Mr. J. W. Grover observed in respect of the head of the Clevedon pier, that considering its horizontal dimensions, 50 feet by 40 feet, it might seem extraordinary that a structure of this kind should have been built on so small a ground-plan. It was not the original intention to have constructed so small a head; but it was consequent on the want of funds to do more. The height, 70 feet, was out of proportion to the width; and in consequence, the head at low tide, from the sea, appeared little more than a tower, and as such, had to be strongly braced, to resist the great force of the impinging tide.

The general idea of a pier was a structure running through water of moderate depth, and a rise of tide of 15 feet or 20 feet at the most; but with a tower 70 feet high in a depth of water of 52 feet, it became a question how to resist the shocks of steamers; and, at the same time, to prevent the structure from offering an undue resistance to the passing volume of tidal water, and to the waves of translation and oscillation to which it was exposed. The fender-piles, therefore, were only 8 inches square, and though that was too large as far as the waves were concerned, it was too small with regard to the vessels. He never intended that the head should be left unprotected, to sustain the shock of sea-going steamers of 500 tons to 600 tons; but had recommended that a lighter should be stationed outside, as at New Passage. This advice had not been followed, and the consequence was the fender piles were constantly being carried away, or the shock from ships soon reduced the front piles to the condition of splinters. Larger ones had been substituted, which had resisted the effects of the steamers; but had the defect of holding the water, and thus bringing an undue strain upon the structure. He thought fenders might be made advantageously of iron where it was desirable to avoid the action of the sea, with spiral springs to act as buffers.

For structures in sea-water more reliance was in his opinion to be placed upon wrought iron than upon cast iron. Although wrought iron rusted rapidly, yet the intensity of the action could be measured, whereas the decay of cast iron was uncertain. He had found cast iron exposed to the sea entirely disintegrated by the action of the water. When the cast-iron skin was unbroken as it left the sand, the coating of silica acted as a natural protection; but in parts where the iron underwent any considerable manipulation, by being turned on a lathe, or being perforated by drilling, that protection ceased to exist. However, the stability of the structure depended upon the wrought-iron ties, which formed the element of its greatest strength.

With regard to this work as a piece of construction, he would draw attention to the employment of arches below as lower members of the girders. The strains on these were proportioned to their depth and angle of thrust. Moreover, it would be possible to saw any girder through at the middle of the span, and yet as cantilevers the halves would stand. He alluded to that because he had heard so many discussions on the question of employing arches as girders, combined with horizontal members, and obtaining the proper strain upon each. He thought it desirable in every case to consider the question as if the girder was intersected at the centre, and that the half-span formed a jib.

The deflection upon each span varied from 1 inch to $1\frac{3}{8}$ inch, the line representing it passing through the structure in a series of waves; and it was greatest at the two extreme ends. The girders were continuous: there was no break at any point, and at no part were they exposed to a strain greater than 4·3 tons per square inch; and that occurred in the fourth span from the abutment.

Mr. ABERNETHY remarked that there was nothing peculiar about the Clevedon pier; many of a similar character had been erected at different parts of the coast. There appeared, however, to be something original in the mode adopted in getting in the foundations of the Cambrian railway viaducts. He understood that at one spot there was a considerable current through that bridge; and that being the case, he conceived that the pier might subside, inasmuch as the mass of stone placed on the sand would be liable to be undermined by the strong tidal flow.

Mr. H. CONYBEARE said that the stones thrown down over the sandy part of the channel had the effect of fixing the sand. The stone was extended beyond the pier, and there had been no signs of subsidence.

Mr. ABERNETHY conceived that the stone should be carried across the channel, and even then he should be apprehensive that subsidence would take place. He thought it would have been better to have dredged out the sand, down to the gravel, and then to have deposited the stone.

Mr. CONYBEARE replied that any sand that might have been dredged out would probably have been replaced the next tide. When the current was rapid, the water looked like pea-soup, from the great quantity of sand in it.

Mr. ABERNETHY said he should have preferred sinking a caisson through the sand; it could then have been filled with stone, and the discs placed upon the top.

Mr. HEMANS, Vice-President, observed that an opening bridge on a railway was, in his opinion, the greatest nuisance an Engineer could encounter. He had built ten or fifteen such bridges across rivers and estuaries; a great proportion of which were wholly unnecessary. There appeared to be at all times a species of mania on the part of those who commanded often the most paltry navigations to demand opening-bridges, which were, in many cases, really of no use. Some that he had built of from 40 feet to 60 feet span had never been opened since the day they were put up, although a great deal both of time and money was expended upon their erection.

The drawbridges on the Cambrian railway had been ingeniously carried out. A similar sliding-bridge he had seen at Rhyl worked exceedingly well; but it was an error to suppose that there were only the two kinds of telescope-bridges mentioned, viz., one rolling over and one rolling under, in addition to the swivel-bridge. There was a third kind, of which he had built specimens over the Severn[1] and the Shannon; and these consisted of a sliding-bridge which did not alter its level; but room was made for it to slide back into a recess behind.

The extreme end of the arm of a swing-bridge moved the length of a quadrant, whereas a sliding-bridge moved only the length of the radius; whatever the weight was, the latter had to be moved two-thirds of the length of the swing-bridge. He was therefore decidedly in favour of the sliding-bridge wherever it could be adopted for moderate spans; besides, it kept the navigation more clear than a swing-bridge with projecting arms, which, unless protected at great expense, as in the bridge over the Ouse, on the North Eastern railway, were likely to interfere with the rigging of vessels in a cross wind.

Mr. REDMAN remarked, that to have erected a landing pier, in an exposed position like Clevedon, 800 feet long and 70 feet high, for the sum of £10,000 was no ordinary achievement. As usual the chief proportion of the outlay and difficulty was incurred on the head of the pier.

There was one question on which he could not help joining issue with Mr. Grover—and that was with reference to the durability of cast iron for works of this nature. He thought the cases referred to, of the rapid deterioration of cast iron, must have arisen from some peculiarity not described, or from the quality of the iron itself. He could refer to works of cast iron on the coast which had been

[1] *Vide* "The Engineer," May 7, 1869.

erected thirty or forty years, and which he knew personally to be perfectly sound and intact.

He considered that the pier at the Barmouth viaduct must depend for its stability upon the series of lower screw-discs used there; and therefore, in all probability, if there were movement of the sand the upper discs would be found rather an encumbrance to the structure than otherwise. The Trinity House authorities, about thirty-five years back, put down cast-iron screws in a rapid tide-way at sea, for the Maplin sand lighthouse, which was built of an octagonal form upon Mitchell's screw piles;[1] and it was there and at the Fleetwood lighthouse that the first application of those piles was made. Some hesitation was felt in adopting that mode of foundation, and to make sure, after the screws had been worked down into the sand, a grating of timber was formed outside, and was floated over the screw-piles. The grating was then sunk by loading it with Kentish rag-stone; but, though intended to protect the lighthouse, it was nevertheless an incubus, for the sand had subsided considerably and the caisson had been rather a dead weight than an assistance. When first placed it was dry at low water, but it was now in a considerable depth of water at that period of the tide.

With regard to the question of the telescope-bridge, in one of the early metropolitan dock works there was a small foot-bridge called Palmer's bridge, which he had been in the habit of walking over, that had precisely the movement described by Mr. Hemans; it was drawn backwards and forwards in a similar manner and was housed below the quay level.

Mr. G. B. Bruce inquired what was the weight of the monkey with which the wooden piles were driven in the case of the Cambrian railway viaducts. He understood that a piece of lead was placed between the top of the cast-iron socket and the monkey; but he could not imagine how a pile under such circumstances could be driven at all, for he thought the monkey would soon crush the lead and knock the cast iron to pieces. He always drove the pile first, then cut it off and afterwards bolted the socket on the top of it.

With regard to the stone-work round the pier, he had no doubt that the stone would sink through the sand, and that the large stones would be found resting upon the gravel in a conical form, and so would afford some strength and protection to the piles.

Mr. G. H. Phipps referred to a drawbridge of somewhat peculiar

[1] *Vide* Minutes of Proceedings Inst. C.E., vol. ix., p. 193.

construction, and the only one upon the London and North Western railway. This bridge, on a plan proposed by Mr. Stephenson, was erected under his directions to carry the railway over a narrow branch canal leading out of the Grand Junction canal at Weedon, in Northamptonshire, into the Government depôt. The peculiarity of that drawbridge was that when required to be opened it was moved in a direction at an angle of 45° with the line of railway, by means of three wheels running upon two lines of rails laid on the masonry beneath the platform of the bridge, and working into racks to keep the motion square on. There was a sufficiency of the platform beyond the side of the canal to ensure the centre of gravity of the bridge being at all times within the triangular space formed by the three wheels, and so to counteract any tendency to tilt over.[1]

Mr. R. P. Brereton observed that the piers at New Passage, which he had erected about 13 miles higher up the Severn, extended farther out from the shore than the Clevedon pier; but the difficulties experienced were very similar, there having been a still greater velocity of current, and a rise of tide of 50 feet; and some of the piles were 85 feet long, and stood 70 feet above the ground. The piers with the construction of which he had been connected were substantial timber structures with spans of 22 feet opening, capable of withstanding rough usage and heavily loaded to resist the sea; and built for the purpose of steamboat and railway traffic, without the light and ornamented appearance which was desirable in the case of a promenade at a fashionable watering-place. The cost of the Clevedon pier, 16 feet wide and 76 feet in greatest height, appeared to be about £40 per lineal yard, which was nearly the same rate as the other piers, but the width being for a double line of rails, about 30 feet, or 50 per cent. wider than at Clevedon, surface for surface, those structures seemed to have been cheaper.

It had been remarked by Mr. Grover, that the most economical span to adopt would be when the cost of the pier was the same as that of the superstructure. This was not necessarily the case and must not be taken as an axiom, as the total cost of a structure might be least when the respective cost of the piers and of the superstructure were very different. It depended materially on the height and number of the piers. Theoretically, the cheapest structure would be where there were no openings and everything was in the piers; on the other hand, a very deep ravine might be crossed advantageously by

[1] *Vide* Papers on subjects connected with the duties of the Corps of Royal Engineers, vol. iii., p. 189.

a bridge without piers, the whole cost being in the superstructure. In an iron structure where the supports could be used to their full advantage without having to fortify them to prevent flexure, as for instance in the rods carrying the floor of a suspension bridge, whatever the height might be, the minimum of material would be found when the suspension rods were close together without any girders between to form the superstructure.

As to the relative durability of wrought iron and cast iron, there were numbers of cast-iron structures in sea-water which had stood for thirty or forty years without symptoms of decay. With regard to wrought iron, no doubt small wrought-iron rods did stand the sea and the effects of salt water very well. He had occasion to use a number of them ten years ago for marking out a line of training bank on the coast of South Wales, and they had not corroded in the least. These rods were about $1\frac{1}{4}$ inch in diameter, standing 10 or 12 feet out of the sand; they extended 2 miles from the shore into the sea, and were frequently subjected to heavy breakers. It seemed that a film or fur, resembling galvanized iron in appearance, formed upon the iron when it was round, but when there were square edges or arrisses to be chafed off, a polished surface, from abrasion of the metal, continually recurred, and fresh coatings of rust were formed, and in time the bars would perish. The commonly-received explanation of the fact that iron articles in salt-water mud remained uncorroded, while they speedily perished in sea-water, was that the iron took from the salt water a certain amount of fixed air which it contained to form rust, but after this the water became comparatively harmless, and when the film of rust was undisturbed the iron remained in its original condition; but where the water was renewed, and the rust got rubbed off, oxidation went on and the article was at length destroyed. In the holds of iron ships there were places where small quantities of bilge water sometimes remained on the plates for months together without increase of corrosion taking place; whereas in other cases, from the action of sulphur and ashes from the furnaces, rivet heads and patches of the plates would be eaten out and the plates would be rapidly destroyed. He did not think there was any peculiarity in the salt water which was particularly damaging to wrought or cast iron; the mischief arose from the surfaces exposed to chafing and the constant renewal of the water.

