Magazine of Concrete Research : Vol. 17, No. 52 : September 1965

Discussion on articles published in the *Magazine of Concrete Research* Volume 16, Number 48 : September 1964

The effect of the elastic modulus of the aggregate on the elastic modulus, creep and creep recovery of concrete*

by Upendra J. Counto, B.E., M.Sc.(Eng.)

Contribution by Dr. Ing. M. A. Chiorino *Istituto di Trasporti e Strade, Turin Politechnic*

Mr Counto's paper is a valid contribution to the problem of expressing mathematically the complex relations between stresses and strains due to creep in concrete, and he should be commended for introducing a rheological model that tries to represent the real physical nature of concrete, and for not indulging in the fascination of abstract models which are not based on accurate analyses of the material.

The model that Mr Counto introduces (model 4) is the same as that assumed by Levi in some previous work on the same subject⁽¹⁾. Levi proposed model 4 as a rheological scheme which could lead to a general interpretation of creep of concrete, and tried to check its effectiveness on some of the mathematical relations that can be derived from it. In particular, the influence of the variation of E_a (elastic modulus of the aggregate) on the expression of creep at a fixed time was compared with the experimental results obtained by Kordina⁽²⁾. However, the expressions obtained by Mr Counto differ from those deduced by Levi; the divergence is due to the fact that the former introduces the " effective modulus " in deriving his equations. In this case, the use of this artifice must be considered too much of an approximation. It is certainly possible to treat every creep phenomenon by the use of an effective modulus, and consider creep strains as elastic strains; but one should remember that if the load is not constant, there must be a different effective modulus for each law of variation of the stress.

For the combined material BB′CC′ of the model in question (Figure I), the effective modulus for a constant stress cannot be taken into account because the stress in the matrix certainly varies with time. In fact, the statically indeterminate distribution of load on the cross-section BB′ is a function of time, because it

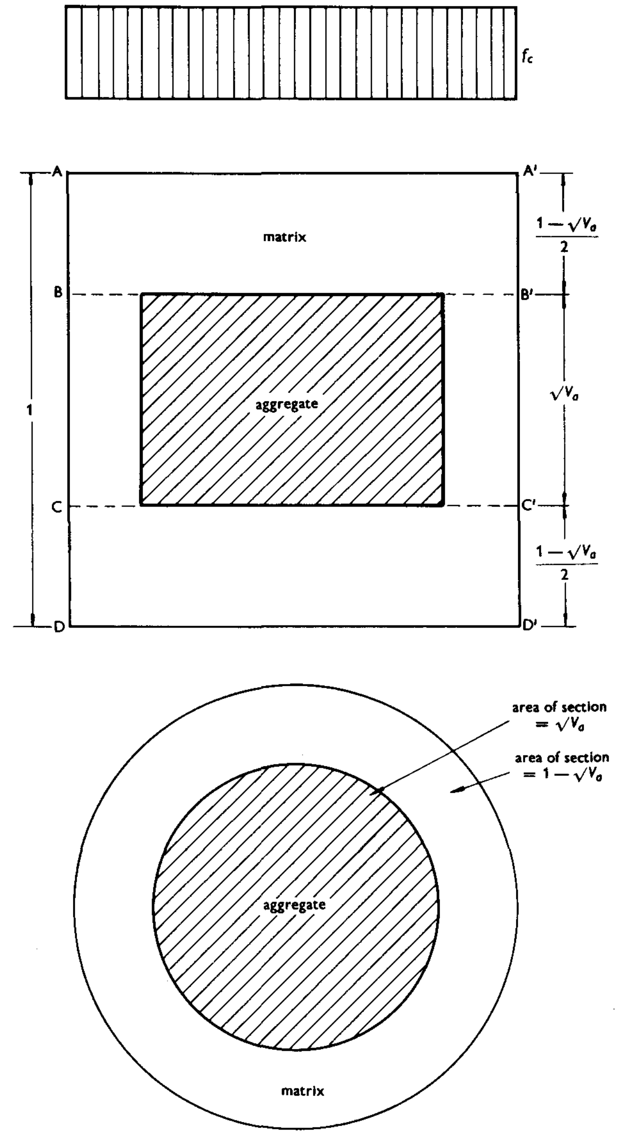

Figure I

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*Pages 129 to 138 of *Magazine* No. 48.

depends on the deformation of the matrix and this is continuously changing because of the viscosity. Without going into the details of the formulae representing the effective mechanical behaviour of model 4 (see Appendix to the present contribution), it is obvious that equation A_{11}, derived by Mr Counto from the concept of the effective modulus, will be completely different from equation XI. By assuming an effective modulus, one attributes, in fact, an initial deformability to the matrix, higher than the real one and equal to the sum of the instantaneous and delayed deformability.

As a result, the matrix of the combined element BB′CC′ will be loaded by an initial stress smaller than the stress that would be derived from a distribution based on the instantaneous deformability. Therefore, as creep is assumed to be proportional to the applied load, the viscous deformation of the matrix, and the creep strain of the element BB′CC′ that depends on it, are certainly under-estimated.

With equation I, the higher initial creep can be taken into account, as also the subsequent tendency to slow down because of the continuous diminution of the stress on the matrix resulting from the constant flowing of the matrix itself. We shall see later the sense and the magnitude of the error induced by the introduction of the effective modulus in the creep formulae.

Equation I is derived from the effective rheological behaviour of the combined material BB′CC′ and is the general relation between stress and strain for an element in parallel. By integrating this equation, the various formulae relating the magnitude of stresses and strains to time are obtained for different boundary conditions applied to the element (i.e. creep for applications of load at various times, creep recovery, stress relaxation etc.). On the other hand, the use of the effective modulus, besides producing the incorrect interpretations of the model mechanical behaviour already mentioned, does not involve this same advantage of leading to a general equation providing interpretations for different cases.

