STRESSES IN DOWNSTREAM PART OF AN EARTH OR A ROCK FILL DAM

by

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INTRODUCTION

It is a well-known fact that, referred to a system of co-ordinates (x, y), the components of a two-dimensional stress tensor, σ_x, τ_{xy}, σ_y, can be derived from a stress function $F(x, y)$, as follows:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad \ldots \quad (1)$$

Stresses which can be derived in this way, and only these, will satisfy the fundamental equation of equilibrium for the elements of a slice having an even thickness and no weight, namely :—

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad \ldots \quad (2)$$

In order to take gravity into account we have to superimpose upon the stress tensor given by equation (1) another tensor satisfying the equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = c\alpha, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = c\beta \quad \ldots \quad (3)$$

where α and β are the direction cosines for the plumb line and c is the weight per unit of area of the slice. In particular, if gravity acts in the direction of the y-axis, we have $\alpha = 0$, $\beta = 1$.

Since the derivatives of any function $F(x, y)$ will satisfy equation (2), obviously no function will have derivatives according to equations (1), which will satisfy equations (3).

We shall now consider possible particular solutions of the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = c \quad \ldots \quad (4)$$

We can obviously obtain such particular solutions by putting either

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \sigma_y = c\{y + f_1(x)\} \quad \ldots \quad (5)$$

or

$$\sigma_x = 0, \quad \tau_{xy} = c\{x + C\}, \quad \sigma_y = 0 \quad \ldots \quad (6)$$

or

$$\sigma_x = f_2(y), \quad \tau_{xy} = \kappa c\{x + C\}, \quad \sigma_y = (1 - \kappa)c\{y + f_1(x)\} \quad \ldots \quad (7)$$

or some more complicated expressions.

TRIANGULAR DAMS

Development of fundamental formulas

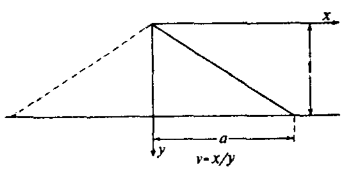

Fig. 124

Let us consider an earth dam or a rock fill dam with a triangular shape, a height equal to unity and a base width of $2a$, as shown in Fig. 124, and let the weight per unit of volume be unity. Further, let us refer the elements of the dam to a system of co-ordinates x, y, as shown in the same figure.

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In such a triangular dam all the stresses must vanish for $y = \pm\frac{x}{a}$. If, in order to avoid confusion, we consider only the downstream part of the dam enclosed by the positive x-axis and the positive y-axis, it will serve our purpose to utilize the possibility given by equations (5) and use as a particular solution :—

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \sigma_y = y - \frac{x}{a} \qquad (8)$$

Let us assume that the stresses are in direct proportion to the radius vector through the origin. To obtain a general solution on this assumption we put

$$F(x, y) = y^3\Phi\left(\frac{x}{y}\right) \qquad (9)$$

Let us substitute

$$v = \frac{x}{y} \qquad (10)$$

and in the following keep y and v as principal variables. We observe that

$$\frac{\partial}{\partial x}f(v) = \frac{1}{y}f'(v), \quad \frac{\partial}{\partial y}f(v) = -\frac{v}{y}f'(v) \qquad (11a)$$

$$\frac{\partial}{\partial x^2}f(v) = \frac{1}{y^2}f''(v), \quad \frac{\partial^2}{\partial y^2}f(v) = \frac{2v}{y^2}f'(v) + \frac{v^2}{y^2}f''(v) \qquad (11b)$$

and

$$\int_{ay}^{x} f(v)dx = y\int_{a}^{v} f(z)dz, \quad \int_{\frac{x}{a}}^{y} f(v)dy = -vy\int_{a}^{v} \frac{1}{z^2}f(z)dz \qquad (12)$$

We can now write

$$F(x, y) = y^3\Phi(v) \qquad (13)$$

and we get, when applying equations (11)

$$\frac{\partial F}{\partial x} = y^2\Phi' \qquad (14a)$$

$$\frac{\partial F}{\partial y} = 3y^2\Phi - y^2v\Phi' \qquad (14b)$$

We thus get the components of the stress tensor

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = y\{6\Phi - 4v\Phi' + v^2\Phi''\} \qquad (15a)$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = y\{-2\Phi' + v\Phi''\} \qquad (15b)$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} + y - \frac{x}{a} = y\left\{\Phi'' + \frac{a - v}{a}\right\} \qquad (15c)$$

and we find that all stresses are in direct proportion to the radius vector through the origin as assumed above.

If we calculate the second derivatives of these stresses in respect to x, we get

$$\frac{\partial^2 \sigma_x}{\partial x^2} = \frac{v^2}{y}\Phi'''', \quad \frac{\partial^2 \tau_{xy}}{\partial x^2} = \frac{v}{y}\Phi'''', \quad \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{1}{y}\Phi'''' \qquad (16)$$

Independent of the value of the function Φ, we thus get

$$\frac{\partial^2\sigma_x}{\partial x^2} = v\frac{\partial^2\tau_{xy}}{\partial x^2} = v^2\frac{\partial^2\sigma_y}{\partial x^2} \quad \ldots \ldots \ldots \quad (17)$$

This equation is of fundamental importance for the practical solving of stress functions under given conditions for triangular dams.

In consideration of equations (15) we can write

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = y\ \xi(v) \quad \ldots \ldots \ldots \quad (18a)$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = y\psi(v) \quad \ldots \ldots \ldots \quad (18b)$$

$$\sigma_y - y\frac{a-v}{a} = \frac{\partial^2 F}{\partial x^2} = y\chi(v) \quad \ldots \ldots \ldots \quad (18c)$$

where ξ, ψ, and χ are functions of the variable v, only. These functions have thus a constant value along any radius vector through the origin.

We shall now endeavour to express any two of these functions as a function of the third. In the first instance we can write

$$6\Phi - 4v\Phi' + v^2\Phi'' = \xi(v) \quad \ldots \ldots \ldots \quad (19)$$

The general solution to this differential equation can be written

$$\Phi = -v^2\int_a^v \left(1 - \frac{v}{z}\right)\frac{1}{z^3}\xi(z)dz + Av^3 + Bv^2 \quad \ldots \ldots \quad (20)$$

From this we get by considering equations (15) and (18)

$$\psi(v) = 2v\int_a^v \frac{1}{z^3}\xi(z)dz + \frac{1}{v}\xi(v) - 2B \quad \ldots \ldots \ldots \quad (21b)$$

and

$$\chi(v) = -2\int_a^v \frac{1}{z^3}\xi(z)dz + 6v\int_a^v \frac{1}{z^4}\xi(z)dz + \frac{1}{v^2}\xi(v) + 6Av + 2B \quad \ldots \quad (21c)$$

If we go over to another system of co-ordinates (x_1, y_1) which is turned over an angle α in the direction from the positive x-axis towards the positive y-axis given by

$$\tan \alpha = \frac{1}{a}$$

it is obvious that in this new system of co-ordinates τ_{x_1, y_1} and σ_{y_1} must vanish for $v = a$. From this we get

$$A = B = 0$$

and hence

$$\xi(v) = \xi(v) \quad \ldots \ldots \ldots \ldots \ldots \quad (22a)$$

$$\psi(v) = 2v\int_a^v \frac{1}{z^3}\xi(z)dz + \frac{1}{v}\xi(v) \quad \ldots \ldots \ldots \quad (22b)$$

$$\chi(v) = 6v\int_a^v \frac{1}{z^4}\xi(z)dz - 2\int_a^v \frac{1}{z^3}\xi(z)dz + \frac{1}{v^2}\xi(v) \quad . \quad . \quad . \quad . \quad . \quad (22c)$$

If instead, we solve the equation

$$-2\Phi' + v\Phi = \psi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and use the expression of Φ obtained as a function of $\psi\ (v)$ we get

$$\xi(v) = -2\int_a^v \psi(z)dz + v\psi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

$$\psi(v) = \psi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24b)$$

$$\chi(v) = 2v\int_a^v \frac{1}{z^3}\psi(z)dz + \frac{1}{v}\psi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24c)$$

We could also have started with equation (18c) and obtained

$$\xi(v) = 2v\int_a^v \chi(z)dz - 6\int_a^v z\,.\,\chi(z)dz + v^2\chi(v) \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

$$\psi(v) = -2\int_a^v \chi(z)dz + v\chi(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25b)$$

$$\chi(v) = \chi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25c)$$

It is thus possible to express any two of the components of the stress tensor as a function of the third.

We shall now consider some important relations. For calculating the derivatives of our functions we use formulas (25). We have

$$\xi'(v) = 2\int_a^v \chi(z)dz - 2\,v\chi(v) + v^2\chi'(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

$$\psi'(v) = -\chi(v) + v\chi'(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26b)$$

$$\chi'(v) = \chi'(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26c)$$

$$\xi''(v) = v^2\chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27a)$$

$$\psi''(v) = v\chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27b)$$

$$\chi''(v) = \chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27c)$$

In the first instance we verify the relation already given by equation (17)

$$\xi''(v) = v\psi''(v) = v^2\chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

We further have

$$\chi(v) + \psi'(v) = v\chi'(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and hence for $v = 0$

$$\psi'(0) = -\chi(0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and for $v = a$

$$\xi'(a) = -2a\chi(a) + a^2\chi'(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31a)$$

$$\psi'(a) = -\chi(a) + a\chi'(a) \quad \ldots \quad (31b)$$

$$\chi'(a) = \chi'(a) \quad \ldots \quad (31c)$$

At $v = a$, that is at the extreme end of the toe, all the stresses must vanish in the case of an earth or a rock fill dam. We thus have $\chi(a) = 0$, and consequently

$$\xi'(a) = a\psi'(a) = a^2\chi'(a) \quad \ldots \quad (32)$$

Further, if there is no shear along the plane $x = 0$, we get

$$\int_0^a \chi(v)dv = 0, \qquad \int_0^a v\chi(v)dv = \frac{1}{6}\xi(0) \quad \ldots \quad (33a)$$

and also

$$\psi(0) = \psi''(0) = \xi'(0) = \xi''(0) = 0 \quad \ldots \quad (33b)$$

Should there be shear along the plane $x = 0$, we get instead

$$\int_0^a \chi(v)dv = +\frac{1}{2}\psi(0), \qquad \int_0^a v\chi(v)dv = \frac{1}{6}\xi(0) \quad \ldots \quad (34a)$$

$$\psi''(0) = \xi''(0) = 0 \quad \ldots \quad (34b)$$

It should be observed that it is not necessary, and not even usual, that the stress function and/or its derivatives are continuous functions. Aristotle's sentence *Natura non facit saltum* does not hold good in this case either.

