

The plastic flow law for reinforced concrete beams under combined flexure and torsion*

Neil Jackson and Romeo A. Estañero

CORRIGENDA

(a) Under **Notation**, for f_w read f_{wy}

(b) In Table 4, Beam No. AU-1, column headed ϕ_{pL} , for 8.3 read 16.6

Contribution by Dr G. S. Pandit†

Malaviya Regional Engineering College, Jaipur (Rajasthan), India

As a result of considerable research endeavour in the past decade, a good deal of understanding of the strength of concrete beams in torsion combined with flexure and transverse shear has been achieved. It is now time for attention to be directed towards its application in the analysis of concrete building frames, grids, interconnected beam systems for bridge decks and other statically indeterminate structures in which significant torsional forces occur. The paper under discussion is a welcome contribution towards this end.

Dr Jackson and Dr Estañero's attempts to establish the plastic potential flow law or the normality criterion for reinforced concrete beams are appreciated. It is possible that the criterion, if established conclusively, may find useful application in the analysis of structural systems in which significant torsional forces exist, particularly in the linear programming technique for collapse load⁽¹⁾. The influence of torsion, axial forces and flexural shear may be incorporated in the analysis by multiplying the fully plastic moment capacities of members by suitable reduction factors⁽²⁾.

I am not, however, convinced that the tests reported by Dr Jackson and Dr Estañero establish the normality criterion conclusively. In the first place, the assumed interaction curve represented in the non-dimensional form by the circular curve $\bar{M}^2 + \bar{T}^2 = 1$ does not reflect the trends which are now generally accepted. Tests at the University of Alberta⁽³⁾ were the first to show that, for unsymmetrically reinforced beams, the torsional strength in combined loading may exceed the

strength in pure torsion, thus producing a hump in the interaction diagram. The gain in torsional strength of unsymmetrically reinforced beams has been recognized by the ACI Committee 438 (Torsion)⁽⁴⁾. The circular interaction curve evidently ignores this effect. The interaction curves of Figures 7 to 11 also reflect the same trends. The hump is more evident for series GU of Figure 11 than for series CU of Figure 9. This is due to the fact that the asymmetry of longitudinal reinforcement is more accentuated in series GU ($A_{sL}'f_y'/A_{sL}f_y = 0.43$) than in series CU ($A_{sL}'f_y'/A_{sL}f_y = 0.61$). It is interesting to note that the trends are valid irrespective of the width/depth ratio. This is evident from the interaction curves of series CU and EU, the width/depth ratios of which are the reciprocals of each other.

Whilst many test data are now available giving interaction curves for strength, very few data exist giving interaction curves for stiffness in combined loading. The tests reported by Dr Jackson and Dr Estañero provide interesting information on this fascinating aspect of the problem of combined loading. The curves shown in Figure 5 and 6 appear to support an observation I made earlier⁽⁵⁾ that the effect of bending moment is to reduce the torsional stiffness in combined loading. The effect of torsion upon the flexural stiffness is not evident from Figures 5 and 6. This effect can be observed clearly only when the bending rotation in combined loading is plotted against bending moment. I hope that the authors will enlighten readers on this question in their reply. Perhaps they will also explain why flexural rotations were observed in pure torsion tests; I feel that any observed flexural rotation in a pure torsion test can only be incidental.

*Page 169 to 180 of *Magazine* No. 77.

†Senior Professor and Head, Department of Structural Engineering.

Contribution by Dr R. Nagaraja*

*Pant College of Technology, G.B. Pant University of Agriculture and Technology,
Pantnagar (U.P.), India*

Dr Jackson and Dr Estañero deserve to be congratulated for investigating a very useful aspect of structural behaviour, particularly relevant to limit design. Structural elements in practice are very commonly subjected to combined loading. One of the common examples of such a structure is the highway bridge grid subjected to unsymmetrical loads.