For the matrix in series (parts AA′BB′ and CC′DD′ of model AA′DD′—see Figure I) under a constant sustained load, the effective modulus gives of course a correct evaluation of its creep strain. Therefore, equation A_{11}, giving the total creep strain of concrete AA′DD′, differs from equation XI only in its second term, which represents the creep strain of the element in parallel.

If these two equations are examined more closely, it can be seen that equation A_{11} generally underestimates the creep strain in comparison with equation XI: in fact, according to the above considerations the determination of the creep strain of the element in parallel by the use of the effective modulus seems always to err on the low side for ordinary values of the volume concentration, V_a. The magnitude of this error, with reference to the total creep of the element AA′DD′, depends of course on the relative importance of the creep of the element in series (first term of equations A_{11} and XI) and the creep of the element in parallel (second term).

For high values of E_a, the modulus of the aggregate, the creep of the element BB′CC′ is very small in comparison with the creep of the matrix in series, since a small fraction of the load is applied to the matrix of the parallel element: then the error in its evaluation is small compared with the total creep of element AA′DD′. However, for low values of E_a, the creep strain of the element BB′CC′ becomes greater than the creep strain of the element in series: the error in its evaluation then becomes large compared with the total creep. Eventually, for very low values of E_a, the error introduced by the effective modulus becomes small again compared both with the total creep and with the creep of the combined element, this last becoming preponderant*.

The above considerations can be easily checked against Figures II and III, representing the relation between creep strain at a fixed time and the modulus E_a. Besides the total creep given by equations A_{11} and XI, the diagrams also show the values of the partial creep strain of the parallel element BB′CC′, represented by the second terms of these two equations. The percentage difference between the two formulae is shown too, for both total and partial creep. The values of E_a, c_u and V_a assumed in Figure II are those given by Mr Counto in his Figure 11 and correspond to his first test series. The values assumed in Figure III correspond to those adopted by Levi(1).

As far as creep recovery is concerned, the effective modulus leads to some other misinterpretations of the effective behaviour of model 4. Mr Counto evaluates the creep recovery, imposing McHenry's principle of superposition of strains to the strains wrongly evaluated by the effective modulus. For the combined material BB′CC′, the expression that he obtains (second term of equation A_{13}) is completely different from ours (VII), which is derived from the general equation (I) by simple integration. It must be noticed that equation I and, consequently, the expression for the creep recovery (VII) are completely in accord with McHenry(4,5).

According to our theory, equation VII represents the whole creep recovery strain of the element AA′DD′. Mr Counto's method, however, attributes a certain amount of creep recovery also to the matrix in series (AA′BB′+CC′DD′) which, from a strictly theoretical point of view, cannot give back any delayed elasticity; in fact, according to the fundamental equation for a visco-elastic element in series (VIII), strain ceases as soon as the load disappears and creep recovery is therefore zero.

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*For low values of E_a the load applied to the aggregate is small and the element in parallel behaves nearly as an element made only of matrix. For this reason, the use of the effective modulus does not lead to a large error.

