

Application of mathematical programming to yield-line analysis*

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Dr Palmer's paper deals with an interesting problem in yield-line analysis: how may the pattern of yield lines which maximizes the ultimate moment be found? Mathematical programming methods can be of great assistance in solving such problems, but we believe that the use of Powell's method in the way suggested in the paper cannot be recommended as a general procedure.

Powell's method is an unconstrained method and cannot handle geometric constraints. Dr Palmer correctly formulates the first example which consists of minimizing equation 1 subject to the constraints 2. He then proceeds to ignore the constraints and minimizes equation 1 by using Powell's method. He is very fortunate that the results obtained in this way happen, coincidentally, to satisfy the constraints which he has ignored.

A similar procedure has been followed in the second example which consists of maximizing equation 7 subject to the constraints 4 and 6. Dr Palmer ignores the constraints and simply uses Powell's method to maximize equation 7. Fortunately again his results also satisfy the constraints, but there is no a priori reason why they should.

The question immediately poses itself: what should the engineer, who has programmed his specific yield-line problem for Powell's method as is suggested in the paper, do if his results do not satisfy geometric requirements? Dr Palmer's answer, towards the end of his paper, is that he must then use some other method such as SUMT or Rosenbrock's method, in which event the time spent on Powell's method will have been wasted.

The fact that Dr Palmer's procedure requires an almost trivially small amount of computer programming effort in addition to Powell's sub-program is of little account if the results obtained from it do not satisfy geometric requirements. This procedure is unsound and, in cases where constraints may or may not be active, they should always be included ab initio in

whatever mathematical programming method is being used. We therefore suggest that any method of constrained optimization is in all cases preferable to Powell's method.

The object of the paper, however seems to be to emphasize how useful mathematical programming can be in yield-line analysis and we support this entirely. We have used constrained optimization techniques in investigating far more complex yield-line problems than those discussed by Dr Palmer. The slab in question formed a bridge deck, curved in plan, containing four variable steel areas: top longitudinal, bottom longitudinal, top transverse and bottom transverse. The slab could carry three loading cases corresponding to different combinations of HA and HB loading. For a given depth of slab it was required to minimize the total volume of reinforcement

For a particular set of reinforcement areas, several different types of yield-line pattern were examined for each loading case. In fact, for the three loading cases, a total of eleven different general patterns were investigated. For each failure pattern, the yield lines were generally specified and a mathematical programming procedure was used to vary the pattern until the ultimate moment was maximized, i.e. a procedure similar to that outlined in Dr Palmer's paper. Each pattern of the eleven, therefore, produced a maximum ultimate moment, the maximum of which was then taken as the slab ultimate moment. Thus, for each set of reinforcement, a slab ultimate moment was obtained.

A second optimization routine was then carried out by varying the four reinforcement variables until the steel areas which gave the minimum volume of reinforcement were obtained. The lower constraint on the individual steel areas was the minimum acceptable area of reinforcement as specified by the Draft Unified Code for Structural Concrete; the upper constraint was generally defined so that an excessive area of steel would not be used.

Mathematical programming techniques may therefore be used to great effect in yield-line analysis and can provide solutions to a much wider range of more complex problems than those outlined in the paper.

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Reply by the author

In their criticism of my paper, Dr Templeman and Mr Slater assert that I “ignore” the geometric constraints on the yield-line patterns. Their assertion is wholly mistaken. In my example 1, for instance, equation 1 defines the collapse load in terms of the parameters describing the yield-line pattern, and inequalities 2 define the constraints. It is surely natural to examine first the corresponding unconstrained problem, that of minimizing the function defined by equation 1 without any constraints, and then to see whether or not the minimizing values of the parameters satisfy the constraints. If they do satisfy the constraints, the problem is solved; if they do not, then the constraints must be taken into account in the minimization process itself, and several ways of doing this are mentioned in the paper.

The point at issue can perhaps be made more plain by a simple example detached from yield-line theory. Consider the elementary problem of minimizing the function

$$f = x^2 + y^2 - 2x + 4$$

subject to

$$x + y \leq 2$$

The most straightforward procedure is to attack the unconstrained problem first, to observe that f is convex (so that only one stationary value exists), to evaluate $\partial f/\partial x$ and $\partial f/\partial y$, to set these derivatives separately equal to zero, and finally to solve the resulting equations for x and y . The unconstrained minimum is at $x = 1, y = 0$, but must now be checked against the constraint. By inspection, these values do satisfy the constraint, and the problem is solved. If, of course, the minimizing values for the unconstrained problem had failed to satisfy the constraint, the problem would not have been solved, and one would have to resort to a constrained minimization technique, such as the use of Lagrangian multipliers. The implication of Dr Templeman and Mr Slater’s comment is that the simple procedure outlined above is somehow improper, and that because there is a constraint one ought to bring in Lagrangian multipliers from the beginning, without pausing to ask whether or not the constraint is operative. To me, this seems to complicate the approach quite unnecessarily.

Dr Templeman and Mr Slater also assert that, if the problem turns out to be constrained, “the time spent on Powell’s method will have been wasted”. This, too, is mistaken. If one is going to use sequential unconstrained minimization, or any equivalent penalty function method, one still has to have a means by which the successive unconstrained minimizations can be effected, and Powell’s method is quite suitable. Suppose, for instance, that, in Example I of the paper—

$$f = 24 \frac{1 - 2v + \left(1 + \frac{m}{M}\right) \frac{uv}{u - v}}{1 - 2uv^2} \dots\dots\dots(1)$$

subject to

$$\left. \begin{aligned} 0 &\leq u \leq 1 \\ 0 &\leq v \leq 0.5 \\ u &> v \end{aligned} \right\} \dots\dots\dots(2)$$

—the unconstrained minimum had turned out to violate the constraints. It would then be simple to modify the function f by adding to it a penalty function which becomes large as the constraints are approached, thus

$$f^* = f + r \left(\frac{1}{u} + \frac{1}{1 - u} + \frac{1}{v} + \frac{1}{0.5 - v} + \frac{1}{u - v} \right) \dots\dots\dots(3)$$

where r is a positive multiplier, and then to use Powell’s method to carry out a sequence of unconstrained minimizations of f^* , starting with a large value of r and examining the sequence of minimizing values of u and v for successively smaller values of r . This is the SUMT technique, and is described in great detail in reference 11 of the paper. As far as a computer program is concerned, only five Fortran statements need to be added to modify the program to use this technique: it seems exaggerated to claim that the earlier effort was “wasted”.

I thank Dr Templeman and Mr Slater for their interest, and am delighted to learn that they have successfully applied the method to more ambitious problems.