We get the following rules directly by considering the stress functions and their derivatives. At a point $v = m$, where the values of the function and their derivatives are marked by index 1 at $v = m - 0$ and by index 2 at $v = m + 0$, we have

$$m^2(\chi_2 - \chi_1) = m(\psi_2 - \psi_1) = \xi_2 - \xi_1 \quad \ldots \quad (35a)$$

$$\psi'_2 - \psi'_1 = -(\chi_2 - \chi_1) + m(\chi'_2 - \chi'_1) \quad \ldots \quad (35b)$$

$$\xi'_2 - \xi'_1 = -2m(\chi_2 - \chi_1) + m^2(\chi'_2 - \chi'_1) \quad \ldots \quad (35c)$$

As will be seen later on, the relations given above will make it possible to solve certain problems by employing the method of " trial and error ".

Practical Applications

(*a*) Hooke's Law

It is obvious that the formulas deducted in the foregoing contain all possible solutions of stress distribution in triangular dams, where the stresses are in proportion to the radius vector through the apex. In order to obtain a definite solution we must also take into account some physical requirements characteristic of the practical problem which we want to solve.

Let us first consider the stresses which would occur in such a dam body, if Hooke's law should hold good. In such a case the stress function $F(x, y)$ would satisfy Airy's equation

$$\frac{\partial^4 F}{\partial y^4} + 2\frac{\partial^4 F}{\partial x \partial y} + \frac{\partial^4 F}{\partial y^4} = 0 \quad \ldots \quad (36)$$

Let us estimate the above derivatives. We have, using equations (18) and (28)

$$\frac{\partial^4 F}{\partial y^4} = \frac{v^2}{y}\xi''(v) = \frac{v^4}{y}\chi''(v) \quad \ldots \quad (37a)$$

$$\frac{\partial^4 F}{\partial x^2 \partial y^2} = \frac{v}{y}\psi''(v) = \frac{v^2}{y}\chi''(v) \quad \ldots \quad (37b)$$

$$\frac{\partial^4 F}{\partial x^4} = \frac{1}{y}\chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37c)$$

Airy's equation will thus be

$$\frac{1}{y}\chi''(v) + 2\frac{v^2}{y}\chi''(v) + \frac{v^4}{y}\chi''(v) = 0$$

or

$$(v^2 + 1)^2\chi''(v) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Obviously, this equation has one solution only, viz.

$$\chi''(v) = 0$$

From this follows, according to equation (28),

$$\xi''(v) = 0, \quad \psi''(v) = 0$$

The components of the stresses are thus given as straight lines within the whole interval $0 \leq v \leq a$.

If there are no shear stresses at the front (e.g. in the case of a concrete gravity dam) equations (33) are valid, and our function must vary as a straight line between the following points

$$v = 0 \ldots \chi(0) = -e, \quad v = a \ldots \chi(a) = +e \quad . \quad . \quad . \quad . \quad (39)$$

where e is a constant to be determined.

Hence

$$\chi'(v) = 2\frac{e}{a}, \quad \chi(v) = -e + 2\frac{e}{a}v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40a)$$

From equations (25) we now get

$$\xi(v) = ea^2 \quad \text{and} \quad \psi(v) = ev \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40b)$$

Considering equations (18) we thus get

$$\sigma_x = yea^2, \quad \tau_{xy} = yev \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41a)$$

$$\sigma_y = ye\frac{2v - a}{a} + y\frac{a - v}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41b)$$

If the load σ_x is given, we get e directly from the first of the above equations. The stress system given above is quite interesting. It is obviously valid for the top portions of triangular concrete gravity dams and of triangular buttresses in buttress dams, since in such top portions the influence of the conditions at foundation level must be insignificant.

The formulas arrived at show, however, that such stresses as are compatible with Airy's stress function do not exist in an earth or a rock fill dam as a whole, since for such dams all stresses must, of necessity, vanish for $v = a$.

From this we can draw the conclusion that a triangular downstream part of an earth or a rock fill dam suffering even the slightest superimposed horizontal loading cannot, theoretically, under any conditions be subject to an elastic deformation only.

Since, however, such a triangular downstream part can usually suffer a considerable horizontal loading, it follows that some part of it must yield plastically. We will, therefore, in the first instance, have to study the conditions for plastic deformation.

(*b*) Plastic Deformation

It is generally accepted, at least as a working hypothesis, that in friction material plastic deformation occurs when the ratio k between the minor principal stress σ_1 and the major principal stress σ_2 decreases to a certain value k_0 characteristic of the material in question.

When this occurs, a sliding will take place in planes where the ratio between the shearing stress τ, and the compressive stress σ_n, normal to the plane, has its maximum value. This maximum value is denoted by tan ϕ, where ϕ is the " angle of internal friction."

This leads to the following equation between k_0 and ϕ

$$\tan^2\left(45^\circ - \frac{\phi}{2}\right) = k_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and also determines the direction of the sliding planes so that they enclose an angle of $45^\circ - \phi/2$ with the major principal stress.

For determining the relation between the principal stresses, when the stresses σ_x, τ_{xy}, σ_y are given, many methods can be used. We can, for instance, estimate the principal stresses by the usual formula

$$\sigma_1, \sigma_2 = \frac{1}{2}\left\{\sigma_x + \sigma_y \mp \sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and thus get the relation k.

Since, in our case, we are not concerned about the actual value of σ_1 and σ_2 but only their relations, we may first calculate

$$\sigma_x < \sigma_y \; ; \; \sigma_x/\sigma_y = \delta \; ; \; \tau_{xy}/\sigma_y = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and we thus obtain

$$\sigma_1, \sigma_2 = \frac{1}{2}\sigma_y\left\{\delta + 1 \mp \sqrt{(1 - \delta)^2 + 4\theta^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

We can also depart from the invariants

$$I_1 = \sigma_x + \sigma_y, \quad I_2 = \sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (46a)$$

and using the relations (45a)

$$I_1 = \sigma_y(1 + \delta), \quad I_2 = \sigma_y^2(1 + \delta^2 + 2\theta^2) \quad . \quad . \quad . \quad . \quad (46b)$$

and further, by definition,

$$I_1 = (k + 1)\sigma_2, \quad I_2 = (k^2 + 1)\sigma_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (46c)$$

obtain

$$\frac{I_1^2}{I_2} = \frac{(k + 1)^2}{k^2 + 1} = \frac{(1 + \delta)^2}{1 + \delta^2 + 2\theta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Resolving this equation in respect of θ we get

$$\theta^2 = \frac{\delta(k^2 + 1) - k(\delta^2 + 1)}{(k + 1)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

As will be seen, this equation represents for every given k an ellipse in respect of θ and δ. In Fig. 125 is shown a chart of the mutual values of θ and δ for different values of k. The practical procedure is thus to estimate θ and δ by dividing the given values of σ_x, τ_{xy}, σ_y by either σ_x or σ_y, whichever is the largest, so as to obtain θ and δ. Thereafter, entering these values into the chart, Fig. 125, k is obtained. For convenience the inclination tan α of the major principal stress and the inclination tan γ of the least inclined sliding plane are also given on the same chart.

From the above it may be gathered that the generally accepted theory pre-supposes a considerable change in the stress tensor before a plastic deformation occurs. This may be correct as far as large (unlimited) plastic deformations are concerned. It has, however, been shown above that an earth or a rock fill dam cannot suffer any superimposed horizontal loading without yielding partly plastically. Since it cannot be imagined that even a small loading

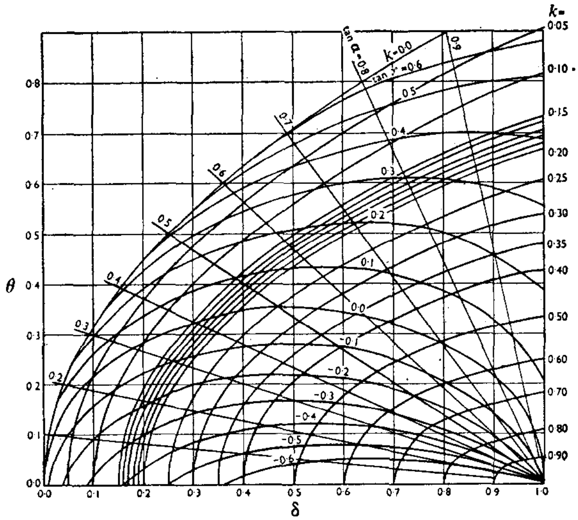

Fig. 125

would cause such a change in the stress tensor as would bring about the relation of the minor principal stress to the major principal stress to reach k_0, there must be some other physical conditions for the occurrence of small (limited) plastical yielding. This question will be reverted to later on.

(*c*) Natural conditions

We know from experience that in a dam of sand, just heaped up, or only slightly compacted, the vertical pressure has a more or less parabolic shape, and that the horizontal pressure in the middle part of the dam is more or less 0·45 times the vertical pressure.