The different varieties of drawbridge that were to be met with on the South Wales railway did not appear to have been noticed by Mr. Conybeare. Between Gloucester and Milford Haven there were seven opening bridges, of which six had been regularly at work for the last twenty years without any accident; and in one instance, as Mr. Hemans had observed, a drawbridge had been put up but had never been opened. Out of these seven bridges only two were of the same description, owing principally to the different circumstances under which they were erected. Some of them had clear openings of 50 feet, and, being skew, required 70 feet of overhanging leaf.

The bridge described by Mr. Conybeare was apparently a modification, and in one respect rather an improvement, upon the rolling bridge across the river Towey, at Carmarthen, erected in 1852; but it had the disadvantage of requiring a separate bridge to carry its weight when rolled back. There were several kinds of opening bridges, including the bascule, the hinge or lifting, the swing or swivel, and the hydraulic lifting pivot description; but the Carmarthen bridge was the only one of the rolling kind at the time that he was acquainted with. It was not telescopic, but the moving part extended over two entire openings and was rolled backwards above the floor of the adjoining viaduct.

He had first suggested a drawbridge of this kind, in 1850, to cross the Rhine at Cologne, where there was to be an opening of 96 feet, and with no facilities for getting in a large centre pier and fenders for the protection of a swivel bridge. The plan was ultimately adopted by Mr. Brunel at the Carmarthen bridge. Mr. Brunel had been much in favour of the hinge system of lifting bridges, which could be raised for opening and lowered upon dead points or solid bearings free from the trembling tendency of the rollers of a swivel bridge under the weights of some hundreds of tons of passing load besides the weight of the bridge itself; but being skew, and with a span between 50 and 60 feet, the raising in the air of so large a surface of double line bridge would have been a formidable matter. The bridge, as constructed, had no running wheels or rollers, and resembled some that had subsequently been put up at the Swansea docks. The piers required were narrow, occupying but little water-way, and had fixed wheels upon the tops of them; the rails were attached to the underside of the girders, which, when closed, rested upon dead points, and when opened were tilted about 1 foot to clear the rails and then rolled back; and there was no necessity for any under structure to carry the roller. He did not think there had been an earlier instance of that construction than the Carmarthen bridge, built in 1852.

Mr. Abernethy remarked that in one situation it might be desirable to construct a swivel-bridge, in another a telescope-bridge, and in a third a hoist-bridge. Mr. Hemans stated that he preferred the draw-bridge to the swivel-bridge; but, when practicable, he preferred the latter, an excellent example of which had been constructed by Mr. Harrison over the river Ouse, near Goole.

It was impossible to adopt swivel-bridges at Swansea, because the width of the river Towey was only 205 feet, and sufficient water-way had to be given for vessels to pass up and down the river, and the pier necessary for a double swivel-bridge would have occupied too much of the water-way. Moreover, there was the danger of vessels running up the river under canvas coming into collision with the open bridge, and that danger could only be avoided by giving considerable additional width to the pier; he therefore divided the space into three spans, two of 60 feet and one of 61 feet. The opening portion of the bridge crossed two of these spans, and passed over fixed rollers on the pier and west abutment. When closed, the bridge formed a gradient of 1 in 56 with the load. In opening the bridge, the end resting on the second pier was released, which caused the bridge to tilt on the centre rollers, raising the end at the abutment sufficiently high to pass over the fixed rollers, and it was then drawn back by hydraulic power.

The bridge across the docks under other circumstances differed materially from that over the river. To avoid interference with the gearing of the lock gates, this bridge, instead of passing over fixed rollers, rolled back upon a roller path, independent of the lines of rail. It was lifted in the centre by a hydraulic ram; at the end of the arch girders there were horns which locked into three slots in the abutment. When the bridge was raised the horns bore upon rollers at the top of the slot, and that brought the end of the bridge sufficiently high to rest upon the roller path, when it was also drawn back by hydraulic power. The weight of this part of the bridge was 200 tons, and of that over the new cut 205 tons: they had been at work since 1864, and were, on the whole, a success; but he should in all cases, where it was practicable, adopt the swivel-bridge, in preference to the draw-bridge, as being worked more easily, having much less friction, and capable of being opened and shut in much less time.

He thought the profession at large was greatly indebted to the firm of Sir W. Armstrong and Co. for the high perfection to which they had brought hydraulic machines, without which it would have been impossible to work bridges of this magnitude in such a short period of time.

Mr. Hemans, Vice-President, observed that he had visited the New Ross bridge, and could bear testimony that it had been carried out under serious difficulties. There was not a more important road and navigation bridge in Ireland at a junction between two counties, and he regarded it as one of the most perfect of its kind.

Mr. W. B. LEWIS asked Mr. Maynard the cost per cubic yard of the cast-iron abutment filled in with concrete, so that it might be compared with that of masonry? also the nature and the proportionate quantities of the materials used in the concrete with which the cylinders were filled, and which was said to have cost 21*s.* per cubic yard.

Mr. BRAMWELL enquired whether the alternative plan of a pier composed of four hollow piles, each having a screw pile within, was more or less expensive than that which was actually adopted, and whether Mr. Maynard knew of any instance in which the plan of the hollow pile with the screw pile within had been executed?

Mr. W. LLOYD questioned whether it was always judicious to adopt screw piles in preference to more solid constructions. It was difficult to keep screw piles secure, and he thought, in sinking the piers of the New Ross bridge, greater difficulties had been encountered at some parts of the foundation than at others. In such a case there was the same chance of failure as he once encountered in a river bed in South America, which in the time of flood was a quicksand. The piles were screwed down to what he thought was a perfectly secure depth; in fact, till they were twisted almost in the shape of a corkscrew. This was in the dry season; but when the rainy season had set in, with 20 feet of water on the sand, it was found that the piles were standing without any foundation, and eventually the pier was swept bodily away. He therefore was averse to using screw piles in preference to the solid pier adopted at New Ross, which he was certain was the safest that could be constructed.

Mr. H. N. MAYNARD stated that since the New Ross bridge was erected, experiments had been carried out to test the sleeve bracing he then proposed, and a great many piers, or rather abutments, had been made on the same principle with satisfactory results. He had now on hand a large number of road bridges, and they were all made upon this principle. The abutments were composed of a number of cast-iron columns or sleeves braced together with plates and angle-iron entirely filling up the spaces between the piles. They were placed in position with wrought-iron screws of sufficient length to reach through the top of the sleeves; and the capstan was fixed on and the screws driven to the required depth. The set of four columns were lowered into the river, and by the capstan the piles were driven down to the hard ground. The number of columns varied, according to circumstances, from three to five or seven. There were four in this case, as giving the proper sectional area to the pier. In some instances this plating was carried all the way up. It was weighted down and the material dredged out from the inside, and after the whole was sunk as far as the dredging would allow, the screws were driven to the requisite depth. The abutments he had now in hand, for the most part, weighed 9 tons, and were about 14 feet in height for roadways that were 18 feet wide. The quantity of iron, and consequently the cost, depended upon the width of the roadway and the height of the abutment. He could not, therefore, give any direct comparison as to the relative cost between an abutment upon this principle and one of masonry; but in a country where masonry was expensive, the facilities for erecting this kind of iron abutment were so great as to counterbalance any trifling difference of cost. Abutments of iron on this principle might in some countries cost a little more than abutments of masonry, but against that must be set the rapidity of execution and the facility of transporting the materials ready to be put together. The people who erected the iron portion of the New Ross bridge were capable of erecting the iron abutments at the same time, and making the work perfect as they went on without the employment of masons.

This sleeve-braced structure might be built to a considerable height, and the braced cluster lowered through a depth of 150 or 200 feet of water, and the screwing process carried on by a capstan above, the structure forming its own staging. It would be found very inconvenient, perhaps impracticable, to sink cylinders through so great a depth; but there was no practical difficulty in carrying out a pier of this kind through that depth of water, and it was intended for such conditions. The braced cluster weighed less than 5 cwt. per foot in height as compared with a cast-iron cylinder of equal strength at 14 cwt. per foot in height. The abutment rested not upon the flanges of the sleeves, but upon the screw blades, and the bracing would be put upon them at any depth, so as to make the whole cluster act the same as one pile of that size. The bracing served as a temporary stage while the screwing in was being done, as well as for a permanent bracing; and the screws might be driven as far as required through the bracing. The casing or sleeves and bracing took the place of the staging while the screws were being driven; if the bracing was attached to the screw the screw must stop where the bracing held it. The cylinders of the New Ross bridge were filled with Portland cement concrete, at a cost of about £1 1*s.* per cubic yard. The structure might be wall-plated instead of being of open bracing, and filled with concrete, and in some instances he had arranged for the plating to be carried up to the top. Every piece of the abutment wall was of a weight easy to be transported to inaccessible parts of the country, and to be fixed with bolts and nuts. In one case he had one hundred abutments to send out for distribution over many miles of country, where there were at present no roads, the bridges having to be made first and the roads afterwards. The locality was one where masonry and building materials of all kinds were very difficult to obtain, and these abutments were speially adapted to a case of that kind.

Mr. W. SHELFORD said that fourteen years ago he was engaged under Mr. Fowler in the completion of some swing-bridges over the river Nene. One of these bridges, on the pivot principle, measured 104 feet from the centre to the longer end, and 52 feet from the pivot to the extremity of the short end; the difference in length being compensated by a weight placed in a balance box. The width between the girders was 28 feet, and the weight 500 tons. It was worked by the Armstrong hydraulic accumulator; the accumulator weighed 50 tons, and the area of the ram under the bridge was $\frac{1}{10}$th of that in the accumulator. There was nothing new about the construction, but it occurred to him to point out the great importance of time in the working of these bridges, an importance which increased in proportion to the traffic over the bridge and the consequent weight to be moved. The bridge he referred to was a great success in that respect. It could be swung over the river in two minutes, and the time would have been less if there had been sufficient power at command; but it required from four to six men to pump for at least an hour in order to get this 50-ton accumulator to a sufficient height to lift the bridge. It was originally intended to have a steam-engine to lift the accumulator in a few minutes to the required height, but that was abandoned. In point of fact the bridge was built by order of the Admiralty at other people's expense, and was never used simply because it was never wanted.

There was another bridge about 5 miles below Peterborough of a similar calibre to that described by Mr. Conybeare. It was a swing-bridge worked by a rack and pinion, and the only occasion on which that was opened was under peculiar circumstances. It was not sufficiently secured to its bearings; a gale sprung up one night and the next morning the bridge was found blown over the river.

That was not the first time that he had known such bridges influenced by wind. A bridge which in calm weather could be worked by one man could not be worked at all by him in a wind. He thought Mr. Conybeare's bridge was free from this objection, but was inapplicable to large spans and weights of 500 tons.

Mr. Brunlees said that Mr. Piercy, while the Engineer of the Cambrian railways, had designed the viaduct for the crossing of the estuary at Barmouth, yet his name had not been mentioned by Mr. Conybeare, either in the Paper or in the course of the discussion. He had with him the whole of Mr. Piercy's drawings for the viaduct, and they differed very slightly from those described by Mr. Conybeare, who said that he had designed the discs represented, and had distributed the bearing on the piles by means of the number introduced.

When Mr. Piercy was about to design the viaduct in question, Mr. Brunlees lent him the drawings of several viaducts for crossing similar estuaries, and also advised him as to the best method of getting over the difficulty arising from the stratum of peat that had been referred to. It would be seen, from an inspection of the drawings, that the piles were designed with a disc, and with the same description of screw at the bottom as was shown by Mr. Conybeare, and were in fact identical with those carried out. He merely wished that Mr. Piercy should have whatever credit was due to him, and he hoped that Mr. Conybeare would award it to him.

