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The effect of compressive stress on the dynamic modulus of concrete*

by R. H. Elvery, B.Sc.(Eng.), A.M.I.C.E. and M. Furst, B.Eng., M.Sc.(Eng.), D.I.C.

CORRIGENDUM

The first equation on page 146 should read:

$$\frac{W_c}{W_e} = \frac{\pi f_2}{f_0}\sqrt{\frac{1}{(ni+1)\,(nk+1)}}\tan\frac{\pi f_2}{2f_0}\sqrt{\frac{ni+1}{nk+1}}$$

Contribution by N. B. Terry, B.Sc., Ph.D., A.Inst.P.
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This paper is of considerable interest to those engaged in applying acoustic techniques to studies of the mechanical properties of materials.

The analysis leading to the equation developed by Timoshenko applies to a rod carrying an inert mass, whereas in the experiments described the end plates were bonded to the concrete and formed an integral part of the vibrating system. Thus, in the absence of the steel rod, the system was roughly equivalent to a vibrating tripartite rod with the component members bonded end to end. The appropriate resonance condition may be shown[1] to be

$$W_c f_0 \tan\frac{\pi f_1}{f_0} + W_a f_a \tan\frac{\pi f_1}{f_a} + W_b f_b \tan\frac{\pi f_1}{f_b} = \frac{W_a f_a\, W_b f_b}{W_c f_0}\tan\frac{\pi f_1}{f_0}\tan\frac{\pi f_1}{f_a}\tan\frac{\pi f_1}{f_b} \quad \ldots\ldots.(1)$$

using the symbols used in the paper and where f_a and f_b are the natural thickness resonance frequencies and W_a and W_b the weights of the two unequal end plates. If the two end plates are equal, and it is assumed that tan $(\pi f_1/f_b) \simeq \pi f_1/f_b$ and tan $(\pi f_1/f_a) \simeq \pi f_1/f_a$, equation (1) reduces to Timoshenko's equation, i.e.

$$\frac{W_c}{W_a} = \frac{W_c}{W_b} = \frac{\pi f_1}{f_0}\tan\frac{\pi f_1}{2f_0} \quad \ldots\ldots\ldots\ldots(2)$$

If, however, the end plates are not equal then, from equation (1),

$$\frac{W_c}{W_e} \simeq \frac{\pi f_1}{f_0}\left(\frac{\tan\frac{\pi f_1}{2f_0} - \cot\frac{\pi f_1}{2f_0}}{W_c^2 f_0^2 - \pi^2 f_1^2 W_a W_b}\right) f_0^2 W_c^2$$

where $W_e = (W_a + W_b)/2$ so that the use of equation (2) would only be justified if $\tan^2(\pi f_1/2f_0) >> 1$ and $W_c^2 f_0^2 >> \pi^2 f_1^2 W_a W_b$; fortunately, these conditions do apply for many of the experiments described.

It is a pity that the authors have not given their theoretical analysis leading to the resonant condition for a concentric composite rod, the validity of which is well supported by their experimental results. A possible application would be the measurement of the elastic properties of very tiny rod-like samples, thinly deposited with a magnetostrictive material to enable magnetostrictive excitation. A simplification of the authors' technique, now that the basic relations have been established, would be to use a nickel prestressing rod and to vibrate the system magnetostrictively.

The results obtained on concrete are both interesting and difficult to explain. Jones[2], using an elastic pulse technique, did not observe any change in elastic wave velocity when stress was applied to concrete specimens in the same direction as the elastic pulses.

It is possible that, since concrete is a porous material, there are two distinct effects due to the application of stress, i.e. the deformation of the solid framework and changes in the elastic properties of the material of the framework. One would expect an increase of frequency (and therefore derived elastic modulus) as the cracks are partially closed at fairly low stress levels; probably the major deformations are accomplished at the lowest stress level used of 400 lb/in^2. In order to explain the main features of the observed results, it appears necessary to assume that the dynamic elastic modulus of the framework material is reduced with stress, although one would in fact have expected the opposite result. If the magnitude of the latter decrease in modulus is larger than the increase due to the closing of cracks, the net reduction in modulus

*Pages 145-150.

observed on application of stress can be explained, as can the increase in magnitude of this reduction with increased stress. Since the modulus of the steel is higher than that of the concrete, the latter is virtually held at constant strain. Possibly there is some relaxation of stress in the concrete, insufficient to allow the cracks to be re-opened, but giving rise to the observed gradual increase in modulus of the framework material. When the restraints are finally removed, the framework deformation is recovered, i.e. cracks are opened giving rise to the observed reduction in modulus which would be expected to be higher than the initial reduction (due to two opposing effects) on application of stress.