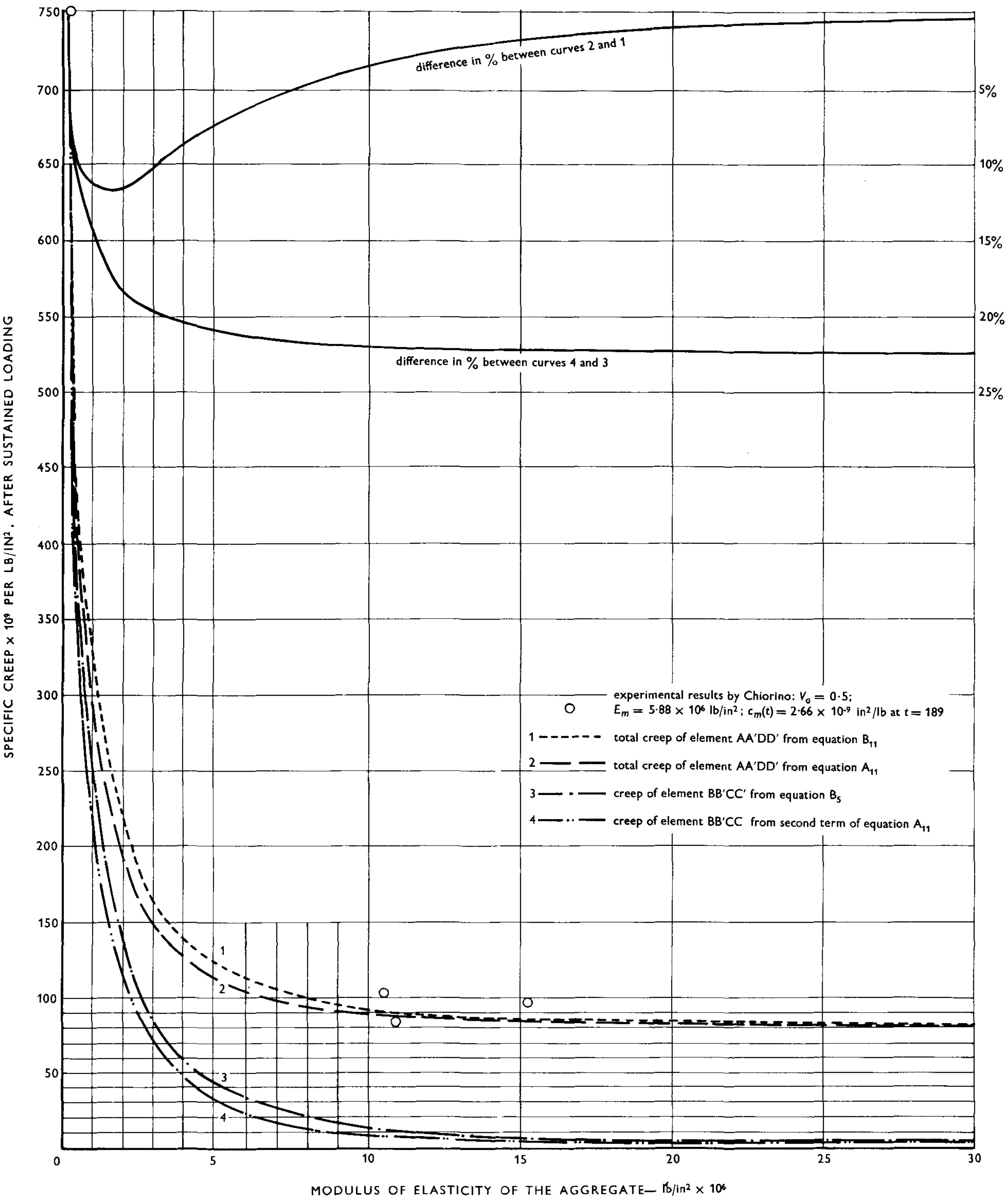

Figure II

It is true, however, that in reality the cement mortar, represented in the model by the matrix in series, is also affected by a certain degree of creep recovery. But, rather than use a false equation as Mr Counto does to eliminate the divergence between theory and practice on this point, it is preferable to overcome the deficiences of the model in other ways.

At this point it may be useful to concentrate our attention on these deficiences and to give some criticism of the structure of model 4, which seems incapable of representing the creep of concrete with a close enough agreement to the experimental results.

As regards creep recovery, we have seen that the model leads to the incongruity of denying, in contradiction of experience, any elastic recovery to the matrix in series (considered as a simple visco-elastic

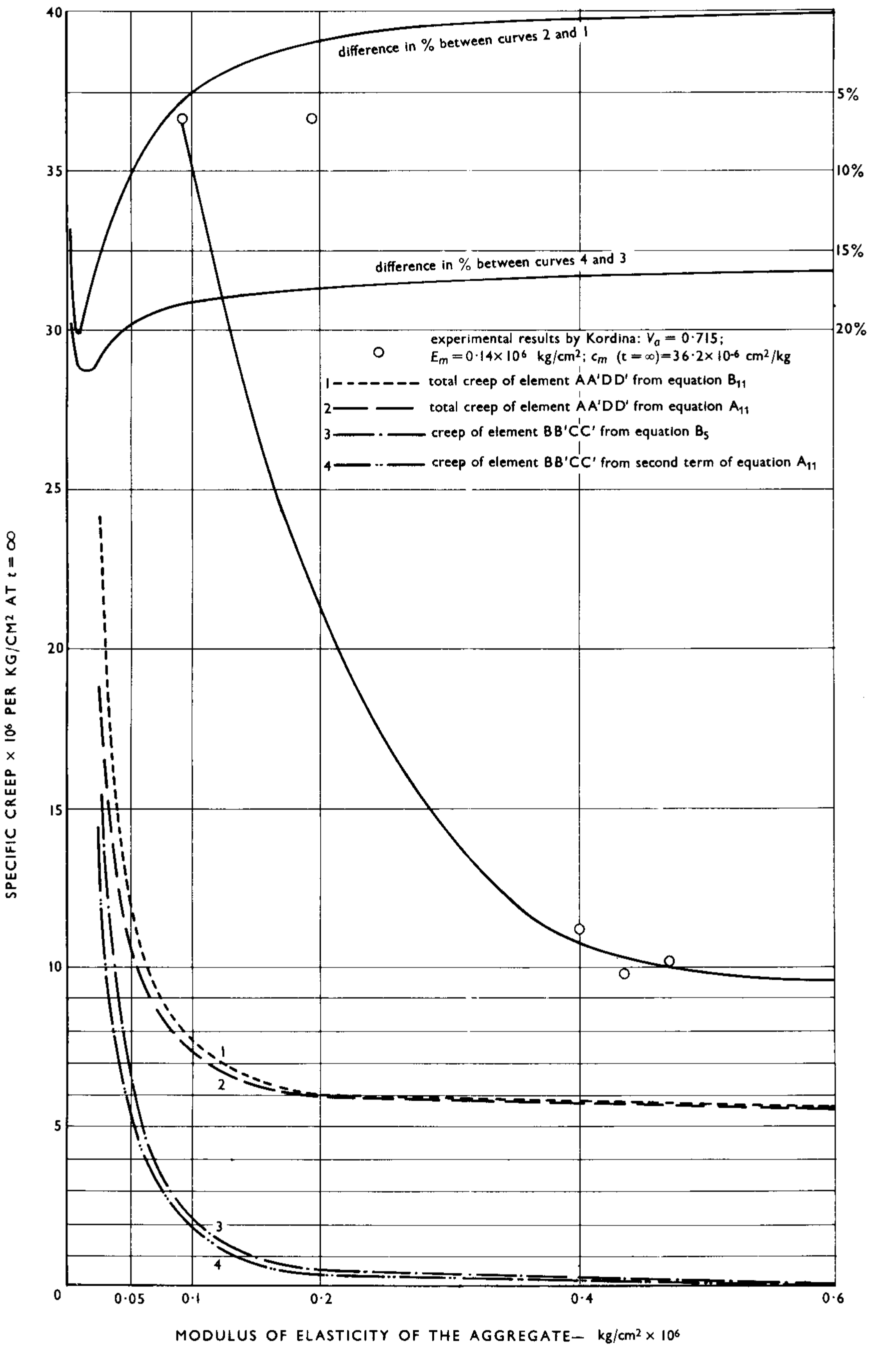

Figure III

element). In addition, the total creep recovery strain given by the proper interpretation of the model (equation VII) is small compared with the total creep strain. This is particularly true in the presence of aggregates with a high modulus E_a. In fact, the creep strain takes place mainly in the element in series (AA'BB'+CC'DD'); creep of the combined element is then very small and so also is the creep recovery, which can be only a small proportion of the creep strain itself.