As regarded the stoning of this viaduct, it was out of the question to suppose that the stone would lie upon fine sand such as was found in the estuary at Barmouth. When the rubble was placed in such a position it presented an obstruction to the tide, which washed away the sand below, and the stone settled down and occupied its place. During the last year he had completed a viaduct across the Solway Frith, nearly 1¼ mile in length, and the stoning was executed in the following manner:—Openings were left at intervals in the superstructure of the viaduct, and through these the stone was 'tipped' in heaps. As soon as the tides acted upon the sand, from the obstruction presented by the stone, it was washed away and the stone took its place. In this manner the whole of the bed of the estuary, in the line of the viaduct, was covered with a platform of stone, to a depth of from 4 to 8 feet, giving great support to the piles. He might mention as a proof of the solidity with which stone so disposed held its place, that the Morecambe Bay viaducts were originally constructed for a single line of railway, but the stone was put in for a double line.

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When the viaducts were widened, large bearing stones were bedded in the rubble and served for the foundations of the columns. The viaducts were thus doubled at a comparatively small cost, and the work had stood without any settlement whatever, showing conclusively how close and compact the stones became after they were once laid down.

Mr. J. W. GROVER had only one observation to make, and that was as to the question of cost, as compared with the piers to which Mr. Brereton had alluded. He found that the actual cost of the body of the Clevedon pier was £6,800, and that divided by 800 feet gave £8 10*s.* per lineal foot, and consequently £25 10*s.* per yard. He understood Mr. Brereton to state that the New Passage piers cost only £40 per yard for a width of 30 feet; but it was necessary to bear in mind that those piers were of timber, with 20-feet openings only, while the structure he had described had the large openings of 100 feet, and was designed to carry a moving load of 16 cwt. to the lineal foot. The actual test weight placed on the centre of one span was 42 tons, a load approaching that induced by ordinary railway traffic. The difference, if any, in cost might be accounted for simply by the fact of iron having been employed, and he need not say that timber was much cheaper than iron in first cost; but notwithstanding that, the New Passage pier seemed to have cost as much per foot, though built in a river, as the Clevedon pier, which was placed in an open sea-way, and exposed to severe gales from the Atlantic.

Mr. VIGNOLES, President, observed that two or three centuries ago, during the great wars in the Netherlands and in Spain, amongst the physical difficulties that were encountered there was particularly that with respect to diversion of streams of water, and obstruction of the passage of troops and artillery across marshy grounds and occasional deep channels. To obviate these difficulties a system was invented, which was still carried out in the north of Europe, of forming fascines composed of osiers, technically named 'matrasses,' of considerable extent, loading them with stones, and sinking them. That plan was carried out for the protection of the low banks of the Elbe, even down to near its mouth, where for a number of years they had constantly required protection against the tides and sea. In the Zuyder Zee this practice was such as to form a remarkable feature there. The original intention was that they should lie nearly flat, while the loading with stones kept them in position; but the action of the water was such that in many cases they gradually sloped down to angles of from 10° to 45°, and at the Helder this had been the case to the extent of as much as 60° to 70°. The Danish engineers in the low grounds on the shores of the Baltic used this system of fortification for banks and foundations to a considerable extent. He had used it himself on a large scale in Russia and in Spain, and had always found it constituted a solid foundation for masonry, and, though the 'matrasses' subsided downwards outside and around the foundation platform to various angles, they always made a good base and perfect protection in depths of water of from 40 feet to 50 feet, and never stirred. With these layers of osier 'matrasses' laden with stones, and sunk around the masonry, whether executed in the shape of an apron across the stream or of piers, the work remained secure, even though the scouring out of the river bed brought them from their original horizontal position to a considerable depression, whether with the stream or otherwise. He believed this was a system which was scarcely known in Britain; but he had found, in dealing with sandy and soft foundations, that it was one of the most perfect kinds of foundation that could be adopted.

Mr. J. R. Mosse remarked, that the large navigable rivers which had to be crossed by railways in the United States obliged American engineers to adopt swing-bridges of timber of considerable spans. Several double timber swing-bridges of 120 feet span each, having a turn-table in the centre, had been constructed with complete success; and Mr. McCallum had designed a timber swing-bridge having two openings of 150 feet each, and resting upon a turn-table 29 feet in diameter, for carrying a railway over the Mississippi river at Clinton, in Ohio.[1] To the best of his knowledge, swing-bridges were more common in the United States than opening bridges of any other description.

Mr. H. Conybeare, in reply to the discussion on his Paper, observed that the foundation of the Aberdovey viaduct consisted of ordinary timber bearing-piles, to the top of each of which, as soon as it was driven down to the level of low-water, a cast-iron splice was affixed and driven home with it until the whole of the timber was sunk below the bed of the channel. The piles of the Dovey viaduct were found to drive quite similarly to the trial piles that had been previously driven as a basis for calculation. The timber piles were driven down to the level of low-water, at which level the cast-iron splices were attached to them by a 21-cwt. monkey, falling 8 feet; the last blow sinking them about ·7 of an inch. The fall and weight required for driving them home, with the

[1] *Vide* Minutes of Proceedings Inst. C.E., vol. xxii., p. 315.

cast-iron splices affixed, below this level, so as to give them the requisite supporting power, were then calculated by the formula of Major Sanders of the United States Engineers, which was based on experiments tried at Fort Delaware, to determine the supporting power of piles, in the year 1849. By this formula $W = \frac{\omega H}{8 H'}$, where W = weight that may be safely placed on the pile; ω the weight of the ram, in tons; H the fall of the ram at the last stroke, in inches; and H′ the distance moved by the pile at the last stroke, in inches. The piles, with the cast-iron splices affixed, were driven home with a 22-cwt. monkey, falling 3 feet; and, for the last two strokes, sunk ·3 of an inch per stroke. This gave, by Major Sanders' formula, 16 tons as the load that might safely be placed on each pile, or 112 tons as the load that might safely be placed on each pier of seven piles. The cast-iron splices were accordingly designed of sufficient strength to bear, without injury, the final stroke required by the formula in this particular case, and they did so. Of course, had a greater supporting power per pile been necessary, and consequently greater stress in driving, the cast-iron splices would have been designed accordingly; for, with proper precautions, it was obvious that an adequately designed cast-iron pile might be driven as hard as a timber one.

The extraordinary divergence of the various formulæ for determining the supporting power of piles was unparalleled in any other branch of engineering.

In the course of the discussion on Mr. McAlpine's Paper on the supporting power of piles,[1] Mr. A. Beazeley exhibited a table of the supporting power of the piles of the Cudgegong bridge, as estimated by various formulæ, showing that the supporting power of each of these piles, as calculated by Rankine's formula, was 5·20 times, and by Weisbach's 5·24 times, that given by Sanders' formula; and that was only $\frac{1}{5}$th the full measure of the divergence of such formula; for Mr. Bruce, in his account of the building of the Royal Border viaduct,[2] appeared to have used a formula that required, in order to obtain a supporting power of 75 tons per pile, a weight of 15 cwt. falling 16 feet, till each pile only moved $\frac{1}{20}$th of an inch for the last fall. Now, this gave by Sanders' formula

$W = \frac{\cdot 75 \times 16 \times 12}{8 \times \frac{1}{20}} = 360$ tons, instead of 75 tons; thus the formula that appeared to have been employed by Mr. Bruce in this

[1] *Vide* Minutes of Proceedings Inst. C.E., vol. xxvii., p. 307.
[2] *Vide* Ibid., vol. x., p. 222.

case gave a supporting power of scarcely more than $\frac{1}{5}$th of Sanders' formula, and Sanders' formula gave only about $\frac{1}{5}$th of Rankin's and Weisbach's; therefore the two latter would give twenty-five times as much supporting power as the formula that appeared to have been employed by Mr. Bruce with reference to the piles of the Royal Border viaduct. He quite concurred in the opinion expressed by Mr. Beazeley, that for practical purposes there was not a better formula than that of Major Sanders of the United States Engineers.

In the particular case of the Dovey viaduct a timber dolly was interpolated between the monkey and the cast-iron splice, instead of the mass of lead which had been specified for, and which he had used with success in another case of an estuary crossing, in which the masonry piers were supported by bearing-piles wholly of cast iron, and driven precisely as timber piles. This was the bridge over the estuary of the Swale on the Sittingbourne and Sheerness railway, which he constructed in the years 1857–58. The length of this bridge was a little over 600 feet, and the width of the platform was 30 feet 3 inches. It carried a single line of railway and a turnpike road. There were twelve openings; eleven of 47 feet span, and an opening span of 60 feet clear in the centre of the channel. The piers were of brick in cement, resting on piled foundations; the depth of the channel at high water was 37 feet, and at low water, 18 feet. The bearing-piles were of cast iron, the cross section of which measured 16 inches by 10 inches, and consisted of a longitudinal web, extending 16 inches, crossed by two transverse webs of 10 inches each; the cross section, which was adopted from some similar works in the north of England, was a very advantageous one, as the disposition of the longitudinal and transverse webs caused it to drive particularly true, and afforded a large periphery of frictional bearing surface. These cast-iron piles were 42 feet in length for the piers of the central opening, and 32 feet for the other piers; the 42-feet piles were driven 25 feet into the bed of the channel, which was of London clay; and thence to their heads, which were driven down level with low-water mark, was 17 feet; the heads of the piles were rectangles of 16 inches by 10 inches, formed by flanges filling up the spaces between the longitudinal and transverse webs. These piles were designed to be driven 4 feet apart from centre to centre, but were really driven somewhat closer. At low-water mark they were surmounted by two courses of Yorkshire flagstones, above which the piers were carried up in brick in cement. In driving the piles a mass of lead was interpolated between the monkey and the pile-head, and not a pile was broken in driving. He had referred to this bridge as an example of a description of piled foundations which, though not very infrequent before cylinders and screw-piles came into such general use, had never before been noticed in the Proceedings.

Several questions were asked regarding the stoning of the channel under the Barmouth viaduct. The stoning was continuous, and had settled down to the bed of gravel. Masses of rough stone when thus thrown down, in a rapid current with a bottom of shifting sand, very shortly found their way to the bottom of the sand if shallow, as in this case; or if the sand were deep, to a depth at which they ceased to be liable to disturbance; and if the stoning were carried to a proper extent, it formed a settled mass incorporated with sand, and incapable of further movement or settlement.

As regarded the doubts expressed as to the utility of the supplementary bearing discs, and the apprehension lest, in case of a settlement of the stoning, their unsupported weight might dangerously affect the stability of the structure, he might observe that the supporting power of the area of the screws was more than ample for the support of the bridge, and therefore that the stability of the structure was wholly independent of these discs, preserving their bearing on the stoning. He applied them, however, because it was the usual practice to bolt balks of timber, resting on the stoning, on each side of the single row of piles which usually formed the piers of estuary viaducts similarly circumstanced; and as the tripoidal arrangement of the pier-piles of the Barmouth viaduct precluded the application of the expedient usual in such cases, he adopted the discs as an equivalent. As regarded the second part, even supposing that the stoning did settle to the extent of depriving these discs of its support, their weight was too insignificant to affect the coefficient of safety allowed in calculating the area of the screws.

Some apprehensions were expressed lest the tripoidal arrangement of the piles should cause more weight to be thrown on one pile than another. He could not understand the force of the objection, for it was manifest that the effect of such an arrangement must be to preclude the possibility of unequal bearings. Each of the three legs of each tripod must necessarily bear one-third of the load at the apex of such tripod, which was formed by a single casting forming one of the two supports of the transverse girders that supported the longitudinal girders of the bridge. Under such an arrangement an unequal bearing on the piles was an impossibility. The primary object, however, of the arrangement of the piles in the pier was to distribute the weight on the piers over a larger area of the thin crust of gravel than could alone be depended on for the support of the bridge.

Mr. Hemans had observed that there was at least a third variety of the telescope draw-bridge, in addition to the under and over arrangements adopted respectively for the opening spans of the Dovey and the Barmouth viaducts, namely, the arrangement in which the sliding portion was of rather more than double the length of the opening, and was protruded horizontally, the gap left in the line by its protrusion being filled up by a section of the way, wheeled in laterally from a recess in which it remained when the bridge was retracted; and there had been also mentioned an old and very cumbrous arrangement in which the opening span was withdrawn and wheeled off, without losing its parallelism, on rails laid at an angle of 45° with the axis of the bridge. He was acquainted with both these arrangements, but considered them obsolete, being, as they were so manifestly, inferior to more modern arrangements, in respect of first cost, space occupied and moved through, and facility and rapidity of manipulation.