Let us assume that

$$\sigma_y = cy(a^n - v^n) \quad \ldots \ldots \ldots \quad (49)$$

Since

$$\int_0^a \sigma_y dv = \frac{1}{2}\gamma a, \quad c = \frac{n+1}{2n}a^{-n} \quad \ldots \ldots \ldots \quad (50)$$

By derivating so as to obtain $\partial^2\sigma_y/\partial v^2$, multiplying by v and v^2, respectively, and then integrating, we get

$$\psi(v) = \frac{n-1}{2n}a\left\{\frac{v}{a} - \left(\frac{v}{a}\right)^{n+1}\right\} \quad \ldots \ldots \quad (51a)$$

$$\xi(v) = \frac{n-1}{2(n+2)}a^2\left\{1 - \left(\frac{v}{a}\right)^{n+2}\right\} \quad \ldots \ldots \quad (51b)$$

We now have the equation

$$\frac{\sigma_x(0)}{\sigma_y(0)} = \frac{n(n-1)}{(n+2)(n+1)}a^2 = 0{\cdot}45 \quad \ldots \ldots \quad (52)$$

for determining the value of n. If we take a at 1·5 we will get $n = 2{\cdot}225$.

Evaluating the functions given by equations (49) and (51) we get the diagram shown in Fig. 126, where the value of k is also shown. In view of the successively decreasing value of k from $v = 0$ towards the toe, where it attains the value of $k = 0{\cdot}235$, which is valid for slightly compacted sand, it appears that the diagram of stresses obtained is reasonably correct.

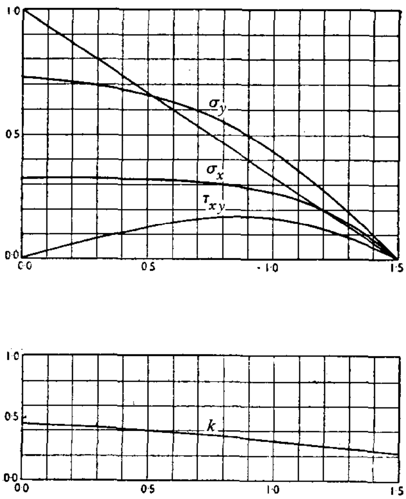

Fig. 126

(*d*) Dam loaded to failure

In the natural state the stress conditions in a dam are approximately as shown in Fig. 126, the k-value being fairly large over the whole base of the dam and decreasing towards the toe. If such a dam is provided with a watertight flexible diaphragm in the middle, i.e. at $v = 0$, and subjected to a gradually increasing hydrostatic pressure, it will have to yield plastically somewhere. Unfortunately, a purely mathematical analysis does not appear to indicate where such a primary yielding will take place, and model tests will have to be made to indicate what actually will happen. It is, however, possible to determine the ultimate stress system when the dam is yielding as a whole. It will be demonstrated in the following that it is possible to obtain a k-value equal to k_0 at the same time in every element of a triangular dam. It is taken that this state represents a loading to failure.

Let us assume that the ultimate load for a dam built of a material having $k_0 = 0{\cdot}18$ is given by $\xi(0) = 1{\cdot}10$. In the interval $0 \leq v \leq 0{\cdot}56$ we will assume $\sigma_y/y = \chi(v) + \frac{a-v}{a}$ to be a polygon where the breaking points satisfy the condition $k = 0{\cdot}18$. Such a polygon would be as follows :

Table 1

$v = 0$	0·1	0·2	0·3	0·4	0·5	0·55	0·56
$\sigma_y/y = 0{\cdot}198$	0·212	0·233	0·283	0·354	0·483	0·610	0·644
$\psi(v) = 0$	0·080	0·161	0·248	0·341	0·457	0·546	0·569
$\xi(v) = 1{\cdot}100$	1·100	1·100	1·101	1·104	1·117	1·139	1·146

The values of the area $F = \frac{1}{y}\int_0^v \sigma_y dv$ and of the moment $M = \frac{1}{y}\int_0^v v\sigma_y dv$ are as follows :

Table 2

v	F	M
0·50	0·1423	0·04108
0·55	0·1696	0·05543
0·56	0·1759	0·05891

At the toe the condition $k = 0{\cdot}18$ will be satisfied by the following system of stresses

$$\chi(v) + \frac{a-v}{a} = 1{\cdot}285(a-v) \qquad (53a)$$

$$\psi(v) = 0{\cdot}927(a-v) \qquad (53b)$$

$$\xi(v) = 1{\cdot}390(a-v) \qquad (53c)$$

If we estimate the area $F(a) - F = \frac{1}{y}\int_v^a \sigma_y dv$ and the moment $M(a) - M = \frac{1}{y}\int_v^a v\sigma_y dv$, we will get

Table 3

v	$F(a)-F$	$M(a)-M$
0·50	0·6425	0·535
0·55	0·5799	0·503
0·56	0·5675	0·495

Comparing Tables 2 and 3 we find that, if we let our function jump at $v = 0{\cdot}55$ from the values given by the polygon, Table 1, to those given by equations (53), the value of the area enclosed between σ_y/y and the base line becomes practically correct (0·7494 as compared with 0·75). We will therefore have to let the function jump in this way.

We shall now check our functions at the point of discontinuity $m = 0{\cdot}55$

$$\xi_1 = 1{\cdot}139 \qquad \psi_1 = 0{\cdot}546 \qquad \chi_1 + \frac{a-m}{a} = 0{\cdot}610$$

$$\xi_2 = 1{\cdot}320 \qquad \psi_2 = 0{\cdot}881 \qquad \chi_2 + \frac{a-m}{a} = 1{\cdot}220$$

$$\xi_2 - \xi_1 = 0{\cdot}181 \qquad \psi_2 - \psi_1 = 0{\cdot}335 \qquad \chi_2 - \chi_1 = 0{\cdot}610$$

We find that $0{\cdot}55(\chi_2 - \chi_1) = 0{\cdot}335 = \psi_2 - \psi_1$.

On the other hand $0{\cdot}55\ (\psi_2 - \psi_1) = 0{\cdot}184$.

We should therefore add 0·003 to the ξ-values given in Table 1. This agrees with the M-value which indicates that $\xi(0)$ should be slightly higher than 1·100.

We shall now consider the angles of turning. We have at $m = 0{\cdot}55$

$$\xi'_1 = 0{\cdot}432 \qquad \psi'_1 = 1{\cdot}779 \qquad \chi'_2 - \frac{1}{a} = 2{\cdot}540$$

$$\xi'_2 = -1{\cdot}390 \qquad \psi'_2 = 0{\cdot}927 \qquad \chi'_2 - \frac{1}{a} = -1{\cdot}285$$

$$\xi'_2 - \xi'_1 = -1{\cdot}822 \qquad \psi'_2 - \psi'_1 = -2{\cdot}706 \qquad \chi'_2 - \chi'_1 = -3{\cdot}82$$

$$+0{\cdot}670 \qquad +0{\cdot}610$$

$$-1{\cdot}152 \qquad -2{\cdot}096 \qquad -3{\cdot}82$$

We see that after addition of $(\chi_2 - \chi_1)$ and $2m\{\chi_2 - \chi_1\}$ we get for ψ the value of $-0{\cdot}55\,.\,3{\cdot}82 = -2{\cdot}10$ and for ξ the value $-0{\cdot}55^2\,.\,3{\cdot}82 = -1{\cdot}155$, which shows a good agreement.

The result is shown graphically in Fig. 127, where the values are also given of

(a) The ratio between shear stress and vertical stress at the base plane . . . . τ_{xy}/σ_y

(b) The slope of the maximum principal stress . . . . $\tan\alpha$

(c) The slope of the less inclined sliding plane . . . . $\tan\gamma$

(d) The possible sliding plane along which the relation between the shear stress and the normal stress is equal to $\tan\phi$.

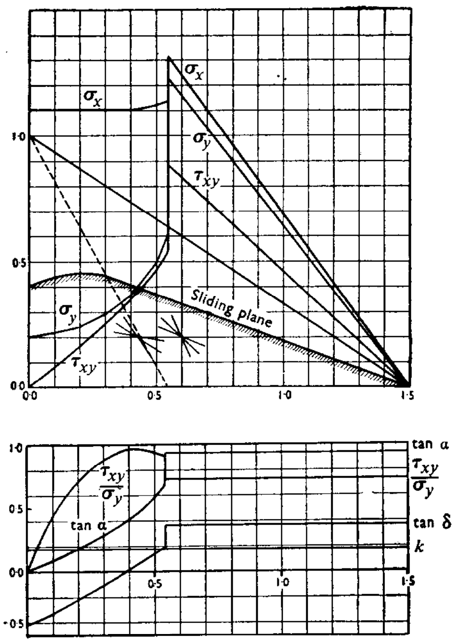

Fig. 127

As may be gathered from Fig. 127, the dam body will not, in the first instance, slide on its base. Instead the upper part will slide out.

It can also be gathered from Fig. 127, that the introduction of a thin layer of sand having a higher k_0-value (up to $k_0 = 0{\cdot}26$) under the downstream part of the dam would not affect the stability, since the coefficient of friction utilized along the downstream part of the base is only 0·72.

The introduction of such a layer under the upstream part of the dam would, however, decrease the stability somewhat, in fact by about 4%, as will be demonstrated below.

