Magazine of Concrete Research : Vol. 17, No. 52 : September 1965

The influence of support conditions on the behaviour of square concrete tanks*

by J. D. Davies, M.Sc.(Eng.), Ph.D., A.M.I.C.E., A.M.I.Struct.E.

Contribution by M. J. L. Hussey, B.Sc.(Eng.), Ph.D. *University of London: King's College*

THE ANALYSIS

Dr Davies's paper has provided approximate solutions for the primary bending of the walls of square tanks under a variety of conditions of support. However, corresponding results for the tank floor are not obtainable by the methods there employed. Further, as he points out, the conditions of compatibility of rotation along the wall-floor junction and of equilibrium at this junction are not satisfied. The account given in reference 8 of the paper, which presents the derivation of the approximate differential equation, also indicates that optimum boundary conditions for equilibrium at the free edges of the walls have not been obtained. These difficulties all stem from two problems. The first is to find some function which may represent the deflected form of the tank floor and at the same time provide for perfect rotational compatibility along each wall-floor junction. The second problem is the proper formulation of statical boundary conditions for a variationally derived differential equation. Both problems can be resolved so that an approximate solution may be constructed which complies rigorously with all compatibility requirements and satisfies all equilibrium conditions in the sense that the minimum total potential energy consistent with the assumed deflexion functions is attained.

A deflexion function for the floor

The deflected form assumed for the wall was $\varphi(x)[(a/2)^2-y^2]^2$, so that the rotation at the wall-base is $\varphi'(-a/2)\ [(a/2)^2-y^2]^2$. The deflexion of the floor may be taken as

$$w_F = C\left[\left(\frac{a}{2}\right)^2 - y^2\right]^2\left[\left(\frac{a}{2}\right)^2 - x^2\right]^2$$
$$-\frac{1}{a}\varphi'\left(-\frac{a}{2}\right)\left[\left(\frac{a}{2}\right)^2 - x^2\right]\left[\left(\frac{a}{2}\right)^2 - y^2\right]$$
$$\left[\frac{a^2}{2} - x^2 - y^2\right]$$

which is even in x and in y, symmetric with respect to x and y, and gives

$$\left(\frac{\partial w_F}{\partial x}\right)_{x\,=\,\frac{a}{2}} = \varphi'\left(-\frac{a}{2}\right)\left[\left(\frac{a}{2}\right)^2 - y^2\right]^2$$

as required. C is determined, in terms of $\varphi'(-a/2)$ and the resultant loading, by the usual Galerkin integral condition.

Statical boundary conditions

When an approximate differential equation is developed by the technique of Kantorovich, it usually happens that only homogeneous kinematic boundary conditions can be satisfied exactly.

For example, the exact statical conditions for a free edge $x = a$ if $w \simeq \varphi(x)\psi(y)$, are

$$\varphi''(a)\psi(y) + \mu\varphi(a)\psi''(y) = 0 \ldots\ldots\ldots\ldots (\text{I})$$

$$\varphi'''(a)\psi(y) + (2-\mu)\varphi'(a)\psi''(y) = 0 \ldots\ldots (\text{II})$$

and, unless $\psi(y)$ and $\psi''(y)$ are linearly dependent, these require $\varphi(a) = \varphi'(a) = \varphi''(a) = \varphi'''(a) = 0$

Dr Davies negotiated a similar obstacle in his treatment of the bending moment at the base of the wall by assigning a particular value (zero) to y, and presumably used the same device at the top edge. While this reduces the boundary conditions to the appropriate number, it no longer represents an optimum approximation to equilibrium.

The correct boundary conditions are found by returning to the principle of virtual work. The relevant variational form of all the equilibrium conditions for thin plates is given in reference 2, (page 91, equations k, l and m). Putting $w = \varphi(x)\psi(y)$ and $\delta w = \delta\varphi\psi(y)$ in equation k leads to the differential equation for φ. The appropriate statical boundary conditions for an edge, $x = c$, must be obtained by using the same substitution in equations l and m, along with $\delta(\mathrm{d}w/\mathrm{d}x) = \delta\varphi'\psi(y)$. Then

$$\int\left[D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)_{x=c} + M_x(y)\right]\psi(y)\delta\varphi'\mathrm{d}y = 0$$

$$\int\left[D\left(\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\partial y^2}\right)_{x=c} + V_x(y)\right]\psi(y)\delta\varphi\mathrm{d}y = 0$$

where $M_x(y)$ and $V_x(y)$ are the prescribed edge moment and total edge force respectively, and integration extends along the boundary. The integration removes the variable y from the boundary conditions, and the physical interpretation is obvious. If several approximating functions are used, the boundary conditions for all of them may be found in this way. To match moment conditions at the junction of two plates, the interactive moment $M_x(y)$ is introduced for each plate and may be formally eliminated between

*Pages 153 to 165 of *Magazine* No. 48.

the boundary conditions for the two plates. It is to be noted that this procedure is barely more arduous than Dr Davies's rather unsatisfactory device.