There was a simple and inexpensive arrangement for small spans, adopted for canal crossings on some of the early Dutch railways.[1] Here each rail was supported on a hinged bracket, the extremity of the brackets being connected by a link, so that the whole formed an articulated parallelogram, capable of being opened horizontally to a rectangle, or completely closed like a parallel ruler. When a boat passed the parallel ruler, so to speak, was closed. When a train was to pass the ruler was extended fully, the extremities of the principal brackets resting on a sill on the farther side of the opening. This arrangement was obviously only applicable for small spans, and would scarcely pass the Board of Trade inspectors in this country. The same might be said of the various contrivances for military draw-bridges.

It appeared to him that for railway purposes only two species of draw-bridge were suitable, the swing-bridge and the telescope-bridge; and that this latter class being the cheapest to construct, and the more facile and rapid in manipulation, should be adopted in all cases where two contiguous openings were not required, except when the span of the opening was exceptionally great; also that of the two varieties of the telescope-draw-bridge, the over and the under, the former was much the best, for the reasons stated in the body of his Paper.

In reply to Mr. Brunlees, he might observe that although

[1] *Vide* Minutes of Proceedings Inst. C.E., vol. iii., p. 194.

Mr. Piercy (whom he had succeeded as Engineer to the Welsh Coast railway) had designed a viaduct for carrying that line across the Mawddach, at Barmouth, the viaduct actually constructed three years later was designed and calculated 'de novo,' and differed from that projected by Mr. Piercy in the following essential particulars: 1st. The sites of the two were not identical. The different positions of the axis of the two designs on plan was trifling, but it sufficed to affect the engineering design of the bridge. For example, the two northernmost piers in Mr. Piercy's design were of screw piles, while in his bridge these piers rested on naked rock, and were consequently composed of cylinders. 2nd. As regarded the treatment of the estuary, Mr. Conybeare's viaduct afforded a water-way of over 2,600 feet, and Mr. Piercy's design not more than half or two-thirds as much. 3rd. As regarded the adaptation of the design to the specialities of the site. In Mr. Piercy's design the piers, the spans, the superstructure, and the character and scantling of the piles were identical, except in the opening span, throughout the entire length of the bridge; whereas in the bridge carried out by him the material, the spans, and the diameter of the cast-iron piles were varied according to the depth of the channel. 4th. The superstructure was altogether different. The three girders which formed the superstructure of each span of the actual viaduct were all three dissimilar from each other, and no one of the three was the same as any girder in Mr. Piercy's design, and the strength of the superstructure as executed was about 50 per cent. in excess of that designed by Mr. Piercy. 5th. As regarded the piers. Each of the piers in Mr. Piercy's design consisted of a number of piles set in a row in the usual stereotyped fashion; while in his there were six piles in each pier arranged in two equilateral triangular groups. It would be impossible to conceive two arrangements more utterly distinct from one another. The position of the screw discs on the piles, 8 feet above the lower extremity of the latter, was not claimed as original, and he was informed that this arrangement was also adopted—at Mr. Brunlees' suggestion—in Mr. Piercy's design. As stated in the Paper, a drawbridge, similar in principle to that adopted at Barmouth, had been previously put up at Rhyl, to carry the turnpike road over the river; and Mr. Brereton had stated that a bridge on the same principle was erected across the river Towey in 1852. The system had answered most satisfactorily at Barmouth, for during the four or five years the bridge had been in use, it had been worked with the greatest ease and rapidity by a single man.

Mr. W. Bell observed, through the Secretary, that when the Clevedon pier was being constructed, he calculated its strength and stiffness, considering it as a continuous beam of eight spans. The investigation of such a beam was unusual, and the diagrams (Plate 12A) showed the stresses and deflections at the different points with each span loaded, or with the whole beam loaded. He had also attempted to estimate the effect of the arches, assuming them to be brackets, but as they wanted the top member to constitute them brackets, their advantage had been rather over-estimated.

This investigation might be divided into three parts:

1. The strength of the girders, assuming them to be continuous beams of uniform section, of eight equal spans, and loaded on any or all of the spans.
2. The stiffness of the piers, considering them as flexible columns with wrought-iron arches or brackets projecting from the upper parts.
3. The combined action of the piers and continuous beams, assuming that these latter rested on the ends of the brackets, and on the uprights of the piers.

The first part of the investigation being general, the formulæ were readily applicable to the case of a continuous beam of any number of equal spans. From the construction of the equations it followed that the stresses and deflections, for two or more spans loaded, were the algebraical sums of the stresses and deflections for each separate loaded span.

In regard to the second and third parts of the investigation, it was necessary to premise, that although the consequences of assuming the continuous beam to be supported on piers and flexible brackets had been worked out in detail, the results were only approximate as regarded the Clevedon pier; since, in that structure, the top members of the brackets must be considered as included in the lower web of the continuous beam itself. The effect of the brackets was therefore probably somewhat over-estimated, though in order to make some allowance for the want of the top member, the sectional areas of the top and bottom members of the brackets had been taken as each equal to only one half the sectional area of the wrought-iron arches.

The diagram of the combined action of the piers and continuous beams, Fig. 2, Plate 12A, showed considerable diminution of the stresses. It might be considered as exhibiting a step in the passage of a continuous beam of eight spans into a continuous beam of twenty-four spans. It was evident that if the brackets were perfectly rigid, the continuous beam supported on their ends and on the uprights of the piers, would be one of twenty-four spans.

1. Stresses and deflections of a continuous beam of any number of equal spans, and of uniform section.

Let l be the length of each of the equal spans, I the moment of inertia of the cross section of the beam, E the denominator of the fraction, expressing the extension or compression of the unit of length of the material of the beam by a force of 1 ton per square inch. Also, let inches and tons be the units of length and weight, and, reckoning from the unloaded end of the beam, let r_0, r_1, r_2, &c., be the pressures on the piers: x and y the running co-ordinates, horizontal and vertical, of the elastic curves, the origin being taken at the commencement of each span, $+ x$ measured towards the loaded end, and $+ y$ downwards.

Supposing the beam unloaded on all spans up to the $(m + 1)$th, then for some span nearer than this to the unloaded end, say the fourth, the equation of stress, expressing that the moment of elasticity was equal to the moment of the outward forces, was

$$\mathrm{E\,I}\cdot\frac{d^2 y}{d x^2} = -(r_0 + r_1 + r_2 + r_3)\,x - (3r_0 + 2r_1 + r_2)\,l.$$

By integrating this equation the value of $\frac{dy}{dx}$ was obtained, a constant, say c_4, being added, which expressed the value of $\frac{dy}{dx}$ when $x = 0$.

A second integration gave y, the ordinate to the curve of deflection, or elastic curve. The tops of the piers being supposed all at the same level, no new constant was introduced, since $y = 0$ when $x = 0$.

In this manner equations were obtained for each span, the undetermined quantities being the constants c_1, c_2, &c., and the pressures on the piers r_0, r_1, r_2, &c.

To determine these, one equation of condition for each span was obtained by making $y = 0$ when $x = l$, and another by making the value of $\frac{dy}{dx}$ when $x = l$ in one span, equal to the value of $\frac{dy}{dx}$ when $x = 0$ in the span immediately following.

The elastic curve of the beam, considered as a single curve, was discontinuous, the separate parts for each span being fitted together by making the ordinates and tangents coincide at the ends of the spans. This discontinuity was the consequence of the abrupt change of the bending moment at the piers. Each span had a separate equation for its elastic curve, and the only part of this curve which was applicable to the physical state of the beam was the part lying within the limits of the span: the branches of the curve lying outside these limits, not being adapted to the change of bending moment, were inapplicable to the actual state of the beam. A different equation thus became necessary, and the equation of the adjoining span, found as above, represented the state of the beam within the limits of that span, and so on for the other spans.

By writing down the equations of condition in a series beginning with the end span, and eliminating, it was found that r_1, r_2, &c., c_1, c_2, &c., could be expressed in terms of r_0.
Thus

$$\left.\begin{array}{ll} r_1 = -6\,r_0 & c_1 = +\frac{l^2}{6}r_0 \\ r_2 = +24\,r_0 & c_2 = -2\,\frac{l^2}{6}r_0 \\ r_3 = -90\,r_0 & c_3 = +7\,\frac{l^2}{6}r_0 \\ r_4 = +336\,r_0 & c_4 = -26\,\frac{l^2}{6}r_0 \\ r_5 = -1254\,r_0 & c_5 = +97\,\frac{l^2}{6}r_0 \\ r_6 = +4680\,r_0 & c_6 = -362\,\frac{l^2}{6}r_0 \\ r_7 = -17466\,r_0 & c_7 = +1351\,\frac{l^2}{6}r_0 \\ r_8 = +65184\,r_0 & c_8 = -5042\,\frac{l^2}{6}r_0 \\ \&c. = \quad \&c. & \&c. = \quad \&c. \end{array}\right\} \quad \ldots\ldots (A).$$

This series might be continued to any extent, the law of successive derivation being,

$$\begin{array}{l} r_k + 4\,r_{k+1} + r_{k+2} = 0 \\ c_k + 4\,c_{k+1} + c_{k+2} = 0 \end{array} \quad \ldots\ldots (B).$$ [1]

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[1] To show this, write down the equations of the $(k+1)$th span,

$$\mathrm{E\,I}\frac{d^2y}{dx^2} = -(r_k + r_{k-1} + r_{k-2} + \&c.)\,x - (r_{k-1} + 2\,r_{k-2} + 3\,r_{k-3} + \&c.)\,l$$

Calling t the tension or compression of the material of the beam per square inch at the distance k_0 from the neutral axis, and R the radius of curvature of the elastic curve, then

$$\frac{\mathrm{E}}{\mathrm{R}} = \frac{t}{k_0};$$

$$\mathrm{E\,I}\frac{d\,y}{d\,x} = -(r_k + r_{k-1} + r_{k-2} + \&c.)\frac{x^2}{2} - (r_{k-1} + 2\,r_{k-2} + 3\,r_{k-3} + \&c.)\,l\,x + c_{k+1}$$

$$\mathrm{E\,I}\,y = -(r_k + r_{k-1} + r_{k-2} + \&c.)\frac{x^3}{6} - (r_{k-1} + 2\,r_{k-2} + 3\,r_{k-3} + \&c.)\frac{l\,x^2}{2} + c_{k+1}\,x.$$

Making $y = 0$ in this last when $x = l$,

$$c_{k+1} = \frac{l^2}{6}(r_k + 4\,r_{k-1} + 7\,r_{k-2} + 10\,r_{k-3} + \&c.) \quad . \quad . \quad . \quad (1).$$

In the equations for the $(k+2)$nd span, the constant introduced, or c_{k+2}, was the value of $\frac{d\,y}{d\,x}$ in the above equation for $\frac{d\,y}{d\,x}$, when $x = l$, or

$$c_{k+2} = c_{k+1} - \frac{l^2}{2}(r_k + 3\,r_{k-1} + 5\,r_{k-2} + 7\,r_{k-3} + \&c.) \quad . \quad . \quad (2).$$

Substituting in this, the above value of c_{k+1}, there resulted

$$c_{k+2} = -\frac{l^2}{6}(2\,r_k + 5\,r_{k-1} + 8\,r_{k-2} + 11\,r_{k-3} + \&c.). \quad . \quad . \quad (3).$$

