*Discussion on Paper No. 6683**

On the influence of curvature, surface tension and viscosity on flow over round-crested weirs

by

G. D. Matthew, B.Sc.(Eng.), Ph.D., A.M.I.C.E.

Dr D. I. H. Barr (Lecturer, Department of Civil Engineering, The Royal College of Science and Technology, Glasgow) wrote that in this excellent Paper an approach was made to the rationalization of scale effect in a gravitational flow situation where viscous and surface tension forces were important. It was concluded that

$$q = C_q\sqrt{(2g)}H^{3/2},$$

where $$C_q = f(H/r, \quad \sigma/Wr^2, \quad \nu^{1/2}/g^{1/4}\, r^{3/4}).$$

49. An alternative approach was as follows. Similarity criteria based on the Froude, Reynolds, and Weber dimensionless groups or numbers (F, R, and W) could be written:

$$\frac{V_p}{V_m} = \frac{g_p{}^{1/2}H_p{}^{1/2}}{g_m{}^{1/2}H_m{}^{1/2}} \quad \text{(Froude similarity criterion)} \quad . \quad . \quad (18)$$

$$\frac{V_p}{V_m} = \frac{\nu_p H_m}{\nu_m H_p} \quad \text{(Reynolds similarity criterion)} \quad . \quad . \quad (19)$$

$$\frac{V_p}{V_m} = \frac{\rho_m{}^{1/2}H_m{}^{1/2}\sigma_p{}^{1/2}}{\rho_p{}^{1/2}H_p{}^{1/2}\sigma_m{}^{1/2}} \quad \text{(Weber similarity criterion)} \quad . \quad . \quad (20)$$

These relations had been written so that the ratio V_p/V_m was to the first power in each case and the notation was as used in the Paper, with the addition of

V = characteristic velocity,
ρ = density.

H was used in the sense of the stillwater head over the weir.

50. It would be recalled that the ratio V_p/V_m compared velocities at corresponding points in accelerative systems assumed to be dynamically similar. The subscripts 'p' and 'm', while suggested by prototype and model, did not imply that a model/prototype comparison was necessarily being made.

51. Eliminating V_p/V_m between (18) and (19) led to

$$\frac{g_p{}^{1/2}H_p{}^{3/2}}{\nu_p} = \frac{g_m{}^{1/2}H_m{}^{3/2}}{\nu_m}$$

or essentially a further dimensionless group $g^{1/2}H^{3/2}/\nu$, for which equality was required as between compared systems affected by both gravitational and viscous forces for similarity to be possible. For convenience, this group might be given the notation $\overline{FR}$.

52. Similarly (18) and (20) led to

$$\overline{FW} = \sigma^{1/2}/g^{1/2}\rho^{1/2}H$$

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* *Proc. Instn civ. Engrs*, vol. 25, August 1963, pp. 511–524.

and (19) and (20) led to

$$\overline{RW} = \frac{\sigma^{1/2} H^{1/2}}{\nu \rho^{1/2}}$$

53. The first of Dr Matthew's dimensionless groups described the geometry of the weir. For a very carefully constructed V-notch weir this group would be omitted; for a round-crested weir with a short crest length, B, an additional group B/r would be necessary. The second group was essentially similar to $\overline{FW}$ and the third group to $\overline{FR}$. Dr Barr, while recognizing that Dr Matthew had made the realistic approach of having a varying head over a particular weir shape, was not entirely convinced of the rationality of taking r as the characteristic length. Supposing that a series of tests was to be undertaken using a range of liquids with differing viscosities and surface tensions, he thought that the similarity premise led to the expectation of a fairly straightforward correlation of C_q for a given H/r ratio on the basis of entering the observed values at the appropriate points on a graph of $\overline{FW}$ against $\overline{FR}$. Each H/r ratio would require a separate correlation, which would be inconvenient. Perhaps this pointed to the V-notch as being the most appropriate starting point for a study where different liquids were to be used.

54. The $\overline{RW}$ dimensionless group appeared less helpful in the present case, although it might be used, as experience was gained, to determine whether, in an individual case, the effect of viscosity or of surface tension was likely to be dominant, while that of the other could be discounted. For example, in the case of the flow of a very viscous liquid over a weir, with H of the order of, say, 1 in., $\overline{RW}$ would be likely to indicate that the surface tension effect was unimportant.