It is important to remark that this theoretical prediction finds very little support in experimental data; on the contrary, high values of the ratio of creep recovery to creep are also found in practice with aggregates that have a high modulus E_a.

Another cause of criticism is the fact that, for a certain range of values of E_a(1), there is little agreement between the theoretical creep, given by model 4, and the experimental values, even if equation XI is applied. We would think therefore that Mr Counto's statement on this proposal should be more cautious.

Proceeding in analogy with Mr Counto's work, Levi tried to control the creep relations derived from model 4, when the real physical (elastic and viscous) properties and volume concentrations of the concrete mix components are attributed to the elements of the model. Levi compared his results with the experimental data obtained by Kordina. From Figure III it can be seen that the agreement with experiment is very limited, both for our equation XI and for equation A_{11} obtained by the effective modulus. In particular the gap between the results is very evident for aggregates with moduli below $5{\cdot}7 \times 10^6$ lb/in^2 (400,000 kg/cm^2). (Mr Counto did not investigate this range of values.)

In our opinion, this important divergence cannot be attributed solely to the effect of differences in the permeabilities of the aggregates, differences which Mr Counto indicates as the second most important factor influencing creep (after the differences in the elastic modulus).

To eliminate the deficiencies of model 4, and its limited agreement with experiment, Levi has suggested two ways of proceeding in order to produce a rheological equation* which will represent, with good agreement with test results, the various relations between stresses and strains in the viscoelastic region.

(1) Use more complex rheological models which represent with better approximation the intimate behaviour of concrete, and derive more complete rheological equations taking into account a larger number of physical factors.(1)

(2) Use a simple rheological model† and determine the rheological constants affecting the equation deduced from the model, disregarding the real nature of the components of the model.(6)

With regard to method 1 above, theoretical speculation has made clear that model 4 should be changed, in the sense of attributing a lower deformability to the matrix. In other words, under the action of the stresses due to the statically indeterminate internal loads, the matrix should present a deformability, both instantaneous and delayed, less than that indicated by tests for the cement mortar alone under an external load. A possible way of obtaining such behaviour would be to assume that the concrete is made of an assemblage of series-parallel elements, similar to model 4, of different sizes (Figure IV).

However, one should be cautious about the difficulties that can arise in assuming models of this kind. The more complex the model is, the larger is the number of parameters affecting the equations; and the keener the theoretical analysis of the intimate behaviour of concrete fibres tries to be, the harder is the determination of these parameters.

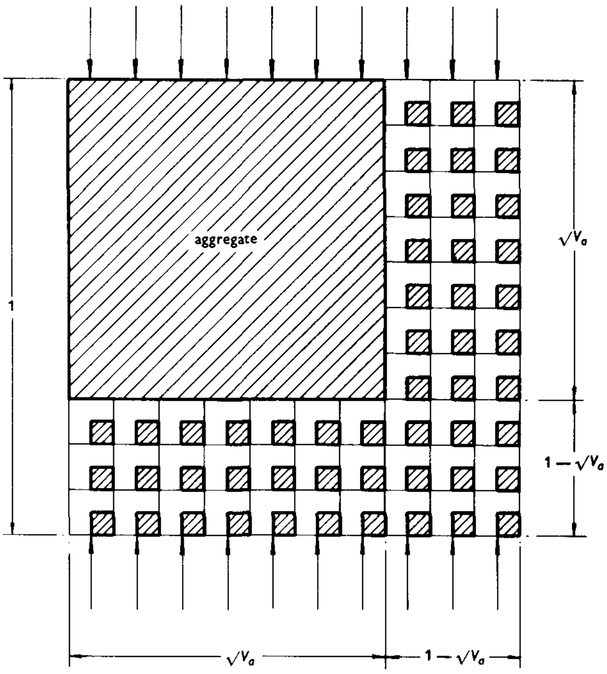

Figure IV

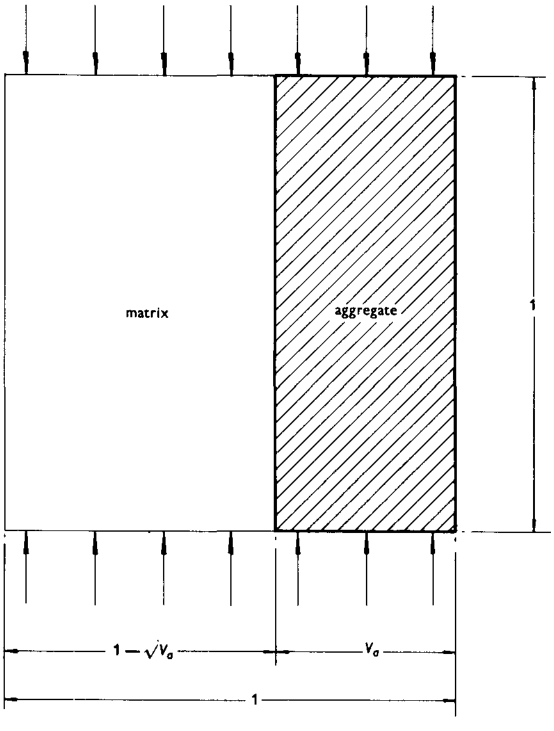

Figure V

In contrast, method 2 represents a typical *a posteriori* procedure. Let us assume, for example, the simple model of Figure V: it has always been accepted that this cannot give an adequate interpretation of the distribution of stresses and strains in the concrete. The constants Q and R of equation I are now given numerical values enabling the equation to provide,

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*The equation must be simple enough for practical application to a wide variety of complex structures, in which the chronological order of loads, of artificial forces and of restraint conditions is conveniently scheduled in order to get well defined statical regimes.