I have myself proposed^(6,7) failure mechanisms for predicting the ultimate load capacity of two-beam and four-beam reinforced concrete bridge grids. These mechanisms consisted of bending hinges in the longitudinals, torsional hinges in the end diaphragms and yield lines in the slabs. The item relevant here is the torsion hinge in the end diaphragm elements which have to develop and twist under simultaneous bending moment; this is similar to the problem investigated by the authors. Undoubtedly, the usefulness of the mechanisms suggested depended to a large extent upon the realization of the necessary twist rotations at the critical torsional hinges located in the diaphragm elements. I tested twelve scale models of bridges to failure under unsymmetrical loading. The twist occurring in each diaphragm element was measured at every load stage up to very close to failure. The twist-torque curves for two typical specimens are shown in Figure I. In all the specimens tested, the suggested mechanism developed fully. The torsion cracks first appeared at about 70% of the ultimate load and further the cracks widened under increasing twists up to the final stage. At no stage did any of the torsion hinges result in crushing of concrete locally. Whilst I agree with Dr Jackson and Dr Estañero that no method is available to calculate the necessary twist rotation required for redistribution of internal forces, my experiments demonstrated that the torsion hinges provided enough twist for the purpose, without causing any local failure, in the presence of a bending moment.

As regards the interaction between torsion and moment, the percentages of longitudinal reinforcement in all categories of symmetrically and unsymmetrically reinforced beams except the GU series are above 1%. For such reinforcement percentages, Cowan⁽⁸⁾ and Mirza and McCutcheon⁽⁹⁾ have found an increase in the torsional capacity of a rectangular section in the presence of bending moments. Even when approximated towards the conservative side, the interaction diagram (non-dimensional) would be a square in contrast to the circle mentioned by the authors. In the GU series, which have a reinforcement of less than 1%, the

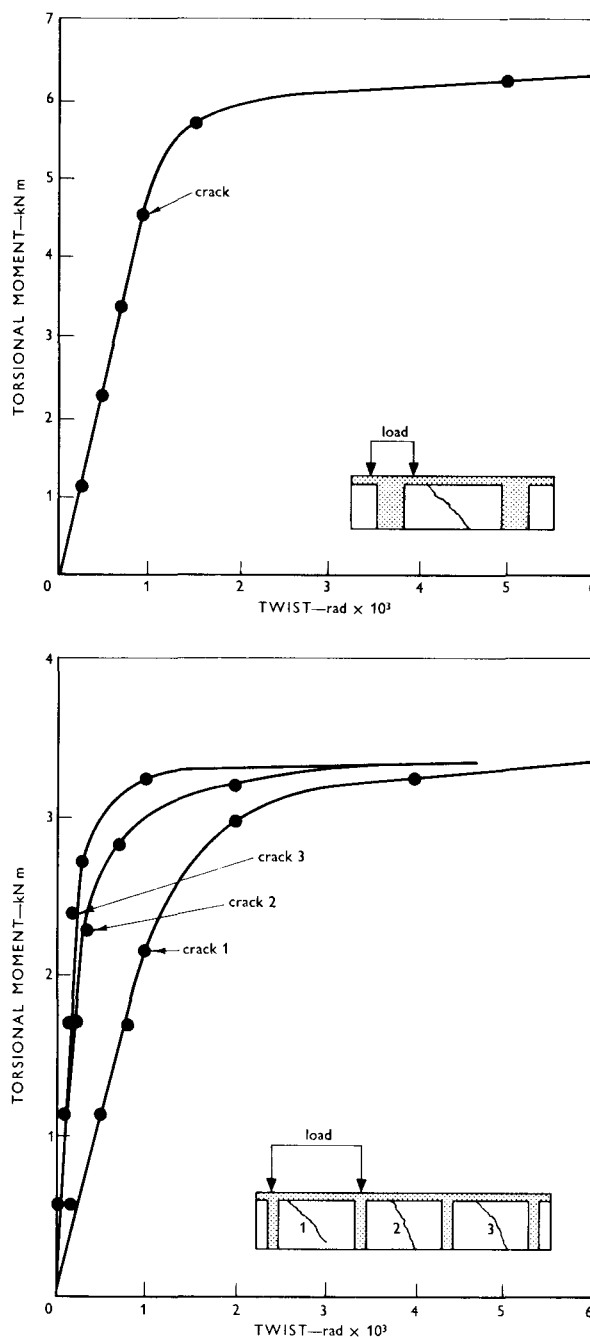


Figure I: Twist-torque curves for typical specimens tested by Nagaraja.

authors' results are somewhat comparable to those of Mirza and McCutcheon.