55. The nomenclature of dimensionless groups appeared to be a somewhat vexed question. For instance, if the densimetric Froude number F_Δ were used instead of the Froude number, then

$$\overline{F_\Delta R} = \frac{(\Delta\rho/\rho . g)^{1/2} H^{3/2}}{\nu}$$

would be obtained as previously. This was of essentially the same significance as the Grashof number, however much the latter was associated with temperature difference. Again, gL^3/ν^2 had been called the Galileo number, where L was the characteristic length.

56. Dr Barr felt that there was some advantage in the notation he had used, even over the writing of the group in full on each occasion, because the relation of the 'paired' active force group to the single active force groups was implied. He hoped to pursue the approach of 'pairing' single active force groups in a paper which was in preparation.[14]

Dr F. V. A. Engel (Consultant, Workington, Cumberland) believed that scale effects might be attributed to an incomplete analysis or presentation of the basic relations. This uncritical approach, often applied in evaluating the results of model investigations, meant that incoherent influences were lumped together. The Author's attempts to disentangle some major influences by rational methods were therefore of great importance.

58. Two aspects might be dealt with which required some elucidation:

(a) the equations applied by Dr Matthew compared with previously-established relations for round-crested weirs; and

(b) 'scale effects' encountered in small-scale models, such as

  (i) surface tension and viscosity influences;

  (ii) other influencing factors, in particular, installation conditions.

59. In some of his basic relations, e.g. equations (4a) and (6), the Author referred to a 'quantity α being in the nature of a "Coriolis coefficient".' In § 13 the special

case of $\alpha=1$ was mentioned. Again in § 14, several references were made to α 'which is required in the evaluation of equation (6)' and later the Author said 'it is possible to obtain first approximations to the derivatives of h and thence to α . . .' But in equation (9) no α-terms occurred. As α was an independent variable depending on installation and flow conditions, some explanation was warranted.

60. Could the Author give a more detailed derivation, presenting some of the intermediate relations which led to equation (9)? Furthermore, how did the term λ/r^2 enter the equation, since no explanation of this term could be traced in the introductory paragraph of the Paper? Did the rate of change of curvature refer to the fact that the curvature of the water-level surface above the weir crest did not truly correspond to the geometrical outlines of the weir crest? Or did this term refer to the geometry of the weir crest itself which might not be formed by a major part of an arc of a circle? The Author could perhaps also give some figures related to λ which he inserted in his calculations when establishing the 'theoretical curves' illustrated in his various diagrams.

61. The 'theoretical' equation (9) was most likely only applicable in cases of small-scale models and should not be used to the full-scale plant, i.e. the prototype. Fig. 5 appeared to imply that the 'asymptote' could represent the performance of a full-scale round-crested weir.

62. Applying only the first two terms in equation (14), i.e.

$$\left[1+\frac{22}{81}\left(\frac{H}{r}\right)\right] \qquad \qquad (21)$$

as a multiplier, the results obtained for $(H/r)\leqslant 1$ were in close agreement with an equation established in 1936 by Lauffer,[15] namely,

$$\text{Multiplier} = 2{\cdot}3(1+r/y)\log(1+y/r) \qquad \qquad (22)$$

Assuming the depth y of the water level over the weir crest to occur at the critical flow section, it might be expressed in terms of the total head H, i.e.

$$y = \tfrac{2}{3}H.$$

63. This multiplier (22) deviated over the range 0–1 for the H/r-values from those resulting from the term (21) by 1% or less, as shown in Table 2. Extending the range to $(H/r)=2$, the multiplier (22) was 1·48 whereas the term (21) was 1·543. This was not surprising, since by omitting the third term in the Author's equation one could not claim a high degree of accuracy. It might be noted that the same relation as established by Lauffer was also derived independently by Knapp (Table 2).[16]

64. It was well understood that the Author wanted to establish an equation for his small-scale models including surface tension and viscosity influences. The engineer mainly concerned with large structures required for his design a relation which was not restricted to small H/r-values, say, below 0·5. He might want to extend the range to $(H/r)=2$. Did the Author recommend relation (22) for practical purposes? Or could he offer any comments on the method of deriving this equation which would seriously restrict its application?

65. The pronounced divergency of the theoretical curve from the curve drawn through the plotted test points, Figs 6–10, might indicate some shortcomings of the theory. Otherwise no explanation could be given for the results of an investigation by Eisner, which would be dealt with in some of the following paragraphs. The Author was therefore asked to comment on this point which appeared to be a crucial one regarding the interpretation of small-scale model tests.