†For instance, model 4 or the even simpler model made up of a parallel element only (Figure V).

through its integrals, a representation of the creep phenomena to fit the experimental data; for normal values of the volume concentration of the aggregate (V_a), this appears to be possible only if we assume, for the constants E_m, $c_{m\infty}$, E_a, of the model*, values having no relation to experience. In other words, the model gives good quantative results only if we attribute a fictitious character to its visco-elastic and elastic components.

In this direction we are conducting a wide programme of research: we have tried to fit the mathematical relations which can be derived from equation I and which represent the different creep phenomena to a large number of experimental diagrams. We have analysed the experimental curves of various authors$^{(2,7,8)}$ and in particular the creep curves of the Comité Européen du Béton$^{(9)}$. Up to now it seems that the following pairs of values should be considered:

$$Q = 2{\cdot}23,\ R = 0{\cdot}49E_c \text{ or } Q = 2{\cdot}89,\ R = 0{\cdot}21E_c$$

for a concrete having a volume concentration of the aggregate $V_a = 0{\cdot}7$, a modulus E_c and a ratio between the specific creep strain at $t = \infty(C_{c\infty}/f_c)$ and the specific elastic strain $(1/E_c)$ of 2·5 ($\varphi_N = 2{\cdot}5$ with *CEB* notation). An adequate law of variation must also be assumed for C_m; preferably, a hyperbolic variation of the type $c_m(t) = c_{m\infty} \times t/(t+\delta)$ should be considered —values of δ around 140 days fit well with the first pair of Q, R values and $\delta = 100$ days fits well with the second pair.

The values of parameters E_m, $c_{m\infty}$, E_a corresponding to the above values of Q and R, no longer have any real meaning; in particular the mortar would appear to have too low a deformability both elastic and viscous and the aggregate to have too high an elastic deformabitity. This result is in accord with method 1 and means that, inside the concrete, the cement mortar is far from being a purely visco-elastic element and, on the evidence, must have all the characteristics of a combined material with a low deformability.

APPENDIX

Theory for model 4

The basic assumptions are the same as those made by Mr Counto, except for assumptions 5 and 6 (assumption 10 is automatically covered by equation I), and the same notation is used.

COMBINED ELEMENT BB′CC′

The general stress-strain relation for an element in parallel such as element BB′CC′ is

$$\frac{\mathrm{d}C_{am}}{\mathrm{d}t} + RC_{am}c_m' - Qf_c c_m' = 0 \qquad \text{(I)}$$

where

$C_{am}(t)$ = specific creep strain of element BB′CC′ under a sustained load f_c at time t

$c_m(t)$ = specific creep strain of the matrix under 1 lb/in^2 at time t

$$Q = \frac{1-\sqrt{V_a}}{\sqrt{V_a}} \times \frac{\dfrac{1-\sqrt{V_a}}{\sqrt{V_a}} + 1}{\left(\dfrac{1-\sqrt{V_a}}{\sqrt{V_a}} + \dfrac{E_a}{E_m}\right)^2} \qquad \text{(II)}$$

$$R = \frac{E_a}{\dfrac{1-\sqrt{V_a}}{\sqrt{V_a}} + \dfrac{E_a}{E_m}} \qquad \text{(III)}$$

By integration with f_c = constant and $C_{am} = 0$ at $t = 0$, equation I gives the equation of specific creep strain under constant unit load for the element BB′CC′:

$$\frac{C_{am}(t)}{f_c} = \frac{Q}{R}(1 - \exp\{-Rc_m(t)\} \qquad \text{(IV)}$$

Since the length of element BB′CC′ is $\sqrt{V_a}$, substitution of equations II and III into equation IV gives the specific creep deformation of the element BB′CC′:

$$\frac{\sqrt{V_a}C_{am}(t)}{f_c} = \frac{1-\sqrt{V_a}}{E_a\sqrt{V_a}}\left(\frac{1 - \exp\left(-\dfrac{E_a}{\dfrac{E_a}{E_m} + \dfrac{1-\sqrt{V_a}}{\sqrt{V_a}}}c_m(t)\right)}{\left(\dfrac{1-\sqrt{V_a}}{\sqrt{V_a}} + \dfrac{E_a}{E_m}\right)}\right) \qquad \text{(V)}$$

The specific creep recovery strain of element BB′CC′ can be calculated by integrating equation I for $f_c = 0$ (see reference 2):

$$\frac{r_{am}(t_3)}{f_c} = \frac{Q}{R}\left(\exp\left\{-R[c_m(t_3) - c_m(t_2)]\right\} - \exp\left\{-Rc_m(t_3)\right\}\right) \qquad \text{(VI)}$$

where

$r_{am}(t_3)$ = (relative) strain in the element BB′CC′ at time t_3 if the sustained compressive stress f_c applied at time $t_1 = 0$ is removed at time t_2 (where $t_2 < t_3$).