Further experiments are certainly justified, for in limit design of most planar and space frames, bending and torsion are invariably encountered together.

*Professor of Civil Engineering

Reply by the authors

We wish to thank Dr Pandit and Dr Nagaraja for their interesting contributions. It appears that both are under the impression that the proposed non-dimensional interaction diagram (Figure 12) does not reflect the experimental evidence which indicates an increase in torsional capacity of unsymmetrical reinforced concrete sections in the presence of small bending moments. We wish to assure them that this is not so. Examination of the definition of the non-dimensional parameter \bar{M} shows that for pure torsion $\bar{M} = -(M_p - M_p') / (M_p + M_p')$. For symmetrical sections $M_p = M_p'$ and the corresponding value of \bar{M} is zero. However, for unsymmetrical sections, $M_p \neq M_p'$ and for pure torsion $\bar{M} \neq 0$. For example, if $M_p' = M_p/2$, then for pure torsion $\bar{M} = -0.333$. In this case, for bending moment M_y corresponding to values of \bar{M} between -0.333 and $+0.333$, the interaction diagram shows that the corresponding twisting moments T_y will be greater than the yield moment in pure torsion T_p . The maximum value of T_y occurs when $\bar{M} = 0$.

Dr Pandit asks for information regarding the observed effects of torsion upon the flexural stiffness. The effect of the T/M ratio upon the post-cracking flexural stiffness is shown in Figure II, where F is defined by $F = (M_{pL} - M_{cr}) / (\varphi_{pL} - \varphi_{cr})$, the subscripts 'pL' and 'cr' signifying values at the proportional limit and at cracking respectively (see Figure 3). F_0 is the theoretical flexural stiffness calculated, by using the elastic

theory, for the transformed concrete, the contribution of the concrete in tension being neglected and a value for the modulus of elasticity of concrete based on the ACI recommendations⁽¹⁰⁾ being assumed. Significant reductions in flexural stiffness occur with increasing values of T/M . The beams of all series exhibited similar behaviour.

We disagree entirely with Dr Pandit's view that in pure torsion any observed flexural rotations can only be incidental. The physical explanation of such rotations, for an unsymmetrical section, is that yielding of the top steel alone results in the longitudinal strains being greater in the top than in the bottom of the beam with a consequent upward curvature and, over a length of the beam, a corresponding (flexural) rotation. As discussed in the Appendix of our paper, the measurement of flexural rotations for beams in pure torsion was not entirely satisfactory. Duplicate tests, in pure torsion, have since been performed on beams of some series; on this occasion, the beams were provided with overhanging lengths at the end of the test length so that flexural rotations could be measured directly from the slopes of these unloaded overhanging ends. Details of these tests and also of tests on three additional beams are given in Tables I and II. These results fully support earlier conclusions regarding the directions of the plastic rotation vectors and, in particular, confirm the existence of flexural rotations in pure torsion.

TABLE I: Details of additional unsymmetrically reinforced beams.