66. The Author had restricted his investigations to heads which were small in comparison with the physical dimensions of the weir, but had stated that the qualitative conclusions were likely to apply beyond that restriction. It might therefore be of interest to refer to scale effects observed by Eisner on round-crested weirs for large H/r-values (Table 3).[17]

67. In view of Reynolds number and/or Weber number influences on the discharge

TABLE 2: COMPARISON OF THE LAUFFER/KNAPP EQUATION (22) WITH THE EXPRESSION (21)

H/r	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	2·0
Equation (22)	1·029	1·060	1·093	1·118	1·148	1·176	1·206	1·227	1·250	1·273	1·480
Expression (21)	1·027	1·054	1·081	1·109	1·136	1·163	1·190	1·218	1·245	1·272	1·543

Note: The figures in this Table are only of slide-rule accuracy.

TABLE 3: SCALE EFFECTS OBSERVED BY EISNER ON MODELS OF ROUND-CRESTED WEIRS

Scale ratio of model	Head, *cm*	Discharge coefficient for similar heads	Deviation of discharge coefficient from prototype %	Discharge coefficient based on equal Froude numbers, $F_r = 0{\cdot}311$	Deviation discharge coefficient from prototype for equal Froude numbers %
1:70	175:70	—	—	0·494	−5·0
1:35	175:35	0·482	−7·3	0·496	−4·6
1:17·5	175:17·5	0·490	−5·8	0·498	−4·2
1:8·75	175:8·75	0·497	−4·4	0·500	−3·85
1:4·375 . . .	175:4·375	0·503	−3·3	0·504	−3·1
Prototype . .	175	0·520 approx.	—	0·520 approx.	—

coefficient on small-scale model weirs, it could not be expected that either similar heads or the same Froude number in the approach channel would be sufficient to characterize similar discharge conditions. This became evident also from the diagrams presented by the Author on an H/r-basis.

68. The fact that the upstream Froude number was not sufficient to characterize similar flow conditions could be demonstrated by reference to a general relation applicable to broad-crested weirs.[18] This relation was

$$F_{r1} = (2/3)^{3/2} C C_{va}[h_1/(h_1+w)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

69. For broad-crested weirs, defined by a length of the weir crest $L \geqslant 2w$, with w being the height of the crest above the channel invert, and the area ratio limited to about $h_1/(h_1+w) = 0{\cdot}5$, a range of flow conditions existed for which the discharge coefficient was nearly constant. There existed a lower limit of the 'constancy' range characterized by small area ratios or more specifically by low Reynolds numbers. The Reynolds number influence for small-scale models might become predominant and the discharge coefficient C might decrease 5–10%, which in accordance with relation (23) would result in a corresponding change of the upstream Froude number. Therefore, equal Froude numbers could not be considered as a sufficient criterion of similarity for long-structure type weirs. In his Fig. 21, Eisner[17] had shown that for square-edged, thin plate weirs, a lower limit of the discharge coefficient characteristics existed when Reynolds number and Weber number influences became predominant. It could be assumed that similar conditions prevailed for round-crested weirs.

70. From the investigations by Eisner it could be concluded that the general shape of the experimental curves (e.g. as shown in Fig. 5 of Dr Matthew's Paper), with the steep increase in the range up to $(H/r) = 0{\cdot}3$, was not limited to this low range. The sharp bend in the coefficient curve might occur for each model size at a different, distinct H/r-value. This would mean that exceeding a certain H/r-value of 0·3 did not present a criterion for the discharge coefficient to correspond closely with the curve given by equation (22). The limit for the onset of predominant viscosity and surface tension influences might depend on a specific Reynolds number (not the so-called 'critical' Reynolds number). The convergency of the flow lines in the approach channel up to the weir crest might produce viscosity influences at Reynolds numbers of 5000 or more. In case of Venturi tubes, considerable deviations from the 'constancy' range of the discharge coefficient depended on the area ratio and the geometry of the entrance nozzle. For this type of device specific Reynolds numbers lay between 100 000 and 250 000.[19]

71. Unfortunately, no tables had been published by Eisner which could be used to clarify this issue. Fig. 4 of Eisner's paper presented the Reynolds number as abscissa and the Froude number as ordinate. A set of curves was given with the discharge coefficient as parameter. From this diagram the predominant Reynolds number influence for both the small-scale models with scale ratios of 1:35 and 1:70 became evident. These model sizes corresponded well with those applied by the Author. One might conclude that Reynolds numbers not smaller than 2000 should be used in small-size model investigations. It would be helpful if the Author would state the Reynolds number range of his investigation; also a Froude/Reynolds number diagram would be a useful addition.

