*Discussion on Paper 6650**

The use of analogue computers for civil engineering problems

by

I. F. Christie, B.Sc., Ph.D., A.M.I.C.E.

Mr T. H. Douglas (Assistant Engineer, Messrs Babtie, Shaw & Morton) observed that unfortunately the computer at Edinburgh University was not available at the early stages of the surge designs for Nant. Dr Christie's work provided a very valuable check on the results obtained from the surge analysis in Glasgow and was, of course, much quicker. Its speed and the ease with which variable sizes could be introduced into the operation suggested that it could be a most useful tool for future analyses. A range of surge-shaft diameters could be investigated and provisions for coping with the varying results evaluated. The most economic arrangement of shaft size to height or overflow could, therefore, be more quickly assessed.

68. Like most schemes, the conditions at Nant did not follow the simple pattern usually illustrated in textbook problems. The junction of sloping surge shaft and tunnel below had an enlargement to allow for the inevitable breakage of rock at the sharp angle at the intersection. A similar enlargement occurred at the surface. The surge pond also served to bring into the tunnel system a hillside aqueduct which almost doubled the Nant catchment. A pond was formed by the construction of a small dam in a stream course but, for topographical reasons, the surge shaft was sited nearly 100 ft away. The two structures were joined by a rectangular channel which included a gate for isolating the aqueduct from the tunnel system and screens. Furthermore, the dam had a spillway which was designed primarily to pass floods from the burn on which it was situated but also to discharge the top wave of maximum surges. Although these features complicated Dr Christie's analysis the adjustments which could be made to the circuits dealt adequately with the problem.

69. The results obtained from the computer analysis shown on Figs 8 and 9 had been compared with those derived earlier in Glasgow and were in close agreement.

70. The office designs were only carried far enough to ensure that maximum or minimum levels had been ascertained. A further advantage of the analogue computer was that any number of waves could be included very simply by allowing the machine to continue the operation; this gave a more complete picture than was available from normal design procedure.

Dr M. B. Abbott (Research Fellow, Mathematical Centre, Amsterdam) congratulated the Author on his very concise introduction to electronic analogue computers, though he regretted that so much had to be omitted, no doubt in the interests of brevity. Analogue computers had much to recommend them to civil engineers, especially as regards their low initial cost and long-term reliability, but much more proof would be required of their broader capabilities before they were really accepted. The same difficulty of presenting a broad front of applications before engineers could be persuaded to apply the machine in practice had arisen with digital computers, and was still not fully overcome.

72. The acceptance of analogue machines could be accelerated, however, if it

* *Proc. Instn civ. Engrs*, vol. 25, July 1963, pp. 267–286.

was realized how much they had in common with digital machines. Thus the difference schemes and flow diagrams given by the Author could be carried over directly to a digital machine, while even the circuit diagrams could be interpreted as schematizations of an autocoded programme. The same problems of breaking down continuous systems into discrete systems arose in both types of machine, and it was probable that the more theoretical aspects of their utilization would increasingly coincide. After all, the same main facility was used in both digital and analogue machines, namely the facility of 'internal repetition'; the ability to repeat a certain process indefinitely and at great speed once that process was defined.[16] Whether the definition proceeded through the intercession of binary systems or not was, from this point of view, comparatively unimportant.

73. However, if analogue machines were to follow a similar path to the digital variety, they could scarcely fail to meet similar difficulties. The most important of these difficulties was associated with the 'stability' of the computation. Thus, if the computation settled down to a genuine solution it was said to be 'stable', but if it generated a spurious solution, generally associated with ever-increasing magnitudes, it was said to be 'unstable'. Instability often arose in the treatment of partial differential equations by difference methods, and its study and subsequent avoidance was one of the major subjects of digital machine utilization. This study could be carried over to the analysis of the finite–difference–differential equations of analogue practice, as follows.

74. Taking as an example the Author's equation (9), the complete operator, transforming one state of the system to another state, could be represented by the set of equations

$$\left\{\frac{du_\mathrm{i}}{d\tau} = (\delta^2 u)_\mathrm{i}\right\}, \qquad (16)$$

which set had a finite number, J, of elements. (In this case $\delta^2 u$ was the second central difference in u, while in the Author's study, $J=9$). The set (16), however, was equivalent to the set of equations

$$\left\{\left.\frac{u_\mathrm{i}^{\mathrm{n}+1} - u_\mathrm{i}^\mathrm{n}}{\Delta\tau}\right|_{\Delta\tau\to 0} = \left.\frac{(\delta^2 u)_\mathrm{i}^{\mathrm{n}+1} + (\delta^2 u)_\mathrm{i}^\mathrm{n}}{2}\right|_{\Delta\tau\to 0};\quad n\Delta\tau = \tau = \text{constant}\right\}, \qquad (17)$$

which set ran over, not only a finite number of $i, = 1, \ldots, J$, but also over a denumerably infinite number of n, as $\Delta\tau$ *tended* to zero. The equivalence of (16) and (17) was obvious for the class of functions normally used in engineering practice, which functions were continuous and differentiable almost everywhere.