(*e*) Dam on sand layer having $k_0 = 0{\cdot}26$, $\tan \phi = 0{\cdot}72$

Let us assume that under the upstream half of the dam there is a sand layer having a coefficient of friction of 0·72. We assume that the dam body is built of material having $k_0 = 0{\cdot}18$ (corresponding to $\tan \phi = 0{\cdot}97$) and that $\xi(0) = 1{\cdot}056$. At $v = 0$ we then get $\sigma_y = 1{\cdot}056 \,.\, 0{\cdot}18 = 0{\cdot}190$. According to formula (30) we have $\psi' = 0{\cdot}81$. If we start with one leg of a polygon at $\sigma_y = 0{\cdot}190$, at $v = 0$, to $\sigma_y = 0{\cdot}225$, at $v = 0{\cdot}2$, we obtain

v	$\xi(v)$	$\psi(v)$	$\chi(v) + \frac{a-v}{a}$	k	τ_{xy}/σ_y
0	1·056	0	0·190	0·18	0
0·2	1·056	0·162	0·225	0·18	0·72

If we project these lines to $v = 0{\cdot}7$, we get

$$v = 0{\cdot}7 \qquad \xi = 1{\cdot}056 \qquad \psi = 0{\cdot}567 \qquad \chi + \frac{a-v}{a} = 0{\cdot}312$$

In order to obtain the required relation between τ_{xy} and σ_y, we have to turn the lines at $v = 0{\cdot}2$ in proportion to 1 to $1/0{\cdot}2$ to $1/0{\cdot}2^2$. If we add

$$0{\cdot}027 \qquad 0{\cdot}132 \qquad 0{\cdot}658$$

we get instead

$$v = 0{\cdot}7 \quad \xi = 1{\cdot}083 \quad \psi = 0{\cdot}699 \quad \chi' + \frac{a-v}{a} = 0{\cdot}970 \quad \tau_{xy}/\sigma_y = 0{\cdot}72$$

If we then project this line to $v = 0{\cdot}72$, we get

$$v = 0{\cdot}72 \quad \xi = 1{\cdot}084 \quad \psi = 0{\cdot}720 \quad \chi + \frac{a-v}{v} = 1{\cdot}000$$

From the toe end we must have the following functions to obtain $k_0 = 0{\cdot}18$.

$$\xi(v) = 1{\cdot}390(a - v)$$
$$\psi(v) = 0{\cdot}927(a - v)$$

$$\chi(v) + \frac{a - v}{a} = 1{\cdot}285(a - v)$$

As will be seen these functions satisfy the relation $\tau_{xy}/\sigma_y = 0{\cdot}72$.

At the value $v = 0{\cdot}72$ we have, for these functions,

$$\xi = 1{\cdot}084 \quad \psi = 0{\cdot}722 \quad \chi + \frac{a - v}{a} = 1{\cdot}000$$

As will be seen, the agreement is good.

The resistance of this dam is therefore $1{\cdot}056/1{\cdot}103 = 0{\cdot}957$, or 4·3 per cent lower than that of the same dam founded on rock.

The diagram of stresses is shown in Fig. 128.

(*f*) Discussion of results

Let us in the first instance consider the stress system shown in Fig. 127. If we draw the radius vector $v = 0{\cdot}55$ shown by a dotted line in Fig. 127, and turn the system of co-ordinates through an angle β, so that the new x_1-axis will coincide with the direction of this radius vector, we will get τ_{x1y1} and σ_{y1} equal on both sides in the immediate neighbourhood of this radius vector. On the other hand, σ_{x_1} is greater on the downstream side than on the upstream side of the dividing line. The inclination of the major principal stress is greater on the downstream than on the upstream side, as shown by the small stars in Fig. 127. The steeper sliding planes on the upstream side have a slightly smaller inclination than the radius vector, and the sliding planes on the downstream side a slightly greater inclination than the radius vector. This is also shown by the small stars referred to above.

When the fill passes from the stress conditions given by Fig. 126 to those given by Fig. 127, it should be observed that at $v = 0$, the σ_y-stress will decrease, and the σ_x-stress increase. Therefore, the distance between the bottom and the top of the dam will increase. This should result in a displacement upwards of the top portion of the dam nearest to the impermeable diaphragm.

As mentioned above, the σ_{x1}-stress parallel to the radius vector drawn, is greater on the downstream than on the upstream side. The fill on the downstream side will therefore have to slide downwards with reference to the upstream part of the dam. Since neither the sliding plane in the upstream part, nor that in the downstream part, is parallel to the dividing line, such a sliding will have to take place in a zig-zag line using alternately the steeper and the more gradual sliding planes. Sliding along the less inclined sliding planes will also make it possible for the fill to obtain the necessary displacement in a downstream direction of the apex of the triangular downstream part of the fill.

Let us now consider the stress system shown in Fig. 128. In this case we can distinguish three parts separated by the radius vectors $v = 0{\cdot}2$ and $v = 0{\cdot}72$. In the following we will refer to these parts as the upstream, the middle and the downstream part. From the stress diagram we realize that in the whole of the upstream and of the downstream parts $k = k_0$, and that thus these parts can deform plastically by sliding along their sliding planes. On the other hand $k > k_0$ in the middle part (except at its boundaries). Therefore, this part can (theoretically) suffer an elastic deformation only. In fact, the stress system superimposed upon the stress system given in Fig. 126 so as to result in the stress system shown in Fig. 128 can, taken *cum grano salis*, be considered as a straight line system, and thus satisfy Airy's equation (36).

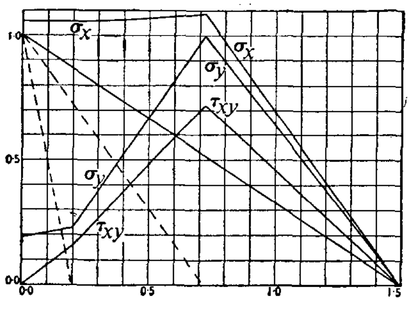

Fig. 128

The upstream and the downstream parts will therefore have to deform plastically so as to accommodate themselves to the displaced boundaries of the middle part deforming elastically.

As mentioned previously, it unfortunately does not seem possible (at least not for the Author) to follow by a pure mathematical analysis the gradual transition from the stress system indicated in Fig. 126 to those given by Figs 127 and 128. It is to be hoped that model tests will indicate how such stress system as those shown in Figs 127 and 128 will develop out of the natural original stress conditions.

For the sake of curiosity we will also treat two other problems, which appear to lead to results that cannot be interpreted by using the generally accepted theory of plastic deformation.

(*g*) Equal coefficient of friction in horizontal layer

We will examine the result at which we arrive if we formulate the condition that

$$\tau_{xy} = C \,.\, \sigma_y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (54)$$

shall hold good in all parts of the dam. This could be realized in a dam made of resistant earth or rock fill ($k_0 = 0{\cdot}18$) placed on a (thin) layer of sand having less resistance than the mass of the fill.

On the above assumption we have

$$\psi(v) = C\left\{\chi(v) + \frac{a-v}{a}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (55)$$

By derivating we obtain

$$\psi'(v) = C\left\{\chi'(v) - \frac{1}{a}\right\}$$

According to equation (32) we have

$$\chi'(a) = \frac{1}{a}\psi'(a)$$

Hence

$$\psi'(a) = C\frac{1}{a}\psi'(a) - C\frac{1}{a} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (56)$$

or

$$\psi'(a) = -\frac{C}{a-C} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (57)$$

and

$$\chi'(a) - \frac{1}{a} = \frac{C}{a(a-C)} - \frac{1}{a} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (58)$$

and

$$\xi'(a) = -\frac{aC}{a-C} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (59)$$

To demonstrate the condition let us assume

$$a = 1{\cdot}5 \qquad C = 0{\cdot}58$$

Hence

$$\xi'(a) = -0{\cdot}946, \quad \psi'(a) = -0{\cdot}630, \quad \chi'(a) - \frac{1}{a} = -1{\cdot}086 \quad . \; . \; . \quad (60)$$

We shall now consider the stresses at the other end, i.e. at $v = 0$. There we have, since $\sigma_y(0) = 0$, according to equation (30)

$$\psi'(0) = 1$$

and thus

$$\chi'(0) = \frac{1}{0{\cdot}58} = 1{\cdot}725 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (61)$$

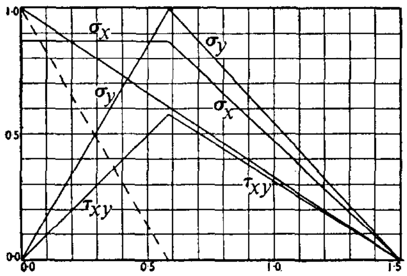

Fig. 129

The straight line for σ_y from $v = a$ reaches unity at $a - v = 1/1{\cdot}086 = 0{\cdot}92$, i.e. at $v = 0{\cdot}58$. The straight line for σ_y from $v = 0$ also reaches unity at $v = 1/1{\cdot}725$, i.e. at $v = 0{\cdot}58$. The diagram for σ_y thus becomes an acute triangle having its apex at $v = 0{\cdot}58$ at the value $\chi = 1$, as shown in Fig. 6. The area of the χ-diagram is thus correct. At $v = 0{\cdot}58$ we get $\psi = 0{\cdot}58$ and $\xi = 0{\cdot}870$. The area of the ψ-diagram is thus $0{\cdot}58 \,.\, 1{\cdot}5/2 = 0{\cdot}435$. This is the total pushing force which is equal to $\xi(0) \,.\, 1/2$. From this we get $\xi(0) = 0{\cdot}870$. Thus the function ξ is a horizontal line from the apex to the x-axis.

We can also check this by considering the turning at the apex. We there have, compare equations (35), $m = 0{\cdot}58$

$$\xi_1 = \xi_2 \qquad \psi_1 = \psi_2 \qquad \chi_2 = \chi_1$$
$$\xi'_1 = 0 \qquad \psi'_1 = 1 \qquad \chi'_1 = 1{\cdot}725$$
$$\xi'_2 = -0{\cdot}946 \qquad \psi'_2 = -0{\cdot}630 \qquad \chi'_2 = -1{\cdot}086$$
$$\xi'_2 - \xi'_1 = -0{\cdot}946 \qquad \psi'_2 - \psi'_1 = -1{\cdot}630 \qquad \chi'_2 - \chi'_1 = -2{\cdot}811$$

Considering the stress system shown in Fig. 129 we find that at $v = 0$ we have $\sigma_y = 0$. This could not be so, if the fill is composed of earth or rock fill. This inadvertence can easily be overcome by dropping the condition (43) for the part $0 < v < 0{\cdot}1$ and substituting for it e.g. $k = 0{\cdot}18$ with reference to the stability conditions of the fill.