72. The range of small Reynolds numbers inherently related to all models of too small geometrical dimensions might show other disadvantages. First of all, somewhere within this range transition from laminar to turbulent flow might occur, which would make it impossible to obtain reliable information for the prototype. Furthermore, the low range of Reynolds numbers might be related to patterns of the velocity profile in the approach channel which differed considerably from those in the prototype with fully-developed turbulent flow. There was no information available on the influence of the velocity profile on weir flow. Some 25 years ago, in co-operation

with A. J. Davies, Dr Engel had dealt with this problem in connexion with flow through pipe contractions.[20]

73. In Fig. 13 of his paper, Lauffer found, in the range of low H/r-values (smaller than 0·5) for increasing rate of flow, coefficients several per cent above the curve given by equation (22) and, with decreasing rate of flow, coefficients well below this curve. This feature might be attributed to disturbances which increased with increasing rate of flow, and vice versa. Changes in the rate of flow might also have influenced the prevailing patterns of the velocity distribution and might not have merely resulted in random disturbances. This was another reason why, as already mentioned at the beginning of Dr Engel's remarks, the Author was asked for his comments on α, namely, the Coriolis coefficient.

74. It would be of interest to know if the Author had considered other influencing factors which might account for scale effects. With a rounded weir crest it was likely that contraction or a separation bubble occurred which might influence those ranges of H/r-values outside the range of predominant viscosity or surface tension effects.

Mr Sergio Montes (Chief Engineer, Laboratorio de Hidráulica, Universidad de Chile, Santiago) remarked that the Author was to be commended for his very interesting Paper. His theory on the subject of round-crested weirs was a definite step forward in the understanding of weir flow.

76. Mr Montes wished to draw attention to certain engineering implications of the Paper. Obviously, the most important point was the formulation of an answer to the problem of what was the limiting size for a reliable reproduction of the discharge coefficient of the model of a weir or spillway.

77. The Author's equation (14) could be the solution, because with different scales the absolute magnitude of r and H could be inserted and the different values of C_q computed. Equation (14) gave good agreement with some very small round-crested weirs tested. Nevertheless, if it was desired to extrapolate to prototype structures the results of equation (14), two points had to be considered:

(*a*) the prediction of a higher C_q value for the prototype was not usually borne out; and

(*b*) as a corollary to (*a*), the values given by equation (14), when extrapolated to prototype conditions, would be excessive.

78. One of the reasons why the C_q value of the model was sometimes higher than the prototype was that the model's relative roughness was higher than that of the prototype. This seemed to have a very definite effect in increasing C_q over its prototype value. Sharp-crested weirs, for example, showed an increase of 1–2% in the discharge coefficient. For other shapes, roughening the upstream faces did not give such conclusive results, but the tests by Johnson[21] of the Upper Narrows Dam model, and the comparison by Hickox and Mavis[22,23] of model prototype data on ogee spillways, indicated a small increase in the value of C_q when the model was roughened (if the scale exceeded a certain value) and when model data were compared with the full size structure data, respectively.

79. This discussion did not take into account several factors which had a distinct bearing on sharp-crested weir coefficients. Recent research had indicated that, for this type of weir, the absolute magnitude of the weir had some influence on the discharge coefficient, apart from the obvious surface tension effects at low heads.

80. The second point, that equation (14) gave an excessive C_q value for prototype conditions (even for the same value of the relative roughness), was open to discussion because no full size structure was exactly similar to models tested by the Author. However, his weir no. 3 came near to an ogee spillway weir. It was interesting then to compare the values of actual experimental data and the values supplied by equation (14). This had been done in Fig. 11, where data by Rehbock, Fawer,