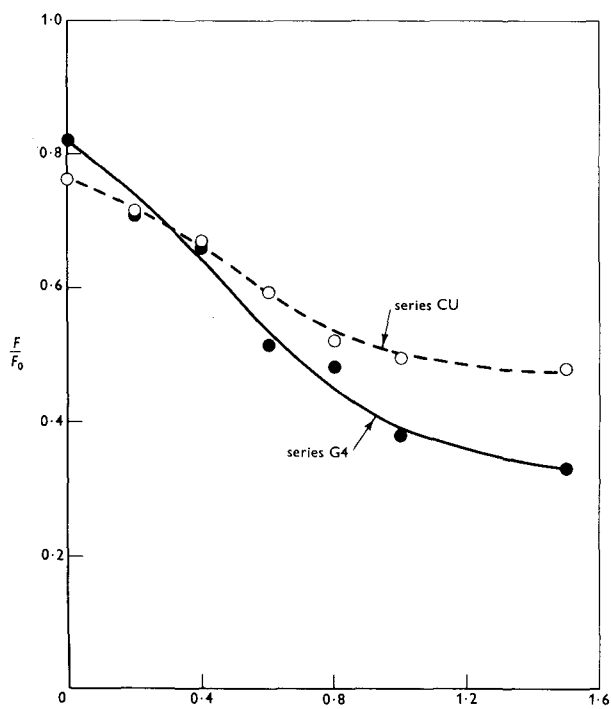
Beam No.	Dimensions (mm)		Concrete properties (N/mm ²)			Reinforcement details (per bar)						
	<i>b</i>	<i>h</i>	<i>f_c'</i>	<i>u</i>	<i>f_t</i>	<i>A_{sL}</i> (mm ²)	<i>A_{sL}f_y</i> (kN)	<i>A_{sL}'</i> (mm ²)	<i>A_{sL}'f_y'</i> (kN)	<i>A_w</i> (mm ²)	<i>A_wf_{wy}</i> (kN)	<i>s</i> (mm)
AU-8(2)	102	203	34.3	38.6	3.1	127	52.1	71	31.6	33	11.4	127
CU-8(2)	152	203	32.6	36.5	2.9	127	52.1	71	31.6	33	11.5	102
EU-8(2)	203	152	30.0	35.4	2.6	127	52.1	71	31.6	33	11.8	102
GU-6I	152	203	41.1	44.6	3.1	33	11.2	82	21.8	28	9.0	51
HU-4	152	203	32.0	35.7	2.7	505	145.5	82	20.5	33	12.0	76
HU-8	152	203	32.6	37.8	3.3	505	145.5	82	20.5	33	11.4	76

TABLE II: Test results for additional unsymmetrically reinforced beams.

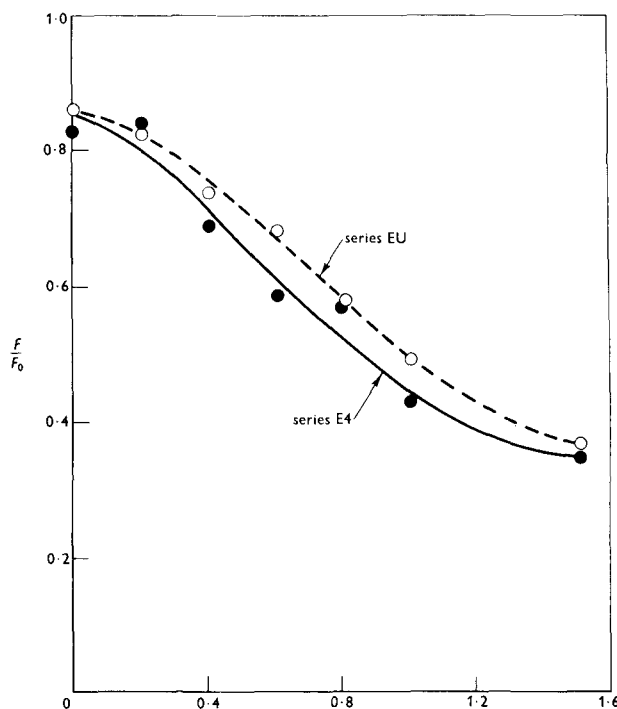
Beam No.	Applied <i>T/M</i>	Experimental moments* (kN m)			Measured rotations over central 762 mm (rad/mm × 10 ⁶)				Axis of rotation (degrees) $\tan^{-1} \theta_p / \varphi_p$
		<i>T_y</i>	<i>M_y</i>	<i>M_t</i>	θ_{pL}	θ_p	φ_{pL}	φ_p	
AU-8(2)	∞	2.82	0	†	52.8	41.8	-1.8	-0.6	91
CU-8(2)	∞	5.29	0	†	45.0	35.0	-3.8	-2.3	94
EU-8(2)	∞	5.06	0	†	43.8	25.3	-3.9	-0.7	92
GU-6I	1.0	2.60	3.26	†	7.2	17.6	5.4	14.3	51
HU-4	0.6	7.70	14.13	†	14.0	51.4	4.0	0.9	89
HU-8	∞	5.81	0	†	42.5	23.7	-6.7	-3.9	100