As the matrix in series gives no creep recovery (according to equation VIII), equation VI represents the whole creep recovery of the model. Since the length of the element BB′CC′ is $\sqrt{V_a}$ the total specific deformation of the model and its specific strain due to the creep recovery are:

$$\frac{r_c(t_3)}{f_c} = \frac{\sqrt{V_a}r_{am}(t_3)}{f_c} = \sqrt{V_a}\frac{Q}{R}\left(\exp\left\{-R[c_m(t_3) - c_m(t_2)]\right\} - \exp\left\{-Rc_m(t_3)\right\}\right) \qquad \text{(VII)}$$

MATRIX IN SERIES (AA′BB′ + CC′DD′)

The general stress-strain relation for the visco-elastic matrix in series is:

$$\frac{\mathrm{d}C_m(t)}{\mathrm{d}t}\mathrm{d}t = f_m\frac{\mathrm{d}c_m(t)}{\mathrm{d}t}\mathrm{d}t \qquad \text{(VIII)}$$

where

$C_m(t)$ is the (relative) creep strain of the matrix under a sustained stress f_m at time t.

By integrating equation VIII and making $C_m(t) = 0$ for $t = 0$ and $f_m = f_c$ we obtain:

$$\frac{C_m(t)}{f_c} = c_m(t) \qquad \text{(IX)}$$

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*c_m is the specific creep strain of the matrix under a unit load. If a fixed law is assumed for the variation of c_m with time, the value $c_{m\infty}$ of c_m at $t = \infty$ characterizes the viscous deformability of the matrix.

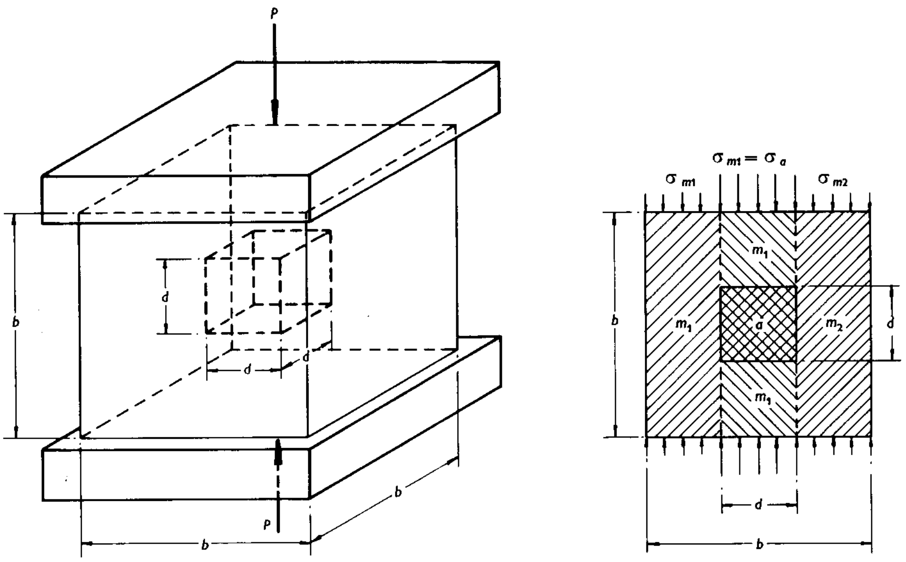

Figure VI: Model representing a heterogeneous element within the concrete mass.

Since the matrix in series has a length of $1 - \sqrt{V_a}$, the specific creep deformation of the matrix becomes:

$$(1 - \sqrt{V_a})\frac{C_m(t)}{f_c} = (1 - \sqrt{V_a})c_m(t) \qquad \text{(X)}$$

TOTAL CREEP STRAIN OF MODEL 4

The sum of equations V and X gives eventually the total specific creep deformation of model 4 [and also its strain (per unit length) since the cylinder AA′DD′ has a unit height] under a constant unit load:

$$\frac{C_c(t)}{f_c} = (1 - \sqrt{V_a})c_m(t) + \frac{1 - \sqrt{V_a}}{E_a\sqrt{V_a}}\left(\frac{1 - \exp\left(-\dfrac{E_a}{\dfrac{E_a}{E_m} + \dfrac{1-\sqrt{V_a}}{\sqrt{V_a}}}c_m(t)\right)}{\left(\dfrac{1 - \sqrt{V_a}}{\sqrt{V_a}} + \dfrac{E_a}{E_m}\right)}\right) \qquad \text{(XI)}$$

Equation XI should be compared with equation A_{11} obtained by Mr Counto and equation VII with equation A_{13}.

Contribution by O. Ishai, B.Sc., M.Sc., D.Sc.

Israel Institute of Technology

Mr Counto's paper is a step forward from the narrow formalistic approach towards a deeper and broader physical analysis. For simplicity, earlier workers constructed inaccurate models of the heterogeneous element comprising an aggregate and a matrix with different elastic properties. Even the most elaborate of these (model 3), however, does not allow for the limiting state, encountered in practice, of a porous elastic mass. This shortcoming is more pronounced in model 1. It is also obvious that in all cases where the rigidity of the aggregate is much lower than that of the matrix, the discrepancy between values calculated according to these models and experimental results will increase, as actually proved in practice.

The shortcoming of model 2 lies in its failure to include the case where the aggregate is much more rigid than the matrix; besides, this model (like model 3) is incapable of representing creep and shrinkage, where the aggregate can be regarded as an inert factor.

Model 4 is self-evident if the object is to represent the actual state of the concrete, in which each grain of gravel or sand is encased in a cement coating.

A similar approach underlies the more accurate multi-phase theories(10) where the heterogeneous material is represented by a dispersion of elastic spheres embedded in an infinite continuous elastic medium, with lateral deformation allowed for. Unfortunately, these solutions, dealing only with low concentrations, are impracticable for the 40–60% C_v range of concrete. The writer has arrived at a model similar to Mr Counto's, in which the aggregate is represented by a small cube embedded in the larger matrix cube, but uniform normal deformation of the specimen surface was assumed (as is the case with any cube compressed between two rigid planes). Mr Counto's assumption of uniform stress distribution is incompatible with the physical boundary conditions in axial loading of heterogeneous material. It is obvious that such a case involves stress variation over the compressed surface, which is easily verified by dividing the element vertically instead of horizontally, as was done by Mr Counto. The conditions are then as follows (Figure VI).