Writing $k+1$ for k in (1),

$$c_{k+2} = \frac{l^2}{6}(r_{k+1} + 4\,r_k + 7\,r_{k-1} + 10\,r_{k-2} + \&c.).$$

Equating these two values of c_{k+2},

$$r_{k+1} = -6\,(r_k + 2\,r_{k-1} + 3\,r_{k-2} + 4\,r_{k-3} + \&c.).$$

Writing in this last equation $k-1$ and $k+1$ for k,

$$-\frac{r_k}{6} = r_{k-1} + 2\,r_{k-2} + 3\,r_{k-3} + \&c.$$

$$\frac{-r_{k+1}}{6} = r_k + 2\,r_{k-1} + 3\,r_{k-2} + 4\,r_{k-3} + \&c.$$

$$\frac{-r_{k+2}}{6} = r_{k+1} + 2\,r_k + 3\,r_{k-1} + 4\,r_{k-2} + 5\,r_{k-3} + \&c.$$

deducting

$$\frac{-r_{k+2} + r_{k+1}}{6} = r_{k+1} + r_k + r_{k-1} + r_{k-2} + \&c.$$

$$\frac{-r_{k+1} + r_k}{6} = r_k + r_{k-1} + r_{k-2} + \&c.$$

deducting again

$$\frac{-r_{k+2} + r_{k+1}}{6} - \frac{-r_{k+1} + r_k}{6} = r_{k+1},$$

or

$$r_k + 4\,r_{k+1} + r_{k+2} = 0.$$

Again, from (1) writing $k-1$ for k,

$$c_{k+1} = \frac{l^2}{6}(r_k + 4\,r_{k-1} + 7\,r_{k-2} + \&c.)$$

$$c_k = \frac{l^2}{6}(r_{k-1} + 4\,r_{k-2} + 7\,r_{k-3} + \&c.)$$

and since $\frac{1}{R}$ might be taken as equal to $\frac{d^2 y}{d x^2}$,

$$E \frac{d^2 y}{d x^2} = \frac{t}{k_0}.$$

Writing down the equations of stress in a series beginning with the end span, and substituting the above value of $E \frac{d^2 y}{d x^2}$, and the values of r_1, r_2, r_3, &c., from equations (A), the equations of stress became

$$\begin{aligned}
&\text{1st span,} \quad t = \frac{k_0}{I} r_0 \cdot x \\
&\text{2nd} \quad ,, \quad t = \frac{k_0}{I} r_0 (+ 5 x - l) \\
&\text{3rd} \quad ,, \quad t = \frac{k_0}{I} r_0 (- 19 x + 4 l) \\
&\text{4th} \quad ,, \quad t = \frac{k_0}{I} r_0 (+ 71 x - 15 l) \\
&\text{5th} \quad ,, \quad t = \frac{k_0}{I} r_0 (- 265 x + 56 l)
\end{aligned} \quad \ldots . (C).$$

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deducting

$$c_{k+1} - c_k = \frac{l^2}{6} (r_k + 3 r_{k-1} + 3 r_{k-2} + 3 r_{k-3} + \&c.) \quad . \quad . \quad . \quad (4).$$

From (3) in the same manner,

$$c_{k+2} = - \frac{l^2}{6} (2 r_k + 5 r_{k-1} + 8 r_{k-2} + 11 r_{k-3} + \&c.)$$

$$c_{k+1} = - \frac{l^2}{6} (2 r_{k-1} + 5 r_{k-2} + 8 r_{k-3} + 11 r_{k-4} + \&c.)$$

deducting

$$c_{k+2} - c_{k+1} = - \frac{l^2}{6} (2 r_k + 3 r_{k-1} + 3 r_{k-2} + 3 r_{k-3} + \&c.) \quad . \quad (5).$$

Adding (4) and (5),

$$c_{k+2} - c_k = - \frac{l^2}{6} r_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Adding the above value of c^{k+2} to the first value of c_{k+1}, and multiplying by 3,

$$3 c_{k+2} + 3 c_{k+1} = - \frac{l^2}{6} (3 r_k + 3 r_{k-1} + 3 r_{k-2} + \&c.);$$

and adding this equation to (4),

$$3 c_{k+2} + 4 c_{k+1} - c_k = - \frac{l^2}{6} . 2 r_k \quad . \quad . \quad . \quad . \quad (7).$$

From (6) and (7),

$$2 c_{k+2} - 2 c_k = 3 c_{k+2} + 4 c_{k+1} - c_k,$$

or

$$c_{k+2} + 4 c_{k+1} + c_k = 0.$$

6th span, $t = \frac{k_0}{I} r_0 (+ 989 x - 209 l)$

7th ,, $t = \frac{k_0}{I} r_0 (- 3691 x + 780 l)$

8th ,, $t = \frac{k_0}{I} r_0 (+ 13775 x - 2911 l)$

&c. &c.

By substituting $k_0 \mathrm{E} \frac{d^2 y}{dx^2}$ for t, and integrating the above equations twice, introducing the above values of the constants (c), the equations of the elastic curves became

$$
\left.
\begin{array}{lll}
\text{1st span,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ 1 - \left(\frac{x}{l}\right)^2 \right\} \\
\text{2nd ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ + 5 \left(\frac{x}{l}\right)^2 - 3 \frac{x}{l} - 2 \right\} \\
\text{3rd ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ - 19 \left(\frac{x}{l}\right)^2 + 12 \frac{x}{l} + 7 \right\} \\
\text{4th ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ + 71 \left(\frac{x}{l}\right)^2 - 45 \frac{x}{l} - 26 \right\} \\
\text{5th ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ - 265 \left(\frac{x}{l}\right)^2 + 168 \frac{x}{l} + 97 \right\} \\
\text{6th ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ + 989 \left(\frac{x}{l}\right)^2 - 627 \frac{x}{l} - 362 \right\} \\
\text{7th ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ - 3691 \left(\frac{x}{l}\right)^2 + 2340 \frac{x}{l} + 1351 \right\} \\
\text{8th ,,} & y = \frac{l^3}{6\mathrm{EI}} r_0 \frac{x}{l} \left\{ + 13775 \left(\frac{x}{l}\right)^2 - 8723 \frac{x}{l} - 5042 \right\} \\
\text{\&c.} & \text{\&c.}
\end{array}
\right\} \quad \ldots \text{(D).}
$$

If the beam were now supposed to be loaded on the $(m+1)$th span, with a weight Q uniformly distributed over the length of the span, the values of r and c, and the equations of stress and deflection for the first m spans, would be the same as those above; but for the $(m+1)$th and subsequent spans, the above equations would require modification. Calling r_m, r_{m+1}, &c., c_{m+2}, c_{m+3}, &c., the values of r and c for the $(m+1)$th and subsequent spans, as given by the above equations, let these quantities after the loading be $r_m + \rho_0$, $r_{m+1} + \rho_1$, &c., $c_{m+2} + \gamma_2$, $c_{m+3} + \gamma_3$, &c.; then, applying the same process as that above described to find r_1, r_2, &c., c_2, &c., there resulted

$$\rho_0 = +\frac{1}{4}Q$$

$$\rho_1 = +\frac{7}{4}Q \qquad \gamma_1 = 0$$

$$\rho_2 = -\frac{18}{4}Q \qquad \gamma_2 = +\frac{1}{24}Ql^2$$

$$\rho_3 = +\frac{66}{4}Q \qquad \gamma_3 = -\frac{5}{24}Ql^2$$

$$\rho_4 = -\frac{246}{4}Q \qquad \gamma_4 = +\frac{19}{24}Ql^2$$

$$\rho_5 = +\frac{918}{4}Q \qquad \gamma_5 = -\frac{71}{24}Ql^2$$

&c. &c.

Except ρ_0 and ρ_1 which were under the direct influence of the weight, the law of successive derivation (B) applied to the quantities ρ and γ.

These additions ρ_0, ρ_1, &c., to the values of r_m, r_{m+1}, &c., would evidently introduce corresponding additional terms into the equations of stress (C), and by writing down these equations, taking into account the action of the new force Q introduced, it would become apparent that there were also terms added by the direct action of Q. For instance, forming the equation of the $(m+1)$ th span, the term added by the alteration of r_m into $r_m + \rho_0$ was $-\rho_0 x$, or $-\frac{Q}{4}x$, and the term added by the direct action of Q was $+\frac{Q}{l}x\cdot\frac{x}{2}$, or $\frac{Q}{l}\cdot\frac{x^2}{2}$. For the $(m+2)$nd span, the terms added by the alteration of r_m and r_{m+1} were $-\rho_0(l+x) - \rho_1 x$, or $-\frac{Q}{4}(l+x) - \frac{7}{4}Qx$, and the term added by the direct action of Q was $+Q\left(\frac{l}{2}+x\right)$. Thus it was found that the added terms were:

$$\text{For the } (m+1)\text{ th span, } +\frac{k_0}{I}\frac{Q}{4}\left(+2\frac{x^2}{l} - x\right)$$

$$\text{,, } (m+2)\text{nd ,, } +\frac{k_0}{I}\frac{Q}{4}(-4x+l) \quad \ldots (C').$$

$$\text{For the } (m+3)\text{rd} \quad ,, \quad + \frac{k_0}{I}\frac{Q}{4}(+14x - 3l)$$

$$,, \quad (m+4)\text{th} \quad ,, \quad + \frac{k_0}{I}\frac{Q}{4}(-52x + 11l)$$

&c. &c.

These terms added to equations (C) gave the complete equations of stress for the load Q.

Integrating these terms twice, having added after the first integration the above values of γ_2, γ_3, &c., for the altered values of c_{m+2}, c_{m+3}, &c., in the equations of the spans to which they corresponded, the following expressions were obtained for the terms to be added to equations (D), to get the complete equations of the elastic curves:

$$\left.\begin{aligned}
&\text{For the } (m+1)\text{th span}, + \frac{Q l^3}{24 E I}\cdot\left(\frac{x}{l}\right)^3\left(\frac{x}{l} - 1\right) \\
&,, \quad (m+2)\text{nd} \quad ,, \quad + \frac{Q l^3}{24 E I}\cdot\frac{x}{l}\left\{-4\left(\frac{x}{l}\right)^2 + 3\frac{x}{l} + 1\right\} \\
&,, \quad (m+3)\text{rd} \quad ,, \quad + \frac{Q l^3}{24 E I}\cdot\frac{x}{l}\left\{+14\left(\frac{x}{l}\right)^2 - 9\frac{x}{l} - 5\right\} \\
&,, \quad (m+4)\text{th} \quad ,, \quad + \frac{Q l^3}{24 E I}\cdot\frac{x}{l}\left\{-52\left(\frac{x}{l}\right)^2 + 33\frac{x}{l} + 19\right\}
\end{aligned}\right\} \quad (D').$$

&c. &c.

The value of r_0 was still undetermined, because for the first span the value of $\frac{dy}{dx}$ when $x = 0$ could not be made equal to the value of $\frac{dy}{dx}$ when $x = l$ in the preceding span. To determine r_0, the pressure on the pier at one end of the beam, the sum of the moments of the forces taken with respect to the other end must be put $= 0$. The pressure on the pier at the other end of the beam was determined by the condition that the sum of all the pressures acting on the beam was equal to zero. These pressures being found, the above equations of stress and deflection contained no undetermined quantities. As an example of the use of the above formulæ, suppose a beam of five equal spans loaded on the middle span with a weight Q uniformly distributed. The pressures r_0, r_1, &c., constituted six unknown quantities, the equations to determine which were thus formed. From equations (A) and (A′), as already explained,

$$r_1 = -6\,r_0$$

$$r_2 = +24\,r_0 + \frac{1}{4}Q$$

$$r_3 = -90\,r_0 + \frac{7}{4}Q$$

$$r_4 = +336\,r_0 - \frac{18}{4}Q.$$

The equation of moments to determine r_0 was

$$r_0 \cdot 5\,l + r_1 \cdot 4\,l + r_2 \cdot 3\,l + r_3 \cdot 2\,l + r_4 \cdot l - Q\frac{5}{2}l = 0\,;$$

and the equation of the sum of the pressures was

$$Q - r_0 - r_1 - r_2 - r_3 - r_4 - r_5 = 0.$$

These gave

$$r_0 = r_5 = +\frac{1}{76}Q$$

$$r_1 = r_4 = -\frac{6}{76}Q$$

$$r_2 = r_3 = +\frac{43}{76}Q.$$

The equations of stress were obtained by substituting the above value of r_0 in equations (C) and (C′). For the two first spans, the two first equations of (C) were applicable without modification. For the third span, the first expression in (C′), or

$$+\frac{k_0}{I}\frac{Q}{4}\left(2\frac{x^2}{l} - x\right)$$

was to be added to the second side of the third equation of (C). For the fourth span, the second expression in (C′) was to be added to the second side of the fourth equation of (C), and so on. It was, however, evident that in this case the state of the beam for the fourth and fifth spans was the same as for the second and first spans.