If we divide the fill into two parts by the radius vector $v = 0{\cdot}58$, we find that, taken *cum grano salis*, each of these parts could deform elastically, since the stress system superimposed on the " natural " stress system more or less is a straight lined system satisfying Airy's equation (36). Yet, the parts thus deformed will not fit together. Furthermore, in both parts of the fill, the k-value is at all places (barring the sector $0 < v < 0{\cdot}1$) far greater than $k_0 = 0{\cdot}18$, which is supposed to be characteristic of the fill and therefore (theoretically) no plastic deformation should take place !

(*h*) Upstream part of dam on silt layer

Let us further consider a dam of resistant material ($k = 0{\cdot}18$) built upon a thin silt layer extending at the base $y = 1$ from $x = 0$ to $x = 0{\cdot}75$, this silt layer being characterized by $\tan \phi = 0{\cdot}39$.

In the first instance we design a polygon from $v = 0$ onwards, where the breaking points satisfy the above condition. Such a polygon is the following :

Table 4

v	$\xi(v)$	$\psi(v)$	$\chi + \dfrac{a - v}{a}$
0	0·810	0	0·150
0·08	0·810	0·067	0·170
0·2	0·812	0·192	0·492
0·3	0·813	0·296	0·756
0·39	0·815	0·394	1·010

From the downstream end we have the same stress system as before ($k = 0{\cdot}18$).

$$\xi(v) = 1{\cdot}390(a - v)$$

$$\psi(v) = 0{\cdot}927(a - v)$$

$$\chi(v) + \frac{a - v}{a} = 1{\cdot}285(a - v)$$

At $v = 0{\cdot}75 - 0$ we must suddenly obtain the relation $\tau_{xy}/\sigma_y = 0{\cdot}39$. This we can obtain only by means of a very deep jump. We have to put

$$\chi_2 - \chi_1 = 0{\cdot}876 \qquad \chi'_2 - \chi'_1 = 1{\cdot}264$$

Hence

$$\xi_1 = 0{\cdot}533 \qquad \psi_1 = 0{\cdot}033 \qquad \chi_1 + \frac{a - v}{a} = 0{\cdot}084$$

$$\xi'_1 = -0{\cdot}777 \quad \psi'_2 = -0{\cdot}992 \quad \chi'_2 - \frac{1}{a} = -2{\cdot}544$$

For $x = 0{\cdot}39$ we obtain

$$\xi = 0{\cdot}813 \qquad \psi = 0{\cdot}391 \qquad \chi_1 + \frac{a - v}{a} = 1{\cdot}000$$

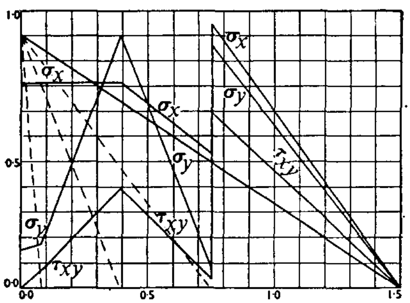

Fig. 130

This tallies sufficiently with the values obtained from the other end as given in Table 4. A diagram showing the stress conditions is given in Fig. 130. The diagram is amazing, yet it is the logical consequence of the assumptions made! It does not seem as if our function "*saltat sobrius*"!

In Fig. 130 we can distinguish four parts divided by the radius vectors $v = 0{\cdot}08$, $v = 0{\cdot}39$ and $v = 0{\cdot}75$. We are up against the same snag here as mentioned with reference to Fig. 129. The second and third parts have each a straight line stress system compatible with Airy's equation, and should therefore deform elastically, yet when deformed in this way they do not fit together.

GENERAL REMARKS

It should be borne in mind that all calculations made are logical consequences of the hypotheses made, namely:

(a) In its deformed state the dam has the shape considered, i.e. a triangle.
(b) The stresses are in proportion to the radius vector.
(c) The k_0-value is a constant.
(d) Sliding will take place along a plane enclosing an angle $45° - \frac{\phi}{2}$ with the major principal stress.

When treating problems of this kind it is generally assumed that the deformations are small in comparison with the linear dimensions of the structure considered. Therefore, it does not matter much whether the stresses are referred to the original shape of the structure or to its deformed shape. In the case of earth fills the deformations can be fairly great and, therefore, strictly speaking, the stress system should be referred to the shape of the fill in its deformed state.

We know that the hypothesis (b) is not exact. At the lower boundary of a fill where it rests on rock or other layers, there will be local stresses. Since, however, the main stress system, where the stresses are in proportion to the radius vector, is in equilibrium with the outer forces, the boundary stress system, which must be superimposed upon the main system to satisfy the conditions of the lower boundary, will have no resultant. The boundary stress system should therefore fade away rather rapidly when moving upwards from the lower boundary. In the top portion of the fill the hypothesis that the stresses are in proportion to the radius vector should therefore be fully valid, and at the bottom part it should be a fairly good approximation.

We also know that the hypothesis (c) is not exact. The k_0-value we know to vary for any material with the " packing." Due to deformations the " packing " of the material will be affected, and this will, in its turn, have an influence upon the k_0-value.

Much more important than the above is, however, the fact that small plastic deformations may take place at k-values far greater than k_0, and also that the sliding planes for such small plastic deformations are not oriented in the way the generally accepted theory given on pages 206 and 207 requires. This theory appears, therefore, to apply to large deformations only, whereas the law for the occurrence of small (limited) deformations is as yet unknown.

If we admit that, besides elastic deformations, small plastic deformations may occur due to small changes in the stress system, the fact mentioned on page 205, namely, that a triangular fill can suffer no horizontal load without a partially plastic deformation, no longer becomes a riddle. Also, the stress diagram shown in Figs 129 and 130 are no longer absurd.

DAMS WITH CREST

Development of fundamental formulas

In the preceding chapters we have dealt exclusively with dams having a triangular shape. It has been demonstrated that, by judicious application of the formulas, it has been possible to solve, in a theoretical way, problems relating to the stresses in such triangular fills subjected to a triangular load. However, dams are generally provided with a crest, and in such dams stresses will, of necessity, differ from those in a triangular dam.

The resistance to pushing forces of a dam provided with a crest should not differ much from the resistance of a triangular dam having the same height, the same base width, and the same total weight. The latter condition can be complied with by assuming an artificially increased weight. Such a rule may be sufficient for a designer of such dams. This is, however, not our present aim, which is to elucidate the stress system that must obtain in such dams with crests.

For tackling this problem let us, as before, refer each element of the dam to a system of co-ordinates (x, y) having its origin at the point of intersection between the slope and the impervious diaphragm. We take the vertical distance from the origin to the bottom plane as unity. Further, let the base width, also as before, be a, and let the crest be at $y = b$.

Even in this case we should prefer to assume that the stresses are in proportion to the radius vector from the origin, less that part of this vector which is above the crest. On this assumption we should write

$$\sigma_x = (y - b)\xi(v) \qquad (62)$$

$$\tau_{xy} = (y - b)\psi(v) \qquad (63)$$

$$\sigma_y = (y - b)\chi(v) + y\frac{a - v}{a} + \left[y\frac{v}{a} - b\right] \qquad (64)$$

where the expression within brackets, here and in the following, should be considered for the interval $0 \leq x \leq ab$, but neglected for the interval $ab \leq x \leq a$.

It can, however, be demonstrated that no stress function $F(x, y)$ exists that will satisfy the above equations. There is only one stress function that satisfies equation (62) for $v = 0$, namely :

$$F(x, y) = (y - b)^3 \Phi(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Introducing, as before,

$$\chi = \Phi'', \quad \Phi' = \int_a^v \chi(z) dz, \quad \Phi = \int_a^v (v - z) \chi(z) dz$$

we get

$$\sigma_x = 2\frac{(y^3 - b^3)}{y^2} v \int_a^v \chi(z) dz - 6(y - b) \int_a^v z\chi(z) dz + \frac{(y - b)^3}{y^2} v^2 \chi(v) \quad . \quad . \quad (66a)$$

$$\tau_{xy} = -\frac{(y - b)^2}{y^2}(2y + b) \int_a^v \chi(z) dz + \frac{(y - b)^3}{y^2} v \chi(v) \quad . \quad . \quad . \quad . \quad . \quad (66b)$$

$$\sigma_y = \frac{(y - b)^3}{y^2} \chi(v) + y\frac{a - v}{a} + \left[-b + y\frac{v}{a} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66c)$$

and for $y = 1$

$$\sigma'_x = 2(1 - b^3) \int_a^v \chi(z) dz - 2(1 - b)^2 (1 + 2b) v \chi(v) + (1 - b)^3 v^2 \chi'(v) \quad . \quad (67a)$$

$$\tau'_{xy} = -(1 - b)^2 (1 + 2b) \chi(v) + (1 - b)^3 v \chi'(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67b)$$

$$\sigma'_y = (1 - b)^3 \chi'(v) - \frac{1}{a}\left[+ \frac{1}{a} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67c)$$

and

$$\sigma''_x = 6(1 - b) b^2 \chi(v) - 6(1 - b)^2 b v \chi'(v) + (1 - b)^3 v^2 \chi''(v) \quad . \quad . \quad (68a)$$

$$\tau''_{xy} = -3b(1 - b)^2 \chi'(v) + (1 - b)^3 v \chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68b)$$

$$\sigma''_y = (1 - b)^3 \chi''(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68c)$$

From the above equations we realize that in the case of dams with crests there is no parallel to equation (17) or (28) that has proved so helpful for solving problems relating to triangular dams.