(a) The axial deformation of the whole element is also that of the central heterogeneous column a, m_1 and of the encompassing box m_2.

(b) The stress on matrix m_1 in the column is equal to that acting on aggregate a.

(c) The resultant of stresses acting on both columns equals the external force P.

The above assumptions yield the following four equations.

$$P = \sigma_{m2}(b^2-d^2) + \sigma_a d^2 \quad \ldots\ldots\text{(XII)}$$

$$\frac{\sigma_{m2}b}{E_m} = \frac{\sigma_{m1}(b-d)}{E_m} + \frac{\sigma_a d}{E_a} \ldots\ldots\ldots\text{(XIII)}$$

$$\sigma_{m1} = \sigma_a \ldots\ldots\ldots\ldots\ldots\ldots\text{(XIV)}$$

$$\frac{Pb}{b^2E_c} = \frac{\sigma_{m2}b}{E_m} \ldots\ldots\ldots\ldots\ldots\ldots\text{(XV)}$$

where σ_a, σ_{m1}, σ_{m2} are the stresses on the aggregate, on the mass above and below it, and on the encompassing mass, respectively;

b, d are the dimensions of the large heterogeneous cube and the aggregate;

E_a, E_m, E_c are the moduli of elasticity of the aggregate, the matrix and the whole heterogeneous cube;

P is the external axial force.

Solution of equations XII–XV yields.

$$E_c = E_m\left[1 + \frac{C_v}{\dfrac{m}{m-1} - \sqrt[3]{C_v}}\right] \ldots\ldots\text{(XVI)}$$

where $m = E_a/E_m$ and $C_v = d^3/b^3$ (volume concentration of the aggregate).

Equation XVI, like the others, enables the modulus of elasticity of the composite material to be calculated by means of the components and their respective volume concentrations. Like Mr Counto's own equation, it yields good agreement with experimental results but its physical basis is more correct.

The writer has checked both equations in experiments with heterogeneous cylinders with epoxy-resin matrix and quartz sand ($m = 20$) at high concentration, and equation XVI was found superior.

Each of the above approaches has a range in which it is accurate and realistic, but it is clear that the closer the approximation of the correct physical state, the wider the range of agreement and the more general the equation.

Mr Counto's attempt at generalization of the heterogeneous equations to include creep is significant, but it should be borne in mind that whilst the total creep under constant load can be assumed proportional to the stress and the effective-modulus technique is applicable, this is not so with the recoverable and residual components. In the writer's experience(11) the same relationship does not necessarily exist between these deformations and the variables time and stress, in view of the difference in mechanism between them and total creep. Application of McHenry's theory is not always justified, as shown in reference 12 and by Figures 7–10 of the paper. In the first place, the residual deformation is too high (about 80% of the total creep) to be attributed to the variations in creep at different ages. Secondly, recovery does not resemble creep under load (even at the advanced age of 189 days); its rate is slower and it stabilizes too early.

Contribution by A. R. Mears, M.A., A.Inst.P.

Cement and Concrete Association

As Mr Counto's proposed equation 4 for Young's modulus of concrete only contains the moduli of the constituents and volume concentration of the aggregate it appears to be a general solution to a fundamental problem in elasticity. Recent theoretical work shows that such a solution cannot be found without additional assumptions on aggregate shape and distribution. I feel that the author could have warned us more explicitly of the restrictions implied in his assumptions so that his equation is not used out of context.

The elastic moduli of materials which are mixtures of two solid phases, such as concrete, have been investigated by workers in the field of crystallography. A closely related problem, that of deriving the elastic moduli of a polycrystalline material from the moduli of its single crystal components has also been considered. Solutions have been sought in terms only of the moduli of the phases and their volume concentrations. The two simplifying assumptions, of equal stress in the two phases and of equal strain, were first made by Reuss and Voigt respectively and lead to the following equations.

$$\frac{1}{E_r} = \frac{V_a}{E_a} + \frac{1-V_a}{E_m} \ldots\ldots\ldots\text{(XVII)}$$

$$E_v = V_aE_a + (1 - V_a)E_m \ldots\text{(XVIII)}$$

It has recently been shown(13) that these two values are lower and upper bounds on the Young's modulus E_c of a two-phase mixture when Poisson's ratio of the phases is equal.

$$E_r \leqslant E_c \leqslant E_v \ldots\ldots\ldots\ldots\text{(XIX)}$$

Closer bounds on E_c can be derived from bounds on the bulk modulus K_c and shear modulus G_c of a two-phase mixture proposed by Hashin and Shtrikman(14).

$$K_1 + \frac{V_2}{\dfrac{1}{K_2-K_1} + \dfrac{3V_1}{3K_1+4G_1}} = K_L \leqslant K_C \leqslant K_U$$

$$= K_2 + \frac{V_1}{\dfrac{1}{K_1-K_2} + \dfrac{3V_2}{3K_2+4G_2}} \ldots\ldots\ldots\ldots\ldots\text{(XX)}$$

$$G_1 + \frac{V_2}{\dfrac{1}{G_2-G_1} + \dfrac{6(K_1+2G_1)V_1}{5G_1(3K_1+4G_1)}} = G_L \leqslant G_C \leqslant G_U$$

$$= G_2 + \frac{V_1}{\dfrac{1}{G_1-G_2} + \dfrac{6(K_2+2G_2)V_2}{5G_2(3K_2+4G_2)}}$$

Where K_1, K_2, G_1, G_2, V_1 and V_2 are the bulk moduli, shear moduli and volume concentrations of the two phases. The bounds on Young's modulus can be calculated from the equations

$$E_L = \frac{9K_LG_L}{3K_L + G_L} \qquad E_U = \frac{9K_UG_U}{3K_U + G_U}$$

The authors of inequalities XX state that these bounds cannot be improved when only the phase moduli and concentrations are known. They conclude also that even if statistical details of the phase distribution could be obtained it is not known how they could be incorporated. It appears then that an equation for E_c in terms only of the moduli and concentrations of the component phases cannot be derived theoretically unless further assumptions on phase distribution or equality of some elastic moduli are made. This approach has been followed by Hill(15) who confirms that the bounds derived by Hashin and Shtrikman are the " best possible ".