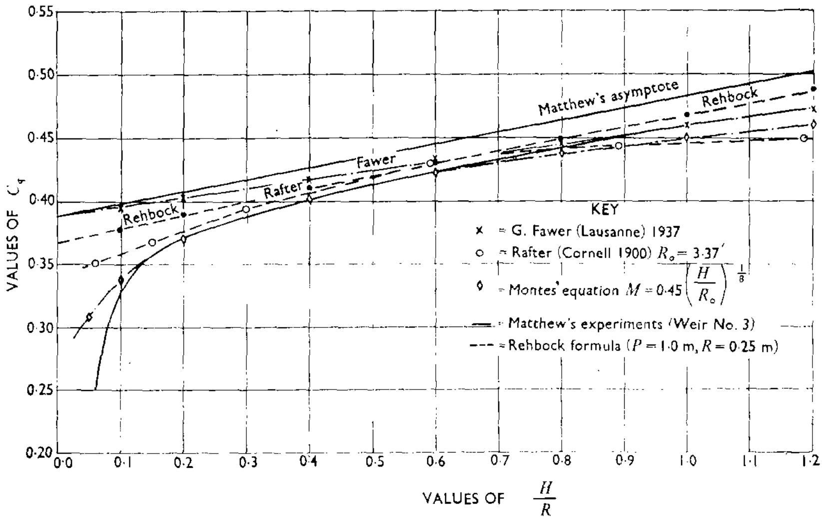

FIG. 11: A COMPARISON OF DATA BY VARIOUS EXPERIMENTS ON ROUND- OR NEARLY ROUND-CRESTED WEIRS

Rafter, and an empirical equation developed from the work of Abecasis Manzanares[24] were shown.

81. In every case, the value of C_q was less than the asymptote indicated for weirs with $r > 1{\cdot}0$, say, and values of $(H/r) > 0{\cdot}10$, while the equation valid for weirs of 1-in. radius seemed to fit rather well data found on these larger structures. Again, a smallest practical size did exist, as shown by Johnson's tests, but viscous effects seemed to disappear faster than equation (14) indicated.

82. Obviously the comparison was incomplete, but the amount of data on model-prototype comparison of the behaviour of weirs was limited. A really interesting development would be the testing of a geometrically-similar series of models of weir no. 3, say with radii of 2, 4, 8, and 16 in. These tests should not present any insuperable difficulty even to a small hydraulic laboratory, and would contribute a great deal to the understanding of weir flow. At the same time, the tests with a roughened upstream face, which the Author found inconclusive, would indicate to what extent relative roughness influenced the discharge coefficient for this type of weir.

Mr P. Ackers (Senior Principal Scientific Officer, Hydraulics Research Station considered that the Author's treatment of the head to discharge relationship for round-crested weirs was a most valuable addition to the literature both on flow measurement itself and on scale effects in hydraulic models. It was particularly valuable in its derivation of theoretical expressions for the two fluid property influences, namely, viscosity and surface tension; this approach was likely to find application in the broader field of flow measurement by weirs.

84. Equation (14) permitted one to evaluate the relative significance of the two relevant fluid properties in influencing the discharge, the ratio of surface tension to viscous effects on a cylindrical-crested weir ($\lambda = 0$) being:

$$\frac{\Delta_\nu}{\Delta_\sigma} = \frac{1{\cdot}05(3/g)^{1/4}(\nu^{1/2}/r^{3/4})}{5\sigma/6Wr^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Assuming water at 15°C to be the fluid,

$$\frac{\Delta_\nu}{\Delta_\sigma} = \left(\frac{1\cdot66\nu^{1/2}}{g^{1/4}\sigma/W}\right) r^{5/4} = 30\cdot3r^{5/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

85. Thus $\Delta_\nu = \Delta_\sigma$ for $r = 0\cdot065$ ft (i.e. approx. $\frac{3}{4}$ in.), and hence if the crest radius was less than this, surface tension was a greater influence than viscosity, whereas for radii above this figure, surface tension became increasingly less important in relation to viscosity in its influence on coefficient values. In typical hydraulic model studies of round-crested weirs, then, both surface tension and viscosity were important, although only the latter might have a significant influence in the prototype.

86. A useful concept that had been put forward for both long-throated structures[25] and thin plate weirs[26] was that of effective head, given by

$$H_e = H - k_h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where k_h was a displacement thickness given by the elevation of the equivalent 'ideal fluid' weir crest above the actual crest level. To first order accuracy, the Author's general equation for a cylindrical weir crest, for example, became:

$$q = C_e\sqrt{(2g)}H_e^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where
$$C_e = \frac{2}{3\sqrt{3}}\left\{1 + \frac{22}{81}\left(\frac{H}{r}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and
$$k_h = \frac{5\sigma}{9Wr} + 0\cdot70(3/g)^{1/4}\nu^{1/2}r^{1/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In terms of the Author's semi-empirical equation (15),

$$k_h \simeq \tfrac{2}{3}\, cr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and hence the experimental k_h values could be compared with their theoretical values as follows (for water at 15°C, ignoring λ, as values for this were not quoted):