From the above equations we get, however, the important relations

$$\sigma'_x(a) = a\tau'_{xy}(a) = a^2\left(\sigma'_y(a) - \frac{y}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and

$$\tau'_{xy}(0) = \left\{ y - b - \sigma_y(0) \right\} \frac{y + 2b}{y - b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Equation (69) is the same as equation (32) valid for triangular dams, and equation (70) substitutes equation (30) for triangular dams.

It is further to be observed that we have similar rules for discontinuities as for triangular dams. In fact, at a point $x = m$, where the values of the functions and their derivatives are marked by index 1 at $x = m - 0$ and with index 2 at $x = m + 0$, the following relations exist for $y = 1$.

$$\sigma_{x_2} - \sigma_{x_1} = m(\tau_{xy_2} - \tau_{xy_1}) = m^2(\sigma_{y_2} - \sigma_{y_1}) \quad \dots\dots \quad (71a)$$

$$\sigma'_{x_2} - \sigma'_{x_1} = -2\frac{1+2b}{1-b}m(\sigma_{y_2} - \sigma_{y_1}) + m^2(\sigma'_{y_2} - \sigma'_{y_1}) \quad \dots \quad (71b)$$

$$\tau'_{xy_2} - \tau'_{xy_1} = -\frac{1+2b}{1-b}(\sigma_{y_2} - \sigma_{y_1}) + m(\sigma'_{y_2} - \sigma'_{y_1}) \quad \dots \quad (71c)$$

Since we cannot build straight line polygons in all three functions, it is found expedient to build straight line polygons in τ. For this purpose we estimate functions of the other variables based upon the assumption

$$y = 1 \dots \tau''_{xy} = 0 \quad \dots\dots \quad (72)$$

We then have

$$-3b\chi' + (1-b)x\chi'' = 0 \quad \dots\dots \quad (73)$$

This equation leads to

$$\chi = c_0 + c_1\frac{1-b}{1+2b}x^{\frac{1+2b}{1-b}} \quad \dots\dots \quad (74)$$

and further to

$$\int_a^x \chi(z)dz = c_0x + c_1\frac{(1-b)^2}{(1+2b)(2+b)}x^{\frac{2+b}{1-b}} + c_2 \quad \dots\dots \quad (75)$$

$$\int_a^x z\chi(z)dz = \frac{1}{2}c_0x^2 + c_1\frac{(1-b)^2}{3(1+2b)}x^{\frac{3}{1-b}} + c_3 \quad \dots\dots \quad (76)$$

Inserting these values into equations (66) we get the system of stresses dependent upon the choice of four independent constants, namely :

$$c_0, \quad c_1, \quad c_2, \quad \text{and} \quad c_3$$

When applying these equations we generally have the following given quantities

$$y = 1 \dots \sigma_x(m) = p, \quad \tau_{xy}(m) = q, \quad \sigma_y(m) = r, \quad \tau'_{xy} = t \quad \dots \quad (77)$$

Determining our constants c_0 to c_3 so as to comply with the above data we get for $y = 1$

$$\sigma_x = p + 2\frac{1-b^3}{(1-b)^2(2+b)}(tm-q)(v-m) - 3\frac{b^2t}{(1-b)(1+2b)}(v^2-m^2)$$
$$-\frac{b}{(2+b)}\frac{1}{m^{\frac{1+2b}{1-b}}}\left\{\frac{1-b}{1+2b}t - \frac{a-m}{a} + \left[-\frac{m}{a}+b\right] + r\right\}\left(v^{\frac{3}{1-b}} - m^{\frac{3}{1-b}}\right) \quad (78a)$$

$$\tau_{xy} = tv - (tm - q) \quad \dots\dots \quad (78b)$$

$$\sigma_y = -\frac{1-b}{1+2b}t + \frac{1}{m^{\frac{1+2b}{1-b}}}\left\{\frac{1-b}{1+2b}t - \frac{a-m}{a} + \left[-\frac{m}{a}+b\right] + r\right\}v^{\frac{1+2b}{1-b}}$$
$$+\frac{a-v}{a} + \left[+\frac{m}{a} - b\right] \quad \dots \quad (78c)$$

Dam loaded to ultimate capacity

(*a*) Downstream part.

In order to obtain $k = 0{\cdot}18$ at the downstream toe we should have for $a = 1{\cdot}5$

$$\sigma_x(a) = \tau_{xy}(a) = \sigma_y(a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79a)$$

$$\sigma'_x(a) = -1{\cdot}390, \quad \tau'_{xy}(a) = -0{\cdot}927, \quad \sigma'_y(a) = -1{\cdot}285 \quad . \quad . \quad (79b)$$

If we use the stress system given by equations (78) and determine the constants so that the conditions given by equations (79) are satisfied and assume

$$b = 0{\cdot}2, \quad a = 1{\cdot}5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

we get for $y = 1$

$$\sigma_x = 2{\cdot}607 - 1{\cdot}960v + 0{\cdot}100v^2 + 0{\cdot}0236v^{\frac{15}{4}} \quad . \quad . \quad . \quad . \quad (81a)$$

$$\tau_{xy} = 1{\cdot}390 - 0{\cdot}927v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81b)$$

$$\sigma_y = 1{\cdot}530 - 0{\cdot}667v - 0{\cdot}260v^{\frac{7}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71c)$$

By evaluating these equations we get:

Table 5

$y = 1$

v	σ_x	τ_{xy}	σ_y	k
1·5	0	0	0	0·180
1·25	0·367	0·232	0·313	0·182
1·00	0·771	0·463	0·603	0·184
0·80	1·113	0·649	0·823	0·184
0·60	1·470	0·834	1·026	0·180
0·50	1·654	0·927	1·126	0·176

As can be seen from the preceding table, the result is satisfactory. It is evident that the tail end of the dam will have the same resistance to pushing forces as the same part of a triangular dam, since in both cases the value of τ_{xy} is the same.

(*b*) Upstream part.

We may suppose that our crest dam at failure will hold a total thrust that is somewhat greater than that which can be suffered by the triangular dam shown in Fig. 128. The thrust suffered by the triangular dam amounts to $1 \,.\, 1{\cdot}10/2 = 0{\cdot}55$. If the same thrust is exerted on our crest dam, we will have for $y = 1$

$$0{\cdot}8 \,.\, \sigma_x(0)/2 = 0{\cdot}55, \quad \sigma_x(0) = 1{\cdot}375$$

With the guidance of the above we assume

$$\sigma_x(0) = 1{\cdot}420 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82a)$$

For $k = 0{\cdot}18$ we then get

$$\sigma_y(0) = 0{\cdot}18 \,.\, 1{\cdot}42 = 0{\cdot}256 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82b)$$

and according to equation (70)

$$\tau'_{xy}(0) = (0{\cdot}8 - 0{\cdot}256)\frac{1{\cdot}4}{0{\cdot}8} = 0{\cdot}952 \quad . \quad . \quad . \quad . \quad . \quad . \quad (82c)$$

We shall now determine the coefficients in the system of equations (78) so as to comply with the following :

$$\tau_{xy}(0) = 0, \qquad \tau'_{xy} = 0{\cdot}952 \quad \ldots \quad (83a)$$

$$\sigma_x(0) = 1{\cdot}420, \quad \sigma_y(0) = 0{\cdot}256 \quad \ldots \quad (83b)$$

These four requirements are, however, not independent of each other, since in fact $\tau'_{xy} = 0{\cdot}952$ follows from $\sigma_y(0) = 0{\cdot}256$. We therefore have to introduce another requirement, which we may choose as follows :

$$x = 0{\cdot}3$$

$$\sigma_x = 1{\cdot}41, \quad \tau_{xy} = 0{\cdot}286, \quad \sigma_y = 0{\cdot}341, \quad k = 0{\cdot}18 \quad \ldots \quad (83c)$$

Using the values given by equations (83) we get

$$\sigma_x = 1{\cdot}420 - 0{\cdot}102v^2 - 0{\cdot}0634v^{\frac{15}{4}} \quad \ldots \quad (84a)$$

$$\tau_{xy} = 0{\cdot}952v \quad \ldots \quad (84b)$$

$$0 < x \leq 0{\cdot}3 \; \sigma_y = 0{\cdot}256 + 0{\cdot}699v^{\frac{7}{4}}$$

$$0{\cdot}3 < x \; \sigma_y = 0{\cdot}456 + 0{\cdot}699v^{\frac{7}{4}} - 0{\cdot}667v \quad \ldots \quad (84c)$$

Evaluating these equations we get:

Table 6

$y = 1$

v	σ_x	τ_{xy}	σ_y	k
0	1·420	0	0·256	0·180
0·1	1·419	0·095	0·269	0·182
0·2	1·416	0·190	0·298	0·182
0·3	1·410	0·286	0·341	0·180
0·4	1·401	0·381	0·331	0·13
0·5	1·390	0·476	0·331	0·09
0·6	1·374	0·571	0·342	0·07

As will be seen from the above Table, the function given by equations (84) can be accepted up to $v = 0{\cdot}3$. For higher x-values it gives k-values which are altogether too low. To remedy this we have to make a hinge at $x = 0{\cdot}3$ so as to obtain about $\sigma_y = 0{\cdot}430$ at $x = 0{\cdot}4$. This implies that we must increase τ'_{xy} to about 1·22.