In a similar manner the equations of the elastic curves were formed from (D) and (D′).

Applying these formulæ to a continuous beam of eight equal spans, the following table was obtained, in which the load was supposed to be placed on each separate span. The numbers in the table multiplied by $\frac{Q}{43456}$ gave the values of r_0, r_1, r_2, &c.

Loaded on the	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1st span only	+18817	+28330	− 4680	+ 1254	− 336	+ 90	− 24	+ 6	− 1
2nd ,,	− 2131	+23650	+24904	− 3762	+ 1008	− 270	+ 72	− 18	+ 3
3rd ,,	+ 571	− 3426	+24568	+24658	− 3696	+ 990	− 264	+ 66	− 11
4th ,,	− 153	+ 918	− 3672	+24634	+24640	− 3690	+ 984	− 246	+ 41
5th ,,	+ 41	− 246	+ 984	− 3690	+24640	+24634	− 3672	+ 918	− 153
6th ,,	− 11	+ 66	− 264	+ 990	− 3696	+24658	+24568	− 3426	+ 571
7th ,,	+ 3	− 18	+ 72	− 270	+ 1008	− 3762	+24904	+23650	− 2131
8th ,,	− 1	+ 6	− 24	+ 90	− 336	+ 1254	− 4680	+28330	+18817
All spans loaded	+17136	+49280	+41888	+43904	+43232	+43904	+41888	+49280	+17136

The equations of stress, and of the elastic curves, were formed from these values, and from equations (C), (C′), (D), (D′), in the manner already explained.

Although the equations of stress and deflection with all the spans loaded could be made by the addition of those for each span loaded, it was easier to construct these equations directly from the above values of r_0, r_1, &c., which, it would be observed, were formed by algebraical addition.

The equations of stress were:

$$\text{1st, or end span, } t = \frac{k_0}{\mathrm{I}} Q\, l \cdot \frac{x}{l} \cdot \left\{ + \cdot 5 \frac{x}{l} - \cdot 394 \right\}$$

$$\text{2nd span} \quad t = \frac{k_0}{\mathrm{I}} Q\, l \left\{ + \cdot 5 \left(\frac{x}{l}\right)^2 - \cdot 5283 \frac{x}{l} + \cdot 1057 \right\}$$

$$\text{3rd ,,} \quad t = \frac{k_0}{\mathrm{I}} Q\, l \left\{ + \cdot 5 \left(\frac{x}{l}\right)^2 - \cdot 4922 \frac{x}{l} + \cdot 0774 \right\} \quad \ldots (\mathrm{E}).$$

$$\text{4th ,,} \quad t = \frac{k_0}{\mathrm{I}} Q\, l \left\{ + \cdot 5 \left(\frac{x}{l}\right)^2 - \cdot 5025 \frac{x}{l} + \cdot 0852 \right\}$$

And for the elastic curves:

$$\text{1st span } y = \frac{Q\, l^3}{24\, \mathrm{E\, I}} \cdot \frac{x}{l} \cdot \left\{ \left(\frac{x}{l}\right)^3 - 1 \cdot 577 \left(\frac{x}{l}\right)^2 + \cdot 577 \right\}$$

$$\text{2nd ,,} \quad y = \frac{Q\, l^3}{24\, \mathrm{E\, I}} \cdot \frac{x}{l} \cdot \left\{ \left(\frac{x}{l}\right)^3 - 2 \cdot 1132 \left(\frac{x}{l}\right)^2 + 1 \cdot 2684 \frac{x}{l} - \cdot 1544 \right\} \quad (\mathrm{F}).$$

$$\text{3rd ,,} \quad y = \frac{Q\, l^3}{24\, \mathrm{E\, I}} \cdot \frac{x}{l} \cdot \left\{ \left(\frac{x}{l}\right)^3 - 1 \cdot 9688 \left(\frac{x}{l}\right)^2 + \cdot 9288 \frac{x}{l} + \cdot 0406 \right\}$$

$$\text{4th ,,} \quad y = \frac{Q\, l^3}{24\, \mathrm{E\, I}} \cdot \frac{x}{l} \cdot \left\{ \left(\frac{x}{l}\right)^3 - 2 \cdot 0100 \left(\frac{x}{l}\right)^2 + 1 \cdot 0224 \frac{x}{l} - \cdot 006 \right\}$$

Applying these formulæ to the Clevedon pier, the values of l, Q, &c., for one girder, were:

$$l = 1{,}200$$
$$k_0 = 21$$
$$\mathrm{I} = 13{,}454$$
$$\mathrm{Q} = 50$$
$$\mathrm{E} = 8{,}000.$$

The value of E for solid wrought iron was about 10,000, but 8,000 had been used, to allow for loss of strength by rivetting.

Subdividing each span into ten equal parts, and making

$$\frac{x}{l} = \cdot 1,\ \cdot 2,\ \cdot 3,\ \&c.,$$

the stresses and deflections at each of these points were obtained from the above formulæ by substituting in them the values of l, k_0, &c.

Fig. 1, Plate 12A, showed the stresses and deflections with any one of the spans loaded uniformly, and also with all the spans loaded. For the sake of comparison, similar results for a continuous beam of two spans only, and for a beam of one span, with the same values of l, k_0, I, Q, and E, were also shown.

Putting $\phi\left(\frac{x}{l}\right)$ for that part of the equations (C), (C′), and (E), which depended only on $\frac{x}{l}$, and $\psi\left(\frac{x}{l}\right)$ for the like part in equations (D), (D′), and (F), the equations of stress and deflection might be thus written:

$$t = \frac{k_0}{\mathrm{I}}\,\mathrm{Q}\,l \cdot \phi\left(\frac{x}{l}\right)$$

$$y = \frac{\mathrm{Q}\,l^3}{24\,\mathrm{E}\,\mathrm{I}} \cdot \psi\left(\frac{x}{l}\right)$$

Denoting by t', y', l', k_0', I′, Q′, and E′, corresponding quantities for another continuous beam of different length, section, loading, &c., but of the same number of equal spans, then

$$t' = \frac{k_0'}{\mathrm{I}'}\,\mathrm{Q}'\,l' \cdot \phi\left(\frac{x}{l}\right)$$

$$y' = \frac{\mathrm{Q}'\,l'^3}{24\,\mathrm{E}'\,\mathrm{I}'}\,\psi\left(\frac{x}{l}\right);$$

and by eliminating $\phi\left(\frac{x}{l}\right)$ and $\psi\left(\frac{x}{l}\right)$,

$$t' = t \cdot \frac{k_0'}{k_0} \cdot \frac{Q'}{Q} \cdot \frac{l'}{l} \cdot \frac{I}{I'}$$

$$y' = y \cdot \frac{Q'}{Q}\left(\frac{l'}{l}\right)^3 \cdot \frac{E}{E'} \cdot \frac{I}{I'}$$

Hence, by applying these ratios to the values of t and y on the diagram, the corresponding values t' and y' for any continuous beam of eight equal spans might be readily found.

Suppose now the beam to have the first n spans unloaded, and the $(n+1)$ th span to be loaded with a weight P, which acted on a point of the beam whose distance from the origin of co-ordinates for that span was (a).

Then the values of r_1, r_2, &c., up to r_{n-1}, c_1, c_2, &c., up to c_n, would still be given by equations (A), and the equations of stress and deflection for the n spans by (C) and (D).

At the $(n+1)$ th span, for that part of the elastic curve between P and the origin, the value of c_{n+1} would also be given by (A), since it was equal to the value of $\frac{dy}{dx}$ for the nth span when $x = l$.

There being an abrupt change of the bending moment at the point where P acted, there was discontinuity in the curve at that point, and the parts of the elastic curve of the beam lying to the right and left of P could not be expressed by the same algebraical equation. When x was greater than a, the term $+ P(x - a)$ was added to the equation of stress; and in integrating this equation to find the value of y, two new constants were introduced: one of these was an addition to c_{n+1}, and the other was the value of y when $x = 0$, since the condition $y = 0$ when $x = 0$ no longer held good for this curve. The two constants were determined by fitting the two pieces of curve together, so that the values of y and $\frac{dy}{dx}$ at the point where the weight P acted, might be the same for that piece of the curve where x was less than a, as for the piece where x was greater than a. The values of r_n, r_{n+1}, &c., c_{n+1}, as already explained, c_{n+2}, &c., and the equations of stress and of the elastic curve for the $(n+1)$ th and subsequent spans, would all be altered. Calling the additions to r_n, r_{n+1}, &c., σ_0, σ_1, σ_2, &c., and the addition to c_{n+1}, when x was greater than a, δ_1, the additions to c_{n+2}, c_{n+3}, &c., δ_2, δ_3, &c., it was found by the process explained above, that,

$$\sigma_0 = \mathrm{P}\left(1 - \frac{a}{l}\right)^3 \qquad = \mathrm{P}\cdot\left(1 - \frac{a}{l}\right)^3$$

$$\sigma_1 = \mathrm{P}\cdot\left\{\begin{matrix} 1 - 7\,\sigma_0 \\ + 3\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)\end{matrix}\right\} = 6\,\mathrm{P}\cdot\frac{a}{l}\left\{+2 - 3\frac{a}{l} + \frac{7}{6}\left(\frac{a}{l}\right)^2\right\}$$

$$\sigma_2 = \mathrm{P}\cdot\left\{\begin{matrix} 1 - 19\,\sigma_0 - 7\,\sigma_1 \\ + 3\left(2 - \frac{a}{l}\right)\left(3 - \frac{a}{l}\right)\end{matrix}\right\} = 6\,\mathrm{P}\cdot\frac{a}{l}\left\{-7 + 12\frac{a}{l} - 5\left(\frac{a}{l}\right)^2\right\}$$

$$\sigma_3 = \mathrm{P}\cdot\left\{\begin{matrix} 1 - 37\,\sigma_0 - 19\,\sigma_1 - 7\,\sigma_2 \\ + 3\left(3 - \frac{a}{l}\right)\left(4 - \frac{a}{l}\right)\end{matrix}\right\} = 6\,\mathrm{P}\cdot\frac{a}{l}\left\{+26 - 45\frac{a}{l} + 19\left(\frac{a}{l}\right)^2\right\}$$

$$\sigma_4 = \mathrm{P}\cdot\left\{\begin{matrix} 1 - 61\sigma_0 - 37\sigma_1 - 19\sigma_2 - 7\sigma_3 \\ + 3\left(4 - \frac{a}{l}\right)\left(5 - \frac{a}{l}\right)\end{matrix}\right\} = 6\mathrm{P}\cdot\frac{a}{l}\left\{-97 + 168\frac{a}{l} - 71\left(\frac{a}{l}\right)^2\right\}$$

&c. &c.

$$\delta_1 = +\,\mathrm{P}\frac{a^2}{2}$$

$$\delta_2 = +\frac{\mathrm{P}}{6}\left(+3\,a\,l - 6\,a^2 + 3\,\frac{a^3}{l}\right)$$

$$\delta_3 = +\frac{\mathrm{P}}{6}\left(-12\,a\,l + 21\,a^2 - 9\,\frac{a^3}{l}\right)$$

$$\delta_4 = +\frac{\mathrm{P}}{6}\left(+45\,a\,l - 78\,a^2 + 33\,\frac{a^3}{l}\right)$$

&c. &c.

The law (B) applying to these quantities, except for the span on which P acted, and the adjoining one.

By these additions to r_n, r_{n+1}, &c., and by the direct action of P, the following terms were added to the equations of stress (C):

$(n+1)$ th span $x < a, -\sigma_0 x$

,, ,, $x > a, -\sigma_0 x + \mathrm{P}(x - a)$

$(n+2)$ nd ,, $\left\{\begin{matrix} -(\sigma_0 + \sigma_1)x - \sigma_0 l \\ + \mathrm{P}x + \mathrm{P}(l - a)\end{matrix}\right\}$ . . . . (C″).