Description	Observed k_h, *ft*	Theoretical k_h, *ft*
A. Parabolic	0·00168	0·00117
B. ,,	0·00166	0·00114
C. Cylindrical	0·00083	0·00127
D. ,,	0·00103	0·00127
E. ,,	0·00101	0·00127

87. The agreement in such small terms was remarkably close, although the observed values were somewhat greater than the theoretical values for parabolic weirs, whereas the reverse held for the smaller radius cylindrical ones. The uncertainty about the coefficient 0·70 in equation (13) perhaps, or the influence of the rate of change of curvature, might explain the small difference.

88. An important consequence of the magnitude of k_h was that, in experiments on this subject, supreme accuracy of zero setting and head measurement was required. A tolerance of $\pm 0\cdot001$ ft would completely mask the fluid property influences, and this might explain why some collections of weir data were of little value in resolving coefficient variations at low heads.

89. Assuming that the crest radius was large enough for surface tension to be much less significant than viscosity, the Reynolds number influence could be separated from the geometrical influence by expressing the weir equation as

$$q = C_e C_\nu \sqrt{(2g)}H^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where

$$1 - C_\nu = 1\cdot38\left(\frac{\sqrt{(gH)}r}{\nu}\right)^{-1/2}\left(\frac{r}{H}\right)^{3/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It would be noted that $\sqrt{(gH)}r/\nu$ was a crest Reynolds number, and that the equation plotted linearly, on a log log basis, with $(1-C_v)$ against H for a given crest geometry and fluid. The equation was analogous to that for a smooth broad-crested weir[6] (with a turbulent boundary layer generally, however, rather than a laminar one as in the present case):

$$1-C_v = 0{\cdot}0690\left(\frac{\sqrt{(\frac{2}{3}gH)}l}{\nu}\right)^{-0{\cdot}2}\left(\frac{l}{H}\right)^{0{\cdot}8} \quad . \quad . \quad . \quad . \quad (32)$$

where l was the crest length. It could also be shown that the coefficient for a triangular profile (1 in 2 upstream face, 1 in 5 downstream face, for example), in the absence of surface-tension effects, should have the form:

$$1-C_v = \alpha\left(\frac{\sqrt{(gH)}H}{\nu}\right)^{-1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

There was some experimental confirmation for this, too.

90. The application of the new theory to hydraulic model tests of round-crested weirs might be as follows:

(a) evaluation of ${}_pk_h$ for the prototype, on the basis of the known values of r and λ for the given fluid;
(b) evaluation of ${}_mk_h$ for the chosen scale of model, using model values of r and λ, of course;
(c) conduction of the model tests, plotting the calibration data against effective head related to a virtual crest level ${}_mk_h$ above the actual crest;
(d) this plot would take account of geometrical influences, and would thus apply to the Froudian scale to the full-scale installation, prototype head being defined by $H = H_e + {}_pk_h$.

91. For example, a one-tenth scale model of a 1-ft radius cylindrical weir would yield ${}_pk_h = 0{\cdot}0014$ ft and ${}_mk_h = 0{\cdot}0012$ ft. It would be apparent in this case, too, that at 1-ft head in the prototype, the fluid properties would affect the discharge by $1{\cdot}5 \times 0{\cdot}14 = 0{\cdot}21\%$, while at the corresponding head of 0·1 ft in the model their influence amounted to $1{\cdot}5 \times 1{\cdot}2 = 1{\cdot}8\%$. For an accurate full-range calibration to be deduced from tests on a small model, allowance should clearly be made for scale effects.

Mr J. F. Cooke (Hydraulics Research Station) wrote that the Author was to be congratulated on a Paper which would be of great value, and not least in the realm of assisting in the choice of scales for hydraulic models.

93. The last term within the bracket in equation (9) included $(r/H)^{1/2}$, whereas in equation (14) the same term includes $(r/H)^{3/2}$. Was one, or other, of these a misprint?