We shall therefore, for the interval $0{\cdot}3 < x$, determine a stress system according to equations (78) based upon the following data :

$$\left.\begin{matrix}\sigma_x(0{\cdot}3) = 1{\cdot}410, \quad \tau_{xy}(0{\cdot}3) = 0{\cdot}286, \quad \sigma_y(0{\cdot}3) = 0{\cdot}341 \\ \tau'_{xy} = 1{\cdot}22\end{matrix}\right\} \quad \ldots \quad (85)$$

We then get

$$\sigma_x = 1{\cdot}410 + 0{\cdot}112(v - 0{\cdot}3) - 0{\cdot}130(v^2 - 0{\cdot}3^2) - 0{\cdot}1785(v^{\frac{15}{4}} - 0{\cdot}3^{\frac{15}{4}}) \; . \quad (86a)$$

$$\tau_{xy} = 1{\cdot}22v - 0{\cdot}080 \quad \ldots \quad (86b)$$

$$\sigma_y = 0{\cdot}301 - 0{\cdot}667v + 1{\cdot}962v^{\frac{7}{4}} \quad \ldots \quad (86c)$$

By evaluating these equations we get:

Table 7

$y = 1$

v	σ_x	τ_{xy}	σ_y	k
0·3	1·410	0·286	0·341	0·180
0·4	1·409	0·408	0·430	0·180
0·5	1·402	0·530	0·552	0·180
0·55	1·397	0·591	0·623	0·178
0·60	1·386	0·652	0·704	0·176

We deem this result good enough, and we shall now consider the areas enclosed between σ_y and the baseline.

(*c*) Middle part.

Let us consider the area of

$$F(x) = \int_0^x \sigma_y dx$$

From the toe end we have

$$F(a) - F(0{\cdot}6) = 0{\cdot}4804 \qquad (87a)$$

$$F(a) - F(0{\cdot}5) = 0{\cdot}5870 \qquad (87b)$$

From the upstream end we have

$$F(0{\cdot}6) = 0{\cdot}2365 \qquad (88a)$$

$$F(0{\cdot}5) = 0{\cdot}1742 \qquad (88b)$$

If we let our function perform a jump at $x = 0{\cdot}6$, we get an area of

$$F(a) = 0{\cdot}7169$$

which is very close to the correct value $F(a) = 0{\cdot}7200$. By interpolation we find that the jump should be performed at about

$$x = 0{\cdot}59$$

For this value we get, by interpolation

$$\sigma_{x_1} = 1{\cdot}388, \quad \tau_{xy_1} = 0{\cdot}639, \quad \sigma_{y_1} = 0{\cdot}688 \qquad (89a)$$

$$\sigma_{x_2} = 1{\cdot}489, \quad \tau_{xy_2} = 0{\cdot}843, \quad \sigma_{y_2} = 1{\cdot}036 \qquad (89b)$$

We thus have

$$\tau_{xy_2} - \tau_{xy_1} = 0{\cdot}204 \qquad (90)$$

and should then have

$$\sigma_{y_2} - \sigma_{y_1} = \frac{1}{0{\cdot}59} \cdot 0{\cdot}204 = 0{\cdot}345$$

In fact, we have $\sigma_{y2} - \sigma_{y1} = 0{\cdot}348$ which is good enough.

We should have

$$\sigma_{x_2} - \sigma_{x_1} = 0{\cdot}59 \,.\, 0{\cdot}204 = 0{\cdot}120 \qquad (91)$$

However, we have only 0·101. We conclude from this that our assumption of $\sigma_x(0) = 1{\cdot}42$ is 0·02 too high. We should, therefore, diminish our values of σ_x in Tables 6 and 7 by this amount.

We shall, however, make a check by estimating the area $\int_0^a \tau_{xy} dy$. We have

$$\begin{aligned} x &= 0 \quad \text{to} \quad x = 0{\cdot}3 \quad & 0{\cdot}286 \,.\, 0{\cdot}3/2 &= 0{\cdot}0429 \\ x &= 0{\cdot}3 \quad \text{to} \quad x = 0{\cdot}59 \quad & (0{\cdot}286 + 0{\cdot}639) \,.\, 0{\cdot}29/2 &= 0{\cdot}1340 \\ x &= 0{\cdot}59 \quad \text{to} \quad x = 1{\cdot}5 \quad & 0{\cdot}843 \,.\, 0{\cdot}91/2 &= 0{\cdot}3830 \\ & & &\overline{0{\cdot}5599} \end{aligned}$$

$$\sigma_x(0) \,.\, 0{\cdot}8/2 = 0{\cdot}5599, \quad \sigma_x(0) = 1{\cdot}400$$

(*d*) The complete system.

We can now, with confidence, give the complete table of our stress function as follows :

Table 8

$y = 1$

v	σ_x	τ_{xy}	σ_y	k
0	1·400	0	0·256	0·182
0·1	1·399	0·095	0·269	0·183
0·2	1·396	0·190	0·298	0·182
0·3	1·390	0·286	0·341	0·180
0·4	1·389	0·408	0·430	0·180
0·5	1·382	0·530	0·552	0·180
0·59 − 0	1·368	0·639	0·688	0·176
0·59 + 0	1·489	0·843	1·036	0·180
0·6	1·470	0·834	1·026	0·180
0·8	1·113	0·649	0·823	0·184
1·0	0·771	0·463	0·603	0·184
1·25	0·367	0·232	0·313	0·182
1·5	0	0	0	0·180

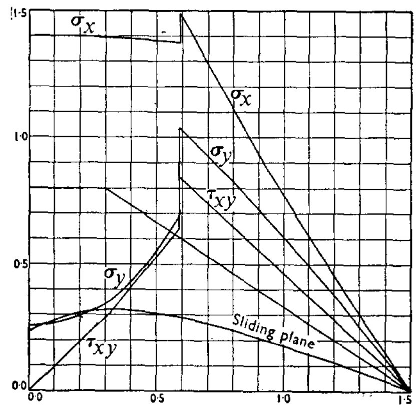

Fig. 131

The small discrepancies between the values given for k in the preceding Table could, of course, be eliminated by a renewed calculation using polygons with shorter legs. Since the object of our analysis, however, is to demonstrate the possibility of obtaining a result by applying the formulas developed, the above result is considered good enough.

The result is shown graphically in Fig. 131.

Stress in interior of dam

In the case of a triangular dam the stresses are throughout in proportion to the radius vector through the origin. The relative proportions between the stresses are therefore the same along any chosen radius vector. The k-value, in particular, is thus constant along any radius vector and the same as the k-value of the stress system obtaining at the intersection of the chosen radius vector with the plane $y = 1$.

As previously mentioned, this is not the case in a dam with a crest. In general, along any chosen radius vector none of the stresses will vary in direct proportion to the length of the radius vector. The k-value may thus be different along a chosen radius vector. We shall estimate this variation in the following:—

If we introduce as parameters

$$\left. \begin{array}{l} x = m \\ y = 1 \end{array} \right\} \sigma_x = p, \quad \tau_{xy} = q, \quad \sigma_y = r$$

we get

$$\sigma_y(m,y) = 2\frac{y^3-b^3}{y^2}\frac{m^2}{(1-b)^2(2+b)}\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]-\frac{q}{m}\right\}$$

$$-(y-b)\left(\frac{(b^2+b+4)m^2}{(1-b)^2(2+b)}\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]\right\}\right.$$

$$\left.-\frac{2m(1-b^3)}{(1-b)^3(2+b)}q-\frac{1}{1-b}p\right)$$

$$+\frac{(y-b)^3}{y^2}\frac{m^2}{(1-b)^3}\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]\right\} \quad . \quad . \quad . \quad \text{(92a)}$$

$$\tau_{xy}(m,y) = -\frac{(y-b)^2(2y+b)}{y^2(1-b)^2(2+b)}m\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]-\frac{q}{m}\right\}$$

$$+\frac{(y-b)^3m}{y^2(1-b)^3}\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(92b)}$$

$$\sigma_y(m,y) = \frac{(y-b)^3}{y^2(1-b)^3}\left\{r-\frac{a-m}{a}+\left[-\frac{m}{a}+b\right]\right\}+y\frac{a-m}{a}+\left[-b+y\frac{m}{a}\right] \quad \text{(92c)}$$

Evaluating these equations we get for different values of y and v the stresses given in the following Table, in which the value of k, and also the inclination $\tan\alpha$ of the major principal stress and the inclination $\tan\gamma$ of the less inclined sliding plane, are given:

Table 9

	v	σ_x	τ_{xy}	σ_y	k	$\tan\alpha$	$\tan\gamma$
$y = 1$	1·5	0	0	0	0·180	0·950	0·370
	1·25	0·367	0·232	0·313	0·182	0·880	0·325
	1·0	0·771	0·463	0·603	0·184	0·825	0·295
	0·8	1·113	0·649	0·823	0·184	0·800	0·275
	0·6	1·470	0·834	1·026	0·180	0·770	0·255
	0·59 + 0	1·489	0·843	1·036	0·180	0·770	0·255
	0·59 − 0	1·368	0·639	0·688	0·176	0·600	0·150
	0·5	1·382	0·530	0·552	0·180	0·485	0·055
	0·4	1·389	0·408	0·430	0·180	0·370	− 0·050
	0·3	1·390	0·286	0·341	0·180	0·250	− 0·160
	0·2	1·396	0·190	0·298	0·182	0·160	− 0·250
	0·1	1·399	0·095	0·269	0·183	0·090	− 0·340
	0	1·400	0	0·256	0·182	0	− 0·425
$y = 0{\cdot}8$	1·25	0·251	0·155	0·230	0·215	0·940	0·320
	1·0	0·542	0·317	0·444	0·215	0·860	0·290
	0·8	0·796	0·450	0·608	0·210	0·810	0·250
	0·5	1·021	0·382	0·458	0·220	0·505	0·035
	0·2	1·04	0·149	0·269	0·225	0·185	− 0·027
$y = 0{\cdot}6$	1·25	0·142	0·085	0·151	0·270	1·050	0·350
	1·0	0·322	0·179	0·293	0·270	0·920	0·275
	0·8	0·488	0·260	0·404	0·265	0·850	0·225
	0·5	0·661	0·238	0·360	0·290	0·560	0·015
	0·2	0·691	0·103	0·226	0·290	0·205	− 0·3
$y = 0{\cdot}4$	1·25	0·049	0·027	0·081	0·350	1·750	0·565
	1·0	0·124	0·061	0·160	0·380	1·351	0·395
	0·8	0·201	0·092	0·222	0·390	0·895	0·170
	0·5	0·306	0·098	0·189	0·365	0·56	− 0·03
	0·2	0·346	0·050	0·151	0·390	0·240	− 0·32

As will be seen from Table 9, the k-value increases, as expected, upwards through the dam fill.