THE EXPERIMENTS

The excellent agreement of Dr Davies's experimental results with his calculations could well provoke accusations of pedantry respecting the above criticisms. But generally a restrictive assumption as to the deformation of a plate will lead to an over-estimate of stiffness and a considerable misrepresentation of curvatures. In most of the given cases this expectation is weakened by the relaxation of the compatibility condition at the junction of wall and floor. However, in case (a), compatibility is maintained with the rigid floor. Here, Dr Davies's computed deflexions at the centre of the wall and at the middle of the top edge are, respectively, about 17% and 9% less than the values determined from a series solution of the problem (see reference 2, page 216, Table 45). Dr Davies has remarked that perfect fixity of the base was not realized in the experiments, so that an experimental deflexion at the centre of the wall about 20% greater than the result of the approximate analysis would be expected. But no discrepancy as large as this is in evidence.

If the absence of the expected disparity is attributed to an incorrect evaluation of the flexural rigidity of the Perspex, which was a common factor in the reduction of all the experimental results, then any correction must be applied to all the results. This leads to the possibility of a large systematic error in Dr Davies's analysis, and it is suggested that failure to comply with a number of boundary conditions contributes to this. It is considered, however, that even using the consistent boundary conditions, a one-term approximation would hardly suffice to confine errors in curvature within ± 10–15%. Timoshenko obtains, by Kantorovich's method, a solution for the much simpler problem of the clamped square plate under uniform load. No approximate static boundary conditions are involved, yet the moments M_x and M_y at the centre of the plate differ by 10%, and the larger is 13% in excess of the result obtained by the series solution.

Reply by the author

Dr Hussey is thanked for his interesting contribution. Unfortunately, he has misinterpreted the main aims of the paper which were specifically to investigate the effects of various support conditions on the magnitude of the bending moments governing the design of the wall and floor sections of concrete tanks. It was not suggested that the method of analysis outlined complied rigorously with all compatibility and equilibrium conditions. As stated in the conclusions, " The method of analysis described represents a middle course between the classical rigorous approach and the semi-empirical approximate approach to the solution of plate flexure problems ".

As is well known, the size of concrete sections for liquid-containing structures is based on the maximum bending moment, and the proportion of reinforcement is small. Thus, what the tank designer requires is a method of analysis which permits a rapid means of assessing the *order of magnitude* of the maximum values which control the design. Whilst the distributions of bending moments are of interest they are not of overriding importance. Furthermore, if magnitudes and distributions can be estimated fairly quickly to within 10%, say, then such a method of analysis should be of direct assistance in design. It is not claimed that the method fulfils all the requirements of classical plate theory and, when the variables in material properties and support conditions are considered, a fully rigorous approach is not warranted.

The deflexion function suggested by Dr Hussey for the floor slab may well prove more accurate for plates under uniformly distributed loads. However, it is restricted to " one-off " problems. One of the advantages of the method described in the paper is its flexibility in coping with various variations in loading and shape. Another advantage of the Kantorovich approach is that it tends to give values which are conservative but not excessively so.

It is doubtful whether Dr Hussey's suggestion for using several approximating functions would only be " barely more arduous " than the single function used in the paper. As the several functions have to satisfy the net boundary conditions simultaneously the computational effort must be increased substantially.

With regard to the experimental work, it has not been the author's experience that curvature cannot be measured with reasonable accuracy, provided the requirements of the small-deflexion theory are maintained. Dr Hussey remarks that the computed values of wall deflexion are not in close agreement with the figures given in Table 45, p. 216 of reference 2. It must be pointed out that the figures in this particular Table are notoriously inaccurate for a plate with unit aspect ratio and are derived from a finite difference solution and not a series solution as suggested by Dr Hussey. For example, the maximum bending moment at the base of a hydrostatically loaded wall with three edges clamped and top edge free is given as $0{\cdot}0299qa^2$ whereas the true series solution gives $0{\cdot}0353qa^2$. The Kantorovich approach yields a solution of $0{\cdot}0372qa^2$ which is 5% more than the correct series solution (and hence on the safe side).