation. We refer Dr Garber to the references, in which the case of stratified soils has been analysed by variational methods and where the numerical technique was shown to be very useful.

P. Friedli and M. Giger suggest applying Janbu's factor f_0 for taking into account the influence of T-forces and they show the changes produced in Fig. 9 by this correction. We thank him for this and certainly agree with the opinion that the differences for flat slopes are due to the simplicity of the selected model.

Nevertheless the purpose of the Paper was to show the power of the calculus of variations technique and Janbu's method was selected as an example of possible application of this technique. The application of the method to $c/0$ soils can be seen in the references.

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