$(n+3)$ rd ,, $\left\{\begin{matrix} -(\sigma_0 + \sigma_1 + \sigma_2)x - (\sigma_1 + 2\,\sigma_0)l \\ + \mathrm{P}x + \mathrm{P}(2\,l - a)\end{matrix}\right\}$

$(n+4)$ th ,, $\left\{\begin{matrix} -(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3)x - (\sigma_2 + 2\,\sigma_1 + 3\,\sigma_3)l \\ + \mathrm{P}x + \mathrm{P}(3\,l - a)\end{matrix}\right\}$

&c. &c.

Integrating these terms twice, having added after the first integrations the above values of δ_1, δ_2, &c., to the second sides of the equations to which they corresponded, it was found that the following terms were added to the equations of the elastic curve (D):

$$(n+1)\text{th span } x < a, -\frac{\mathrm{P}\,a^3}{6}\cdot\left(1-\frac{a}{l}\right)^3\left(\frac{x}{l}\right)^3$$

$$\text{,,} \qquad x > a, +\frac{\mathrm{P}\,a^3}{6}\left\{\left(\frac{x-a}{l}\right)^3-\left(\frac{l-a}{l}\right)^3\left(\frac{x}{l}\right)^3\right\}$$

$$(n+2)\text{nd span}, +\frac{\mathrm{P}\,a^3}{6}\cdot\frac{x}{l}\left\{\begin{array}{l}(1-\sigma_0-\sigma_1)\left(\frac{x}{l}\right)^2 \\ +3\left(1-\sigma_0-\frac{a}{l}\right)\frac{x}{l} \\ +3\left(1-\frac{a}{l}\right)^2-3\,\sigma_0\end{array}\right\} \quad \ldots\ldots (\mathrm{D}'').$$

$$(n+3)\text{rd span}, +\frac{\mathrm{P}\,a^3}{6}\cdot\frac{x}{l}\cdot\left\{\begin{array}{l}(1-\sigma_0-\sigma_1-\sigma_2)\left(\frac{x}{l}\right)^2 \\ +3\left(2-2\,\sigma_0-\sigma_1-\frac{a}{l}\right)\frac{x}{l} \\ +3\left\{1-4\,\sigma_0-\sigma_1+\left(1-\frac{a}{l}\right)\left(3-\frac{a}{l}\right)\right\}\end{array}\right\}$$

$$(n+4)\text{th span}, +\mathrm{P}\frac{a^3}{6}\cdot\frac{x}{l}\cdot\left\{\begin{array}{l}(1-\sigma\ -\sigma\ -\sigma_2-\sigma_3)\left(\frac{x}{l}\right)^2 \\ +3\left(3-3\,\sigma_0-2\,\sigma_1-\sigma_2-\frac{a}{l}\right)\cdot\frac{x}{l} \\ +3\left\{4-9\sigma\ -4\sigma_1-\sigma_2+\left(1-\frac{a}{l}\right)\left(5-\frac{a}{l}\right)\right\}\end{array}\right\}$$

&c. &c.

These expressions might be reduced by substituting the values of σ_0, σ_1, &c., but they were as easy of computation in their present form.

It was worthy of remark that with the exception of the first equations in (C), (C',) (D), (D'), the coefficients of the terms followed the law (B). For example, taking the 4th, 5th, and 6th equations (D):

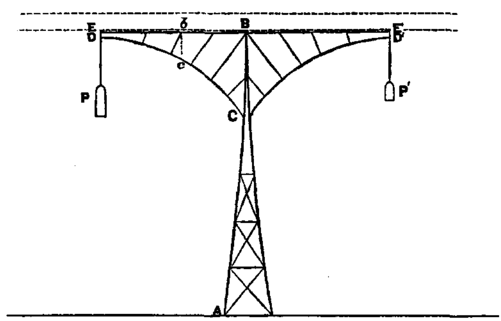
$$+71+4(-265)+989 = 0$$
$$-45+4\times168-627 = 0$$
$$-26+4\times97-362 = 0$$

Probably this could be shown to be a consequence of (B). It meant that for the same values of $\frac{x}{l}$ in the different spans t and y themselves followed this law, as would be seen on examination of the diagrams Fig. 1, Plate 12A.

2. Stiffness of the piers and arches.

If, in the annexed Fig. 1, the bracket B C D E were considered as fixed along B C, so as to be immovable there, the deflection at E from the action of a weight P could be ascertained as follows:

Fig. 1.



Suppose the top and bottom webs B b E and C c D to be of equal sectional areas, and the space between these so filled in with trussing that the whole acted as an ordinary girder. To form the equation of stress, let the area of each of these webs be A, and let B E = a, B C = d, and E D = e. Also let B b, the horizontal distance from B of any point b, be called x, and the deflection, or ordinate of the elastic curve there, y.

If $b\,c$ were the distance between the centres of the top and bottom webs, the value of I, the moment of inertia of the cross section, might be taken as equal to

$$2\,A\left(\frac{b\,c}{2}\right)^2, \text{or } \frac{A \cdot (b\,c)^2}{2},$$

and the equation of stress was

$$E \cdot A \cdot (b\,c)^2 \frac{d^2 y}{d\,x^2} = 2\,P\,(a - x).$$

Assuming the curve D c C to be a portion of a parabola whose vertex is D,

$$b\,c - e : d - e : : E\,b^2 : E\,B^2 : : (a - x)^2 : a^2,$$

and

$$b\,c = \frac{d - e}{a^2}(a - x)^2 + e.$$

Substituting this value in the equation of stress, it became

$$\frac{d^2 y}{d\,x^2} = \frac{2\,P}{E \cdot A} \cdot \frac{a - x}{\left(\frac{d - e}{a^2} \cdot (a - x)^2 + e\right)^2}$$

Integrating this equation twice, and making y and $\frac{d\,y}{d\,x} = 0$ when $x = 0$, the deflection f of the point E, or the value of y when $x = a$, was found to be

$$f = \frac{P\,a^3}{E \cdot A \cdot (d - e)} \cdot \left\{\frac{1}{\sqrt{e\,(d - e)}} \tan^{-1} \sqrt{\frac{d - e}{e}} - \frac{1}{d}\right\} \; . \; . \; (G).$$

Taking for the Clevedon pier

$$\begin{aligned} a &= 360 \\ d &= 252 \\ e &= 9 \\ A &= 8 \\ E &= 8000 \end{aligned}$$

it was found by this formula that

$$f = \cdot 0765\,P;$$

so that the deflection of the point E by ·0765 inch would, on the above supposition, give an upward reaction against the bottom of the continuous beam at the point E of 1 ton.

Had D C been assumed a straight line instead of a parabolic arc, the deflection would have been ·0195 P, the formula in that case being

$$f = \frac{2\,P\,a^3}{E\,A\,(d - e)^2} \left\{\frac{d + e}{d} + \frac{2\,e}{d - e} \log \frac{e}{d}\right\}.$$

But the upright of the pier A C B might be regarded as a girder, and the point C would yield sideways by the action of the weight P. Calling h the height A B, and g the deflection at C by a horizontal force F, g might be found by the ordinary formula for a beam supported at the ends and loaded at a point in its length with a weight F.

Hence $$g = F \cdot \frac{d^2\,(h - d)^2}{E \cdot I \cdot 3\,h}.$$

Taking $h = 720$, $d = 204$, $E = 8000$, and $I = 2064$,

$$g = \cdot 307 \text{ F},$$

or a horizontal force of 1 ton would move the point C through ·307 inch.

By this flexibility of the upright A C B, the point E′ would rise by the action of P, and if another weight P′ were supposed to act at E′, the deflection that would exist there if A C B were perfectly rigid would be modified by the action of P. In like manner, the deflection at E would be modified by the action of P′.

The bracket B C D E being attached to the upright, the horizontal force at C caused by the action of P was $P \cdot \frac{BE}{BC}$. In like manner, the horizontal force caused by the action of P′ was $P' \cdot \frac{BE}{BC}$ in the opposite direction. Hence, calling F the resultant of the two acting together,

$$F = (P - P')\frac{BE}{BC};$$

and the horizontal movement at C in consequence of this was g or ·307 F.

Neglecting for a moment the deflections of the brackets, the movements at E and E′, caused by this movement at C, would be $\cdot 307 \text{ F} \frac{BE}{BC}$; and, by substituting the value of F, the movements at E and E′, irrespective of the deflections of the brackets themselves, were

$$\text{for E}, + \cdot 307\,(P - P')\frac{BE^2}{BC^2};$$

$$\text{for E}', - \cdot 307\,(P - P')\frac{BE^2}{BC^2}.$$

Taking into account the deflections of the brackets, and calling f_1 and f_1' the deflections of E and E′,

$$f_1 = \cdot 0765\,P + \cdot 307\,(P - P')\frac{BE^2}{BC^2}$$

$$f_1' = \cdot 0765\,P' - \cdot 307\,(P - P')\frac{BE^2}{BC^2}.$$

But B E = 30 feet, B C = 21 feet, and, after reduction,

$$f_1 = \cdot 702\,P - \cdot 626\,P'$$
$$f_1' = \cdot 702\,P' - \cdot 626\,P. \quad \ldots\ldots (K).$$

3. Combined action of the piers and continuous beams.

If the continuous beams were supposed to be resting on the piers and brackets and to be afterwards loaded, the deflection of the end of any one of the brackets must be the same as the deflection of the continuous beam at the point where it was in contact with the end of the bracket.

The resistances of the brackets to deflection would cause upward pressures against the continuous beam, and the effect of any one of these pressures on the stresses and deflections of the continuous beam could be ascertained by equations (C), (C″), (D), (D″), bearing in mind that the signs there corresponded to downward pressure, as P was supposed to act downwards. The effect of the whole of the brackets was the sum of the effects of each separately.

Considering the effect of only one bracket, the deflection of the continuous beam at the point in contact with the end of it was the difference between the deflection which would be caused by the load if no bracket existed, ascertained by equations (D), (D′), and the deflection caused by the pressure at the end of the bracket, as ascertained by equations (D), (D″). But the resulting deflection being the same as that of the bracket itself, which might be found by the formula, by putting these two deflections equal to one another, an equation of condition was obtained from which the pressure acting at the end of the bracket could be determined; and from this, the stresses and deflections all along the beam, by forming the equations due to the load, and those due to the now known pressure.

When the effect of all the brackets was considered, the pressure at the end of each bracket would modify the deflection at every point of the continuous beam, and therefore at the ends of all the other brackets. The equation of condition made by equating the deflection of the beam to the deflection of the end of the bracket would therefore contain as many unknown pressures as there were brackets, but since an equation could be formed for the end of each bracket, there were as many equations as unknown quantities.

These equations of condition were simple equations, and by their solution the pressures at the ends of the brackets became known, and from these, by equations (C), (C′), (C″), (D), (D′), (D″), the stresses and deflections at any point of the beam.