75. Once (17) was established it could be analysed in the usual manner, by considering a Fourier series solution with orthonormal basis

$$\left\{\frac{e^{\mathrm{im}(\mathrm{j}\Delta x)}}{\sqrt{2\pi}}\right\} \qquad (18)$$

Then

$$u_\mathrm{j}^\mathrm{n} = \sum_\mathrm{m} A_{(\mathrm{m})}\, \xi_{(\mathrm{m})}^{\mathrm{n}}\, e^{\mathrm{im}(\mathrm{j}\Delta x)} \qquad (19)$$

and by substitution in (17) the Fourier coefficients were found to be related by

$$\xi_{(\mathrm{m})}^{\mathrm{n}+1} = \frac{1-\alpha}{1+\alpha}\,\xi_{(\mathrm{m})}^{\mathrm{n}}, \qquad (20)$$

where $\alpha = \Delta\tau(1-\cos m\Delta x)$, and for all m. Equation 20 showed that

$$|\,\xi^{\mathrm{n}+1}\,| < |\,\xi^{\mathrm{n}}\,|, \quad \text{for all } \alpha, 0 < \alpha, \qquad (21)$$

so that, regardless of the ratio $\Delta t{:}\Delta x$ chosen, no component of the Fourier series (19) could be amplified by the operator (17), even as $\Delta\tau\to 0$, i.e. $n\to w$ with w denumerably infinite. By this reasoning the system (16) was found to be unconditionally stable, and to correspond, in effect, to the implicit finite–difference operator used in a similar connexion by the writer.[17]

76. If, however, one was to employ the simpler, but much less automated circuit, corresponding to the explicit finite–difference operator

$$\frac{u_{l,n+1}-u_{l,n}}{\Delta\tau} \simeq u_{l+1,n}+u_{l-1,n}-2u_{l,n}$$

(a variation on the Author's equation (9)), instability would occur as soon as

$$\Delta\tau > \tfrac{1}{2}$$

or

$$\Delta t > \frac{(\Delta z)^2}{2c_v}$$

77. That the Author's implicit scheme avoided instability in this example was no guarantee that it could always be avoided with analogue machines, and in fact it would almost certainly arise in later developments.[18] This had already been found in the most complex analogue machine so far developed for civil engineering applications, namely that of the Rhine–Maas Delta region, constructed under Dr Ir. J. C. Schönfeld by the Netherlands Rijkswaterstaat. In this machine, instability occurred, whenever the domain of dependence of the forward point of the computational net, as given by the difference relations, fell within that point's domain, as determined by the differential equations, which condition again corresponded to the criterion governing instability in the equivalent class of difference schemes.[19,20]

78. It therefore seemed very necessary to establish a close relation between analogue and digital machines, which would not only help in the application of the former, but could also contribute to the development of the latter. This was especially the case as development moved beyond the study of linearized systems, with their matrix formulations, which systems were not of so much interest in engineering practice, and began to investigate the more realistic world of non-linear systems. If, as seemed likely, this led to a new emphasis on iterative schemes with their associated analysis, especially by continued fractions,[21,22] analogue machines might very well prove more effective than digital machines. This could make a very big difference indeed to their development, and it gave preliminary studies, such as those described by the Author, a special significance at the present time.

The Author, in reply, wished to thank Mr Douglas and Dr Abbott for the interesting supplementary information which they had provided.

80. For the purpose of investigating the rate at which the oscillations of water level in a surge chamber would decay, as suggested in § 70, the process could easily be accelerated by altering the time scale on the computer.

81. The Author agreed with Dr Abbott that some of the techniques of analogue computation could be converted to digital computation, and vice versa, but considered that some of the remarks in § 72 might be misleading. In general, analogue computation involved continuous and parallel operation of elements compared with discrete and serial operation in a digital computer.

82. In the special case of problems defined by partial differential equations, some degree of conversion to a discrete system was required even with an analogue computer because it could integrate with respect to only one variable. However, this conversion was incomplete. In the examples based on Terzaghi's theory of consolidation the space derivatives had been replaced by finite–difference approximations but the time derivative had been retained. Consequently, the analogue computer was used to solve a set of simultaneous finite–difference–differential equations, which were inherently stable in this case.

83. While the finite–difference equation given in § 76 had formed the basis of some work which the Author had carried out on a digital computer, it would never be used with a general-purpose analogue computer because it would be inefficient to use such a machine without taking advantage of its facility for direct integration.

References

16. M. B. Abbott. Structural analysis by digital computer. *Engineering*, vol. 187, 22 May, 1959, pp. 666–667.
17. M. B. Abbott. One-dimensional consolidation of multi-layered soils. *Géotechnique*, vol. 10, no. 4, December 1960, pp. 151–165.
18. R. D. Richtmeyer. 'Difference methods for initial value problems.' Interscience, New York 1957, Chapters I and VI.
19. R. Courant, E. Isaacson *and* M. Rees. On the solution of non-linear hyperbolic equations by finite differences. *Comm. Pure and App. Math.*, vol. 2, 1952, p. 243.
20. M. B. Abbott. 'Tidal wave propagation and the method of characteristics.' Ph.D. thesis, University of London, 1961.
21. P. Wynn. Continued fractions whose coefficients obey a non-commutative law of multiplication. *Archive for rational mechanics and analysis*, vol. 12, no. 4, 1963, p. 273.
22. P. Wynn. Acceleration techniques in numerical analysis, with particular reference to one independent variable. I.F.I.P. Congress, March 1962, p. 149.

Corrigenda

In the original Paper:

§ 48: $T_v = \frac{C_v t}{H^2}$ should read $T_v = \frac{c_v t}{H^2}$

Fig. 14, equation (*a*): $\frac{du_1}{d\tau}, j \simeq$ should read $\frac{du_{1,j}}{d\tau} \simeq \ldots$

equation (*b*): $\frac{du_1}{d\tau}, 0 \simeq$ should read $\frac{du_{1,0}}{d\tau} \simeq \ldots$