In the face of this situation, empirical equations for E_c can still be found and used confidently within their range of validity. This approach has been made by Mandel and Dantu(16). Their work confirmed the inequalities XIX but not inequalities XX over a range of E_a/E_m from 2·4 to 73 and V_a from 0·2 to 0·67. Only when using an Araldite matrix for which $E_a/E_m > 7$ were inequalities XX confirmed. It can be remarked that the condition of equality of the Poisson's ratios was not satisfied.

Their empirical equation

$$E_c = \frac{V_aE_a + (1 - V_a)E_m}{1 + \frac{E_a}{E_m}(V_a - 2{\cdot}8V_a^2 + 3V_a^3 - 1{\cdot}2V_a^4)}$$

can be regarded as the Voigt value multiplied by a reducing factor

$$E_c = \frac{E_v}{1 + \frac{E_a}{E_m}f(V_a)}$$

The equations given by Hirsch and the author are nonetheless good fits to experimental data. Equation 3 can be written

$$\frac{1}{E_c} = \frac{1}{2}\left(\frac{V_a}{E_a} + \frac{1 - V_a}{E_m}\right) + \frac{1}{2}\left(\frac{1}{V_aE_a + (1 - V_a)E_m}\right)$$

by putting $z = 0{\cdot}785$. This is simply

$$\frac{1}{E_c} = \frac{1}{2}\frac{1}{E_r} + \frac{1}{2}\frac{1}{E_v}$$

So Hirsch's value is the geometric mean of the two bounds, which might be expected to be a good estimate.

The author's equation introduces the additional assumption that the aggregate has a particular shape. Otherwise his equation 4 is a modification to the Voigt value.

It can be written

$$\frac{1}{E_c} = \frac{\sqrt{V_a}}{E_v} + \frac{1 - \sqrt{V_a}}{E_m}$$

It would be interesting to treat the power of V_a as unknown and derive it experimentally as this might show what the shape factor of the aggregate is for any one type of aggregate.

Reply by the author

I would like to thank Messrs Chiorino, Ishai and Mears for their contributions which I have read with interest.

Most of Dr Chiorino's criticism seems to be directed towards a model which would predict creep and creep recovery of concrete in the most general case. This was, however, not the intention of the author; the tests reported in the paper were aimed at finding the sole effect of the elastic modulus of the aggregate on the creep and creep recovery of concrete. As explained in the paper, natural porous aggregates could not be used for this purpose, and the investigation unfortunately could not include E_a values between $0{\cdot}0425 \times 10^6$ and $10{\cdot}5 \times 10^6$ lb/in^2. There is no doubt that the investigation has revealed the sole effect of the elastic modulus of the aggregate. An attempt was made to explain the experimental results by means of a simple model wherein the effective modulus technique was employed. It is true that there are deficiencies in this technique, but the results obtained by this method are not much different from those obtained from the more elaborate analysis of Dr Chiorino.

Dr Chiorino in his Figure II compares the creep equation A_{11} with the experimental data on concretes prepared with permeable aggregates. However, it must be emphasized that equation A_{11} does not take into account the effects of permeability of the aggregate. Differences in the permeabilities of aggregates can result in concretes with different rates of drying. Also the interactions between cement paste and aggregate are influenced by phenomena located in the cement paste close to the surface of contact with the aggregate. Because of their water absorption and permeability, aggregate can considerably modify the interface between cement paste and aggregate. The effects of these factors on creep of concrete are difficult to determine.

Dr Ishai has proposed a model with ends loaded by a rigid plate, whereas in the model 4 which I proposed, the matrix layers at the ends have been assumed to be rigid. The actual end conditions for a concrete in the interior of a specimen would be somewhere between these two assumptions. Dr Ishai's model was compared with a wide range of experimental data(17-21) and it was found that the average deviation between the experimental and theoretical values of E_c was $+8{\cdot}47\%$ and $-8{\cdot}94\%$. Comparing these with the deviations obtained for model 4, shown in Table 1 of the paper, it is seen that model 4 is a better approximation.

The expressions derived from Dr Ishai's model and model 4 neglect the effect of Poisson's ratio. This simplifying assumption is justified for normal concretes, where the Poisson's ratio for aggregate is nearly equal to that of matrix. However, for a mixture of quartz sand and epoxy resin, used by Dr Ishai, whose constituents have widely different Poisson's ratios, the expressions should be modified to take the Poisson's effect into account.

Mr Mears suggests that the power of V_a in equation 4 be treated as unknown and derived experimentally to get an indication of the influence of the shape of the aggregate on the elastic modulus of concrete. Indeed the aggregate shape and distribution have an influence on the internal stress distribution in concrete. Mandel and Dantu(16) conducted tests on concretes prepared with aggregates having different shapes. They found that the influence of aggregate shape on the elastic modulus of concrete was smaller than the normal spread of results on identical concrete specimens. It is therefore doubtful whether the power of V_a determined experimentally would vary significantly with the shape of the aggregate.

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