Supposing the end span loaded, and that the beam rested on the piers only, the brackets not being in action, the deflections at the ends of each of the first five brackets, reckoning from the loaded end of the beam, by equations (D), (D′), were

$$+6{\cdot}04,\ +5{\cdot}29,\ -2{\cdot}54,\ -1{\cdot}58, \text{ and } +0{\cdot}68 \text{ inches.}$$

Now assume that the beam rested on the uprights of the piers and the ends of the brackets, and was loaded on the end span. Reckoning from the loaded end, let z, y, x, v, and t be the unknown downward pressures at the ends of the first five brackets. It was not necessary in this case to consider more than these, as beyond the fifth bracket the action was too small to be worth taking into account. By what had been explained, the following were the five equations of condition, the first corresponding to the end of the first bracket, the second to the end of the second bracket, and so on.

$$+\ 6{\cdot}04\ -{\cdot}1845\,z-{\cdot}1275\,y+{\cdot}0555\,x+{\cdot}0332\,v-{\cdot}0125\,t={\cdot}0765\,z$$
$$+\ 5{\cdot}29\ -{\cdot}1275\,z-{\cdot}1443\,y+{\cdot}0727\,x+{\cdot}0441\,v-{\cdot}0180\,t={\cdot}702\,y-{\cdot}626\,x$$
$$-\ 2{\cdot}54\ +{\cdot}0555\,z+{\cdot}0725\,y-{\cdot}1246\,x-{\cdot}0914\,v+{\cdot}0394\,t={\cdot}702\,x-{\cdot}626\,y$$
$$-\ 1{\cdot}585+{\cdot}0347\,z+{\cdot}0453\,y-{\cdot}0739\,x-{\cdot}1215\,v+{\cdot}0625\,t={\cdot}702\,v-{\cdot}626\,t$$
$$+\ 0{\cdot}679-{\cdot}0148\,z-{\cdot}0194\,y+{\cdot}0400\,x+{\cdot}0628\,v-{\cdot}1201\,t={\cdot}702\,t-{\cdot}626\,v$$

The first terms on the first side of these equations were the deflections given above for the continuous beam if rested only on the uprights of the piers. Any one of the other terms on this side of the equations, as for instance $+{\cdot}0725\,y$ in the third equation, expressed the deflection in inches of the continuous beam at the point corresponding to the end of the third bracket by the action of the force y as found by equations (D), (D″).

The second sides of the equations were the deflections of the ends of the brackets by formulæ (G) and (K).

It was not difficult to find a sufficiently approximate solution of these equations. For this purpose, assume y, x, v, and t each $= 0$.

Then $$z = 23{\cdot}14 \text{ tons.}$$

Next assume only v and t each $= 0$, and using this approximate value of z, it was found that

$$z = 21{\cdot}05 \text{ tons.}$$
$$y = 5{\cdot}64 \quad ,,$$
$$x = 3{\cdot}11 \quad ,,$$

By using these values as approximations, it was soon found by a few trials that a sufficiently accurate solution was

$$z = +\ 20{\cdot}8 \text{ tons.}$$
$$y = +\ 5{\cdot}6 \quad ,,$$
$$x = +\ 2{\cdot}9 \quad ,,$$
$$v = -\ 2{\cdot}0 \quad ,,$$
$$t = -\ 1{\cdot}2 \quad ,,$$

By means of these values and that of Q the load, which was taken at 50 tons, the values of r_0, r_1, r_2, &c., could be determined, and from them the equations of stress and deflection.

The curves corresponding to the end span loaded were shown at the right-hand side of Fig. 2, Plate 12A.

If all the spans were loaded uniformly, each with the weight Q, and the continuous beam rested on the piers, the brackets being supposed removed, the deflections at the points corresponding to the ends of each of the brackets, reckoning from the end of the beam, were by equations (F),

4·69, 3·45, 0·63, 0·93, 1·69, 1·61, 1·47, and 1·58 inches.

Now if the beam were supposed to be rested on the uprights of the piers and ends of the brackets before being loaded, each bracket would take a share of the load, and the action of none of them could be neglected. Reckoning from the end of the beam, let $z, y \, . \, . \, . \, . \, q$ be the unknown downward pressures at the ends of each of the brackets up to the middle of the beam. There would be a corresponding and equal set of pressures for the brackets on the other half of the beam, and these must be taken into account in forming the equations of condition.

In this manner the following equations were obtained, where, as before, the first corresponded to the end of the first bracket, the second to the end of the second bracket, and so on.

$$4{\cdot}69 - {\cdot}1845\,z - {\cdot}1275\,y + {\cdot}0555\,x + {\cdot}0332\,v - {\cdot}0125\,t - {\cdot}0130\,s + {\cdot}0003\,r + {\cdot}0001\,q = {\cdot}0765\,z$$
$$3{\cdot}45 - {\cdot}1275\,z - {\cdot}1443\,y + {\cdot}0727\,x + {\cdot}0441\,v - {\cdot}0180\,t - {\cdot}0135\,s + {\cdot}0018\,r + {\cdot}0034\,q = {\cdot}702\,y - {\cdot}626\,x$$
$$0{\cdot}63 + {\cdot}0555\,z + {\cdot}0725\,y - {\cdot}1246\,x - {\cdot}0914\,v + {\cdot}0394\,t + {\cdot}0254\,s - {\cdot}0067\,r - {\cdot}0027\,q = {\cdot}702\,x - {\cdot}626\,y$$
$$0{\cdot}93 + {\cdot}0347\,z + {\cdot}0453\,y - {\cdot}0915\,x - {\cdot}1215\,v + {\cdot}0625\,t + {\cdot}0395\,s - {\cdot}0128\,r - {\cdot}0014\,q = {\cdot}702\,v - {\cdot}626\,t$$
$$1{\cdot}69 - {\cdot}0148\,z - {\cdot}0194\,y + {\cdot}0400\,x + {\cdot}0628\,v - {\cdot}1201\,t - {\cdot}0868\,s + {\cdot}0312\,r + {\cdot}0124\,q = {\cdot}702\,t - {\cdot}626\,v$$
$$1{\cdot}61 - {\cdot}0093\,z - {\cdot}0215\,y + {\cdot}0244\,x + {\cdot}0383\,v - {\cdot}0848\,t - {\cdot}1128\,s + {\cdot}0442\,r + {\cdot}0094\,q = {\cdot}702\,s - {\cdot}626\,r$$
$$1{\cdot}47 + {\cdot}0010\,z + {\cdot}0053\,y - {\cdot}0110\,x - {\cdot}0145\,v + {\cdot}0329\,t + {\cdot}0535\,s - {\cdot}0974\,r - {\cdot}0560\,q = {\cdot}702\,r - {\cdot}626\,s$$
$$1{\cdot}58 + {\cdot}0024\,z + {\cdot}0032\,y - {\cdot}0067\,x - {\cdot}0060\,v + {\cdot}0139\,t + {\cdot}0220\,s - {\cdot}0494\,r - {\cdot}0575\,q = {\cdot}702\,q - {\cdot}626\,q$$

The explanation of the terms of these equations was the same as that given above for the beam loaded on the end span only; but it was to be remarked that, for instance, the term $+ \cdot 0329\, t$ in the seventh equation, which expressed the deflection of the beam at the end of the seventh bracket, was the sum of two terms, one, $+ \cdot 0394\, t$, for the action of the bracket on the right-hand side of the centre of the beam, and another, $- \cdot 0065\, t$, for the action of a corresponding bracket on the left-hand side of the centre, and so on for the other terms.

Proceeding with the solution as in the former case, the successive approximate solutions were

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DIAGRAM SHEWING ESTIMATED DEFLECTIONS AND STRAINS, ARISING FROM LOADING THE STRUCTURE WITH 65 TONS Pr SPAN, UNIFORMLY DISTRIBUTED; SUCH DEFLECTIONS AND STRAINS BEING INCLUSIVE OF ANY CAUSED BY ITS OWN WEIGHT OF 35 TONS.

REFERENCE TO LINES.

- A ——— *Loaded on all the spans*
- B — — — *Loaded on end span only*
- C —·—·— *Loaded on second span only*
- D —··—··— *Loaded on third span only*
- E —·—·— *Loaded on fourth span, or span nearest middle only*
- F - - - - *Girder supposed to be severed at piers and loaded uniformly*
- H — — — *Girder supposed to be continuous over two spans only & loaded on each span*
- I —·—·— *Girder supposed to be continuous over two spans only & loaded only on end span.*
- K —·—·— *Girder supposed to be "Fixed" at both ends, and loaded.*

Fig. 1. — Girders supposed to be continuous, without Iron Arches & resting on fixed Abutments.

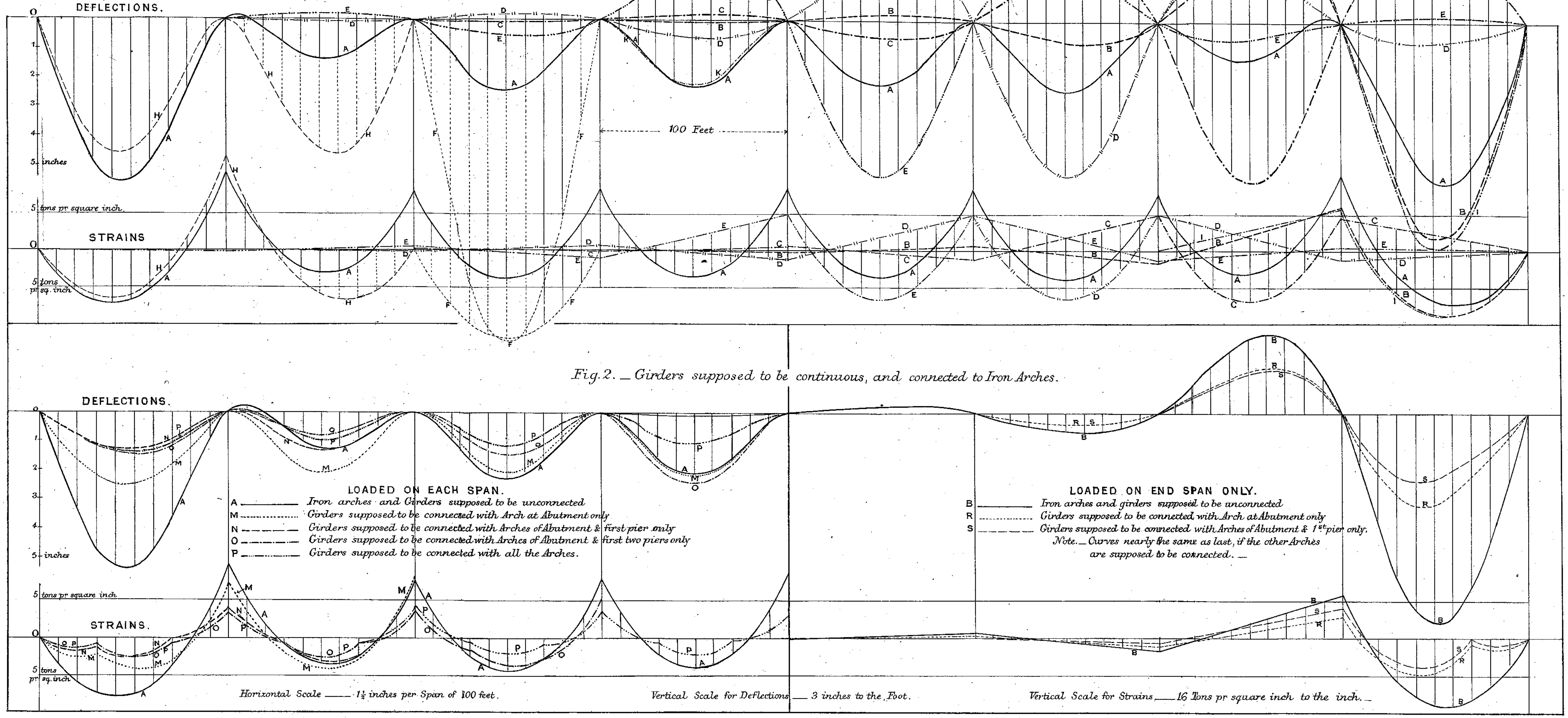

Fig. 2. — Girders supposed to be continuous, and connected to Iron Arches.

Horizontal Scale — 1½ inches per Span of 100 feet. Vertical Scale for Deflections — 3 inches to the Foot. Vertical Scale for Strains — 16 Tons pr square inch to the inch.

W. BELL DELT.

KELL BROS LITHRS CASTLE ST HOLBORN, E.C.

z =	14·4,	15·1,	16·2,	15·7
y =		10·8,	9·6,	9·5
x =		10·9,	9·3,	9·2
v =			10·7,	9·9
t =			11·0,	9·9
s =				9·1
r =				9·2
q =				10·5

For these last values of z, y, x, &c., the residual errors of the several equations were each less than ·1, or the elastic curve was determined within a probable error of ·1 inch. Using these values, r_0, r_1, r_2, &c., were determined, and from them the equations of stress and deflection.

The curves calculated in this manner were shown on the left-hand side of Fig. 2, Plate 12A.