*Total moments

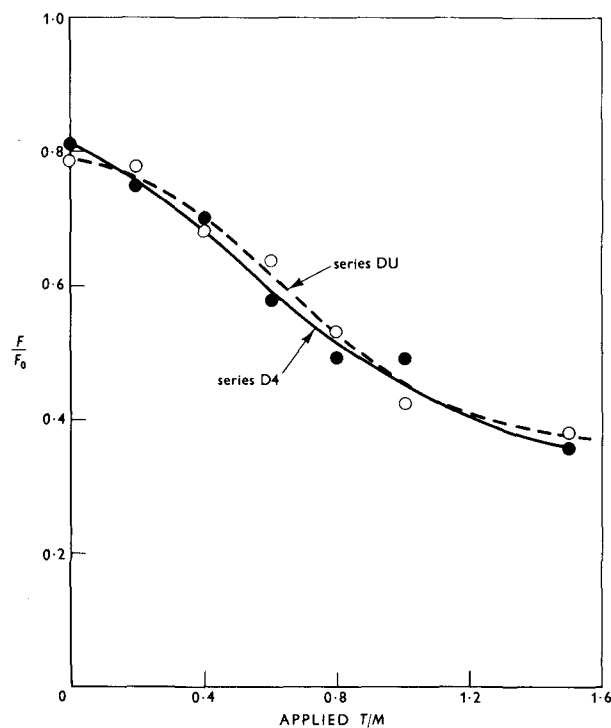
† $M_t = M_y$



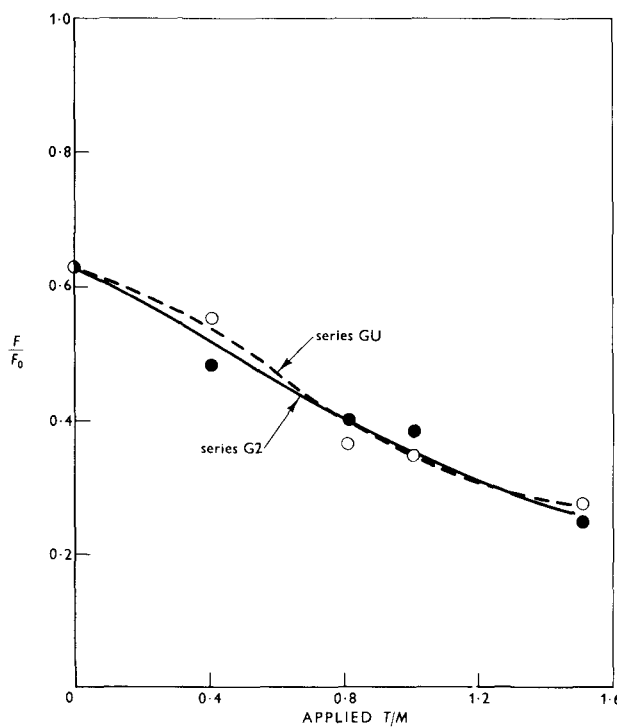
(a)



(b)



(c)



(d)

Figure II: Reduction of post-cracking flexural stiffness in the presence of torsion-experimental trends.

Tests of the type described by Dr Nagaraja are valuable in so far as they indicate that adequate rotational capacity for the development of a collapse mechanism is available for the particular structure and loading considered. A more generalized approach⁽¹¹⁾ is needed, however, to permit estimates to be made of the rotation capacities required for any structure and loading. We are currently investigating the means by which estimates of the inelastic behaviour of trans-

versely loaded reinforced concrete frames might, most conveniently, be made.

Dr Nagaraja's use of the upper-bound theorem of limit analysis in his work implies acceptance of the validity of the plastic potential flow law for reinforced concrete members. His apparent success in predicting the modes of collapse for certain bridge deck structures thus provides interesting circumstantial evidence in support of our conclusions in this respect.

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