In Fig. 131 is indicated a sliding surface cutting out at the downstream toe, and based upon interpolated values of tan γ taken from Table 9. It will be realized that there is the snag that along this "sliding surface" the k-value is higher than 0·18. According to our assumptions with reference to the material, it should not therefore slide along this "sliding surface." Since at the base the value of τ_{xy}/σ_y is smaller than tan ϕ, it should not slide there either. The conclusion might be drawn that the dam will have a greater resistance than that arrived at above.

To obtain a sliding surface where the k-value is equal to 0·18 throughout, it would be necessary to design a new stress system based upon a suitable variation of the k-value along the base, the k-value being in the neighbourhood of 0·14 in the middle of the base.

We shall, however, make no attempt to carry out such a calculation here.

DISCUSSION

Opening the discussion, Mr A. Lazard said that it had been apparent to him that the "overall" geometrical method of stability analysis (*methode geometrique globale*) was not capable of dealing with certain basic problems. He had, however, by a geometrical method been able to determine (a) the forces on any arbitrary surface in an earth mass on the point of sliding, and (b) the Fellenius factor of safety of a slope consisting of two different soils. This method would be published in the French journal, *Travaux*.

Dr E. Nonveiller referred to the interesting Paper presented by Mayer and Habib who had found that the pore pressure measured in the centre of clay samples took an appreciable time to dissipate after removal of the hydrostatic head. Results of similar tests on larger samples (80 cm dia., 60 cm high) reported by Drs Breth and Kückelmann[1] with a water head of 20 m showed that whilst there was some delay in the build-up of the pore pressure when this load was applied, the pore pressure followed the unloading immediately. This was contrary to the findings of Mayer and Habib. Dr Nonveiller then pointed out differences between the two sets of tests which might account for this.

The records of pore pressure from the Alcova dam showed that the flow net after rapid drawdown altered to satisfy the new boundary conditions on the water surface of the upstream slope, which was again contrary to the conclusions of Mayer and Habib. Records from the Anderson Ranch Dam of the pore-pressure during the first filling of the basin agreed with neither the results of Breth's nor of Mayer's tests, and he thought that the main reason for this difference between the laboratory and field tests was the small size of the samples. In reply, Mr Mayer thanked Dr Nonveiller for drawing attention to the importance of the experimental procedure and said he believed that the differences between the two sets of tests could be explained mainly in those terms. He pointed out that it was the practice of the U.S. Bureau of Reclamation to compact soils below the optimum water content leaving a considerable proportion of air in the material. This air might not be enclosed in the form of bubbles but might fill cavities and be able to escape quite rapidly. It was possible, he believed, that similar conditions obtained in the tests described by Breth and Kückelmann.

Referring to Dr Samsioe's Paper, Dr Nonveiller said that the inclination of the sliding surface where it intersected the boundaries (at the toe and the core) could be found using Krey's tables. He had found that the slip surface was in general convex upward, being straight only in the case of $\beta = -\delta$. He had, moreover, carried out several model tests on sand slopes proving that the shapes of the sliding surfaces were in full agreement with the calculations by Dr Samsioe, but these surfaces developed only after large deformations.

Dr Samsioe emphasized that the assumptions on which his calculations had been based were valid only after large deformations had taken place, as had been shown by Dr Nonveiller's experiments. Therefore a dam could not be designed with a factor of safety close to unity because it would then be so deformed as to be useless. It would, he felt, be more appropriate to use "working stresses" or "working deformations" than "factors of safety."

Mr Sten Odenstad presented a proof which showed that the discontinuities in the solutions described in Dr Samsioe's Paper, were inevitable. By combining the equations of equilibrium of a small element of soil with the failure criterion, he demonstrated that the resulting second order differential equation could satisfy only two and not the three boundary conditions necessary for a continuous solution.

Dr A. W. Bishop believed that Professor Reinius's treatment of the effect of drawdown on the stability of compressible fills led to an over-estimate of the residual pore-water pressure, because it failed to take account of the shear stress changes during drawdown. The stress changes on a typical element of soil on a potential slip surface were, first a decrease in total major principal stress $\Delta\sigma_1$, and secondly an increase in shear stress $\frac{1}{2}(\Delta\sigma_1-\Delta\sigma_3)$, the total minor principal stress undergoing a greater reduction than σ_1. If such stress changes were simulated on partially saturated samples consolidated anisotropically in the triaxial apparatus to correspond to the stress and pore pressure conditions in the steady state before drawdown, it was found that $\frac{\Delta u}{\Delta\sigma_1}<1$ for zero change in shear stress—a typical value being 0·6. If the corresponding change in shear stress was also applied, $\frac{\Delta u}{\Delta\sigma_1} \simeq 1$, a typical value being 1·2.[2] This difference was due to dilatancy and it meant that residual pore pressures would, in practice, be considerably less than those indicated by Professor Reinius's method. In examining the stability of a free-draining fill supporting a narrow central core he had made use of the earth pressure tables published by Caquot and Kerisel (1949). Here the passive resistance of a wedge-shaped frictional fill was calculated on the basis of the Rankine state in the outer part of the slope and of an intermediate zone in which the stresses are determined by a step-by-step method. The failure surface implied, in the case of a steep outer slope and zero shear stress on the vertical face, was similar to that shown by Professor Reinius in Fig. 115, being an inverted curve followed by a plane.

Dr Bishop emphasized the importance of the shear force transmitted to the vertical face of the wedge from the core. He had found, in the analysis of the stability of a typical zoned fill dam, little difference in the factors of safety given by using either the circular arc method or a combination of an arc with the passive resistance method in the rock-fill zone, provided the effect of the vertical shear force was allowed for.

Dr N. Janbu described a routine procedure which had been developed at the Norwegian Geotechnical Institute for calculating the factor of safety of an earth mass against failure along composite slip surfaces of various shapes. The method had been developed after studying Dr Bishop's Paper.[3]

By considering the equilibrium of each slice in the vertical (y) and horizontal (x) directions Dr Janbu obtained a general expression for the factor of safety of the form :

$$F=\frac{\sum\frac{[c'+(p+t-u)\tan\phi']\Delta x}{on_\alpha}}{Q+\Sigma(p+t)\tan\alpha\Delta x}=\frac{\Sigma A}{\Sigma B}$$

The notation is the same as that used by Dr Bishop with the following additions :

$$n_\alpha=\cos^2\alpha\left(1+\frac{\tan\alpha\tan\phi'}{F}\right)$$

$P=\frac{\Delta W}{\Delta x}$; W being the full weight of the soil above, or the reduced weight below, an external free water surface. $t=\frac{\Delta T}{\Delta x}$; T being the vertical shear force at any point.

Q = any external horizontal force acting on the earth mass. From the moment equation for each slice the following expression for the vertical shear at any point was obtained :

$$Tx=-\tan\alpha_t\sum_0^x\left(B-\frac{A}{F}\right)$$

where α_t is the slope of the line of thrust.

In practice, an initial value of F was obtained by putting $t=0$, and more accurate solutions were then found by successive approximations, the vertical shear forces being calculated using the second equations. The solution was not sensitive to the exact location of the line of thrust, which was normally assumed to pass through or slightly above the lower third points of the slices.

Professor E. Schultze mentioned that slag heaps in the Rhineland mining industry consisted of mixtures of non-cohesive and medium-cohesive soils tipped in a loose state, and they were found to have a fairly constant angle of repose of about 37°. Occasional slips had taken place mostly following heavy showers. With the increased depth of opencast working, which was contemplated in the future, it would be necessary to increase the height of the tips above the present 60 m. In order to investigate the stability of the slopes, various types of penetration tests had been carried out which showed that the present tips were rather loose, having a porosity of about 45% with inappreciable increase of compactness with depth, and that the angle of friction of the soils was about 31°. Their stability, he believed, must be partly attributed to capillary cohesion.

A few experiments had also been carried out to determine whether more compact and stable slopes could be constructed by tipping the slag from a height in horizontal layers instead of on the face of the slope. These showed that an increase in the height of the drop from 4 m to 25 m led to a 9% increase in compaction in sand and gravel, and a 22% increase with clay. It was appreciated that other factors, such as water content, were also important and further investigations were planned.

Dr Rosenquist asked Dr Keil to give full information about the chemicals used in the " Hydraton " process, as the results obtained were of general scientific interest.

Professor Keil, in reply, said that sodium silicate was absolutely necessary together with clay minerals and water-soluble alkaline salts. The most suitable salts were those containing sodium and potassium only.

[1] BRETH, H., *and* KÜCKELMANN, B., 1954. " Der Porenwasserdruck in Erddämmen." *Die Bautechnik,* 31 : 1 : 25.

[2] BISHOP, A. W., 1954. The use of Pore-Pressure Coefficients in Practice. *Géotechnique,* 4 : 4 : 148.

[3] BISHOP, A. W., 1954. The use of the Slip Circle in the Stability Analysis of Slopes. *Géotechnique,* 5 : 1 : 7.