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A mathematical model for the failure of cement paste and mortars *

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By using statistical analysis, Mr Dougill has successfully illustrated the effects of heterogeneity upon the mechanical behaviour of concrete subjected to tensile stresses. He assumes that stresses are statistically distributed at the microscopic level and this results in a gradual progressive failure at that level. No attempt was made to correlate the microscopic unit or element with concrete physically. Because of the lack of a physical resemblance between the mathematical model and concrete, only *a posteriori* explanation of mechanical behaviour was possible. In addition, Mr Dougill has attempted to apply this analysis to both hardened cement paste and mortar, in spite of the fact that both the physical structure and the mechanical behaviour of paste and mortar are significantly different.[1] The same limitations plague most of the structural models which have formerly been proposed, including that of Brandtzaeg. For a clearer understanding of the behaviour of the material and its prediction, it seems desirable to have a mathematical or a structural model which physically resembles concrete so that the necessary parameters can be obtained *a priori* and experimentally.

Shah and Winter[2] have attempted to develop a behaviour theory of concrete subjected to uniaxial compression which is based upon a micro-element (or a structural unit) which physically resembles concrete. The parameters necessary for predicting the mechanical behaviour of this element can be and have been obtained experimentally and *a priori.* The statistical variations among these elements are taken into account by an analysis which is not too unlike that used by Mr Dougill. Regarding his statistical analysis, Mr Dougill mentions that no previous theory based upon the weakest link concept has used the idea of the progressive fracture of the micro-elements. A. I. Johnson[3] in 1953 has very extensively developed the same concept. Shah and Winter's theory is in part based on Johnson's work. Some of the pertinent aspects of this theory are briefly described in what follows.

The micro-element in this theory consists of a circular cylindrical piece of aggregate embedded in a matrix of mortar (see inset of Figure 1). The relative dimensions of the matrix and the particle (d/r) depend upon the relative amounts of mortar and aggregates in the mix. With an increasing load on the element, it is assumed that the unit behaves elastically until the unconfined bond strength of the most unfavourably oriented part of the interface between aggregate particle and mortar matrix is exceeded. At this location an incipient microcrack is formed which will spread in both directions as the load is increased. This assumption was verified with a separate study of the specimens containing isolated interfaces. It was observed during this study that the strength of the interface can be predicted by the Coulomb-Mohr failure theory, and that the cracked interface behaves pseudo-plastically.

* Pages 135–142 of *Magazine* No. 60.

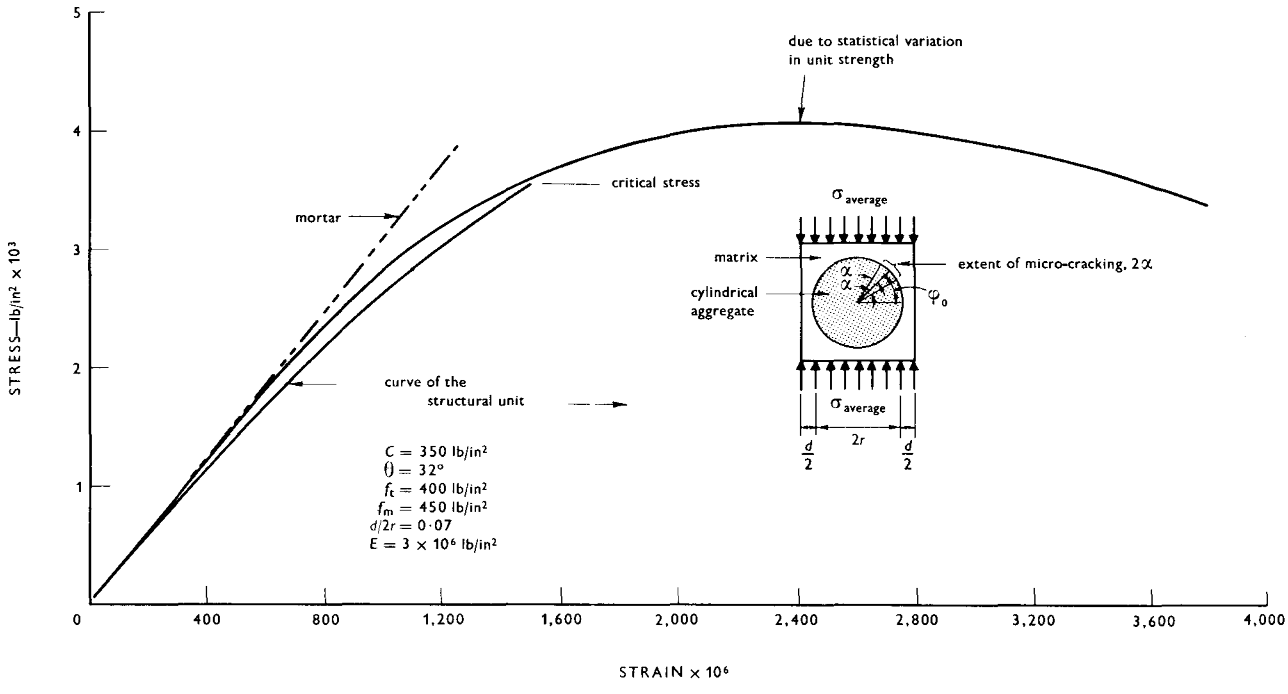

Figure 1: Calculated stress-strain curves.