Jones[2] observed a decrease in longitudinal pulse velocity when unloading and, in agreement with the paper under discussion, found that the magnitude of the decrease increased with the number of load cycles, these effects being attributed to the opening of cracks.

Possibly experiments involving the impregnation of the concrete with a cement might give information about the behaviour of the framework; such experiments, using a pulse technique, have been carried out at the Mining Research Establishment for coal, a porous material for which the dynamic behaviour is strongly dependent on the condition of the framework.

Contribution by R. Jones, B.Sc., Ph.D.

(*Road Research Laboratory*)

The authors are to be congratulated on producing a very interesting paper on a difficult and comparatively unexplored part of the field of dynamic testing. Their careful experimentation has produced a technique which represents a considerable improvement on the work of Obert and their results are obviously more reliable. Obert's results were obtained very early in the development of dynamic testing and were unlikely to be questioned since they appeared to give qualitative agreement with the results obtained from measurements on the change in static modulus with load obtained for porous rocks.[3] About 10 years ago we made some experiments at the Road Research Laboratory[4] to study the effect of loading on rocks and concrete using a pulse technique; our results indicated substantial increases in the dynamic modulus with load for sandstone but no significant change in the case of concrete, except when cracks were being formed. However, the initial changes in dynamic modulus on stressing found by Elvery and Furst would hardly have been significant in our experiments.

The main difficulty in the authors' technique appears to lie in making a satisfactory allowance for the presence of the central steel rod, and I presume that their first equation is obtained by the use of simplifying assumptions. Incidentally, there appears to be a printing error in this equation since, when there is no steel bar, i.e. when $n = 0$, the equation does not reduce to their second equation. An exact equation for the complete system would obviously be very complex and the authors are to be commended for recognizing and making due allowance for discrepancy in f_0 when deduced from f_1 and f_2.

It was interesting to observe that the frequency equation derived for end loading with equal masses gave good agreement with experimental results when the masses were not equal. I have endeavoured to show below under what conditions this will apply by working from the frequency equation for unequal masses at each end, as given by Donkin[5], i.e.

$$(C_1+C_2)\frac{\pi f_1}{f_0} = -\left(1-C_1C_2\frac{\pi^2 f_1^{\,2}}{f_0^{\,2}}\right)\tan\frac{\pi f_1}{f_0} \quad \ldots\ldots(1)$$

where $C_1 = (W_e/W_c)_1$ and $C_2 = (W_e/W_c)_2$ and the other nomenclature is as given by Elvery and Furst. The corresponding equation for equal end masses used by Elvery and Furst is:

$$\frac{1}{C} = \frac{\pi f_1}{f_0}\tan\frac{\pi f_1}{f_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

It can easily be shown that when $C_1 = C_2 = C$, equations (1) and (2) are equivalent.

Let us compare equations (1) and (2) for the worst possible case of unequal masses, i.e. the mass concentrated at one end ($C_2 = 0$). To retain the same total mass, loading $C_1 = 2C$ and equation (1) becomes:

$$2C\,\frac{\pi f_1}{f_0} = -\tan\frac{\pi f_1}{f_0} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

It can be shown that equations (2) and (3) will give approximately the same value of f_1/f_0 provided

$$\frac{f_0^{\,2}}{\pi^2 f_1^{\,2} C^2} >> 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Taking the highest value of C for which unequal weights were used by Elvery and Furst, i.e. $W_c/W_e = 8{\cdot}77$, it can be shown that f_1/f_0 deduced from equation (3) is only about 1% different from the value given by equation (2). Thus, within the range of end loading used by Elvery and Furst, considerable discrepancies between the two end weights would not cause significant deviations from equation (2). Even when the total loading becomes equal to half the weight of the concrete beam, the discrepancy between equations (2) and (3) only amounts to about 4%.

The effect of cyclic loading on the dynamic modulus is somewhat similar to the effect produced by cycles of freezing and thawing, i.e. the first cycle generally causes a much larger decrease in modulus than do subsequent cycles. The initial decreases in modulus may perhaps be due to localized failures at the weaker bonds of the concrete and the subsequent increase in modulus under sustained load could be accounted for by re-formation of the bonds or by volume flow of concrete into the voids. Reiner[6] has shown the latter to occur in the creep of concrete and this would perhaps explain the somewhat anomalous result in Figure 2 where there was ultimately an increase in the dynamic modulus on application of stress of 400 lb/in^2. However, it seems difficult at this stage to provide a hypothesis which fits all the experimental facts and that is no doubt the reason why the authors wisely refrained from comment.

Reply by the authors

We were very interested in the theoretical analyses for the effect of unequal end masses which Dr Jones and Dr Terry have both shown. We had not explored the theoretical aspect of this except for the case of equal masses. Since our experimental arrangement was unlikely to have exactly equal masses at the ends, it was decided that the preliminary tests should include cases of unequal masses and these tests showed that Timoshenko's simple formula was reasonably satisfactory.