In the structural unit, after a part of the interface is cracked, there will be a redistribution of stresses and the matrix and the uncracked portion of the interface will carry more than their elastic share of the total load. Owing to progressive interfacial microcracking, the average stress-strain curve of the unit will be non-linear. It is possible to calculate the stress-strain curve of the unit on the basis of equilibrium, compatibility, experimentally observed behaviour of the isolated confined interfaces, and certain simplifying assumptions. The quantitative information necessary for this calculation includes known properties of the material such as interfacial cohesion and friction, tensile and compressive strengths of mortar matrix, and modulus of elasticity of mortar and aggregates. Such a curve for an indicated set of realistic material constants is shown in Figure 1. The peak of the curve corresponds to the critical stress at which it has been found that significant continuous microcracking begins and that the volume of the concrete begins to expand rather than continuing to contract.(1) The analytical part of this theory is similar to that used by Brandtzaeg(4), except that physical correlations now exist for analytical parameters. Like Brandtzaeg's theory, this too predicts higher biaxial and triaxial than uniaxial compressive strength of concrete.

In this theory, the micro-behaviour is extended to macro-behaviour by assuming that the bulk concrete consists of large numbers of such micro-elements, the individual strengths of which show a certain amount of scatter. It has been assumed that the strengths of these units have a Weibull type of distribution. When concrete containing these units is loaded, those with the lowest strength will fail first, while those of higher strength will continue to function, producing stress redistribution and a progressive failure process. The statistical analysis used was similar to that described by Mr Dougill, except that he assumed that micro-elements had the same strength but that their stresses were normally distributed. In addition, the parameters for the distribution of the strengths of the micro-elements were rationally calculated from the statistical properties of the bulk concrete in the case of Shah and Winter's work, based upon Johnson's statistical analysis. A stress-strain curve for a macro-element (bulk concrete) is shown in Figure 1.

The theory proposed by Shah and Winter has many limitations and is based upon many simplifying assumptions. It is only an attempt to combine rationally statistical analysis similar to that described by Mr Dougill with experimentally observed micro-behaviour and realistic micro-elements for the case of concrete subjected to short-term uniaxial compression.

Reply by the author

I am grateful to Dr Shah for presenting some interesting data on the deformation and failure of concrete in uniaxial compression. Although this is not directly concerned with the subject of my paper, it does illustrate an alternative approach to the type of problem considered by myself, and it is instructive to examine our separate approaches in some detail.

If a complete formal analysis of the behaviour of a heterogeneous material were to be attempted, it would be necessary to assume or obtain the constitutive relations for each type of micro-element together with a full description of the morphology of each macro-element. With this information, it would theoretically be possible to predict the behaviour of a macro-element and, if it is assumed that all macro-elements are identical, this would give the behaviour of the complete material. In real materials there may be significant variations between the macro-elements and these should be included in the description of the material, with a consequent increase in analytical complexity.

Although the analytical and computational difficulties of such an approach are not negligible, the more fundamental difficulties are those of specifying the structural properties and morphology of the material. As yet these difficulties have not been overcome and the formal study of composite materials has been limited to specialized or idealized geometries.

One approach to these difficulties is to assume that each micro-element is deformed identically. On this basis, the stresses in each micro-element can be obtained, using some assumed constitutive law, and the average stress in the material as a whole follows if the volumetric proportions of the various types of micro-element within the material are known. This is the most usual approach. It was used by Brandtzaeg[4] in developing his model and by a number of other workers to devise models exhibiting plasticity and strain hardening. Experimental results by Dantu[5] and Anson[6], as well as those of Shah and Winter[2], suggest that the assumption of uniform micro-strain is unlikely to be suitable in a study of concrete and I took the view that it would also be unsuitable in a study of mortar or hardened cement paste.

Shah and Winter have attempted the formal solution. In my terminology their structural unit is essentially a macro-element with an idealized geometry. They use Johnson's[3] procedures, which include the uniform deformation assumption, to determine the behaviour of a material comprising many macro-elements of varying strength. Of greater immediate interest is the way in which the properties of the structural unit were obtained. Unfortunately, this is not fully described in their paper[2] and, although it is clear that an important contribution has been made to the understanding of interfacial effects, it is not clear how accurately these are reproduced quantitatively in the derived behaviour of the macro-element.

My own approach to cement paste is concerned more with fracture than deformation and is based on the concept that a material fails, under a particular loading, when there is no possible equilibrium state for that situation. Only the stresses are needed to establish the equilibrium condition and thus, rather than adopt an idealized over-all geometry and a constitutive law for each type of micro-element, I assumed a rule for stress redistribution during progressive failure. Clearly, such an assumption, taken together with the assumed distribution of micro-stress, implies some form of structural behaviour and geometry. Possible the assumption used was too restrictive to define completely all the classes of material considered. A better approximation might have been to take the heterogeneity parameter α as some function of the amount of stress redistribution, so that an increase in heterogeneity would accompany stress redistribution and micro-cracking. However, this modification would not influence the qualitative trends shown in the paper and it is these trends that I consider to be fundamentally more interesting than a quantitative comparison with experimental data. Most interesting is the agreement with the experimental trends obtained by Hughes and Chapman for mortars, together with the *a priori* suggestion that this behaviour may possibly be observed in cement pastes, if a suitable experiment can be devised.

Discussion of behaviour under more general states of stress will be left for another occasion.

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