The theoretical analysis concerning the resonance of the concentric composite rod with end masses was, as Dr Jones has suggested, carried out using simplifying assumptions which proved to be justified.

We agree with Dr Terry that the analysis of a tripartite rod would have provided a more rigorous solution for the effect of end masses, but, in the circumstances, would have been unnecessarily complicated.

To correct the omission of the theoretical analysis for the resonance of the composite concentric rod, the outline of the method is given below. This is done for two cases: (1) the composite concentric bar only and (2) the composite concentric bar with end masses added.

It was assumed that both materials could be regarded as being bonded together so that, in any cross-section, strain in the axial direction was the same in both materials.

Let A_s = area of steel
A_c = area of concrete
ρ_s = density of steel
ρ_c = density of concrete
E_s = elastic modulus of steel
E_c = elastic modulus of concrete

(1) The equation of motion for an elemental slice normal to the axis of the beam would be:

$$(A_cE_c+A_sE_s)\frac{\partial^2 u}{\partial x^2}\,\delta x = \frac{(A_c\rho_c+A_s\rho_s)}{g}\,\delta x\,\frac{\partial^2 u}{\partial t^2}\ldots(1)$$

where x is measured along the axis, u is the displacement of the element and g is the acceleration due to gravity.

This replaces the corresponding expression for a single material, namely:

$$(AE)\frac{\partial^2 u}{\partial x^2}\delta x = \frac{A\rho}{g}\,\delta x\,\frac{\partial^2 u}{\partial t^2}\ldots\ldots\ldots\ldots(2)$$

Equation (1) leads to the solution

$$f_c = \frac{1}{2l}\sqrt{\frac{(A_sE_s+A_cE_c)}{A_s\rho_s+A_c\rho_c}g}\ldots\ldots\ldots(3)$$

where f_c is the resonance frequency of the composite rod.

Equation (2) leads to the solution

$$f_0 = \frac{1}{2l}\sqrt{\frac{Eg}{\rho}}\ldots\ldots\ldots\ldots\ldots(4)$$

From equations (3) and (4)

$$\frac{f_c}{f_0} = \sqrt{\frac{(A_sE_s+A_cE_c)\,\rho_c}{(A_s\rho_s+A_c\rho_c)\,E_c}}$$

Putting $n = A_s/A_c$, $i = \rho_s/\rho_c$ and $k = E_s/E_c$,

$$\frac{f_c}{f_0} = \sqrt{\frac{nk+1}{ni+1}}\ldots\ldots\ldots\ldots\ldots(5)$$

(2) With end masses added, the appropriate equation of motion is

$$(A_sE_s+A_cE_c)\frac{\partial u}{\partial x}\,\delta x = -\frac{W_e}{g}\frac{\partial^2 u}{\partial t^2}\,\delta x$$

which leads to the solution

$$\frac{W_c}{W_e} = \frac{\pi f_2}{f_c(ni+1)}\tan\frac{\pi f_2}{2f_c}$$

Substituting for f_c, as in equation (5),

$$\frac{W_c}{W_e} = \frac{\pi f_2}{(ni+1)f_0}\frac{1}{}\sqrt{\frac{ni+1}{nk+1}}\tan\frac{\pi f_2}{2f_0\sqrt{\frac{nk+1}{ni+1}}}$$

$$= \frac{\pi f_2}{f_0}\sqrt{\frac{1}{(ni+1)(nk+1)}}\tan\frac{\pi f_2}{2f_0}\sqrt{\frac{ni+1}{nk+1}}$$

We agree that the results are not easy to explain and, as Dr Jones suggested, we did not offer any hypothesis although we had made various guesses. The values of the changes of modulus due to stress were generally too small to be easily detected by the ultrasonic pulse method as Dr Jones has pointed out.

Dr Terry's explanation of the behaviour of a porous material is probably somewhat too simplified to apply to concrete. Concrete may be regarded as being made up of a relatively rigid aggregate embedded in a weaker and more porous cementing material. Its behaviour might be expected to follow that of a spring and dashpot in parallel, the aggregate providing the more elastic part and the cementing material the viscous part. Uniaxial compression stress would be likely to produce localized high tensile and shear stresses in random directions in the porous cementing material. This would lead to the formation of microcracks when the load is applied initially but subsequent relaxation of stress in the cementing material would partially close the cracks. When the load is removed, the elastic strain recovery of the aggregate would produce initial tensile stresses in the cementing material which would diminish with creep relaxation.

We believe that Dr Terry's technique of impregnating coal might not be so easy to apply to concrete owing to the different nature of the pores in concrete which include capillary channels of very small diameter.

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