94. It would have been of added interest if the experiments with the parabolic weirs had been extended to include values of H/r up to 1·4, which was the approximate value associated with a spillway with no velocity of approach and no negative pressure under the nappe. For such conditions this would represent an ultimate value for H/r, and the value of C_q would be 0·496.[27]

95. Should not the value of 'b' in Table 1 for the Lausanne formula be 0·026 instead of 0·027 as quoted?

The Author, in reply, thanked the contributors to the discussion of his Paper. Many important topics had been raised, one or two errors pointed out, and a new slant given to some of the original theoretical results. In addition, comparison with some other sources not mentioned in the Paper had been made.

97. Dr Barr had approached the subject by a dimensional argument and shown that the groups which the Author had isolated in his theory were, naturally enough, related to the Reynolds, Froude and Weber numbers as defined in § 49. The

Author's equation (14), taking for simplicity the case of a symmetrical weir crest for which $\lambda=0$, could easily be recast in terms of the $\overline{FR}$ and $\overline{FW}$ groups introduced by Dr Barr to give:

$$C_q = \frac{2}{3\sqrt{3}}\left[1+\frac{22}{81}\left(\frac{H}{r}\right)-\frac{5}{6}\,\overline{FW}\left(\frac{H}{r}\right)-1\cdot 38\,(\overline{FR})^{-1/2}\left(\frac{H}{r}\right)^{-1/4}\right] \quad . \quad (14a)$$

98. Dr Barr had queried the rationality of taking r as the characteristic length. There had been no question of selecting r as characteristic since it had simply emerged as such in the analysis. But, in any case, in an essentially two-dimensional flow problem like this, there was no alternative but to take one of the physical dimensions of the cross-section of the weir as 'characteristic', the crest radius of curvature being the obvious choice.

99. The Author defended his original arrangement of equation (14) on the grounds that it was normal practice to present coefficients of discharge for round-crested and ogee weirs as functions of H (or of H/r in a dimensionless plot). The disadvantage of Dr Barr's groups $\overline{FR}$ and $\overline{FW}$ was that they themselves were dependent on H so that no simple C_q versus H/r plot was possible.

100. The case of a V-notch weir had been mentioned but, as Dr Barr had himself said, this was essentially different from the geometrical point of view. Even here, however, the effect of absolute scale was not entirely removed. Ideally, all V-notch weirs having the same V-angle were identical but in reality the precise nature of the lip, its thickness and roughness, did enter the problem to produce small but discernable scale effects.

101. Finally the Author agreed with Dr Barr that the definition of Froude, Reynolds, and Weber numbers was by no means a straightforward matter in a problem of this kind.

102. Dr Engel had opened his contribution with a general comment about 'scale effects'. Surely 'scale effects' were simply the disparity between predicted and actual prototype behaviour resulting from the fact that the modelling criterion used did not relate to all the phenomena present in the problem but only what was reckoned to be the dominant one.

103. Dr Engel had gone on to ask for further clarification of the theory summarized in the Paper, especially in connexion with the quantities α and λ. The Author felt that, in the interests of brevity, he had condensed the theory perhaps beyond the point where the intermediate steps could easily be filled in by the reader. This was to be regretted and so some additional explanation might be of help.

104. Returning to equations (4) and (4a), it was seen that α had been defined by

$$\alpha = 1+\frac{2}{3}h\frac{d^2h}{dx^2}+h\frac{di}{dx}-\frac{1}{3}\left(\frac{dh}{dx}\right)^2-i\frac{dh}{dx}-i^2 \quad . \quad . \quad . \quad . \quad (4c)$$

It was therefore a function of the depth h, the bed slope i, and their derivatives with respect to x. It had been likened to a Coriolis coefficient in traditional open channel flow equations because it appeared as a multiplier in the term $(\alpha q^2/2gh^2)$ or $(\alpha\bar{v}^2/2g)$, where $\bar{v}$ was the mean velocity. Here, however, the 'Coriolis coefficient' α existed, not because of the influence of internal friction, but purely because of the curvilinear nature of the flow.

105. One of the central problems of the theory had been the calculation of first approximations to the quantities α and $d\alpha/dh$ which featured in the critical flow equation (6). Looking at (4c) this meant obtaining first approximations to h, dh/dx, d^2h/dx^2 and d^3h/dx^3 at the critical section. It was explained in § 14 that these first approximations had been derived from the simple open channel equation (8), in which surface tension and curvature effects were ignored altogether by the following argument.

106. Differentiating (8),

$$\frac{dh}{dx} = -\frac{i}{1-q^2/gh^3}$$

so that in order to have $dh/dx \neq 0$ where $i=0$ (at the crest) the denominator had to be simultaneously zero, giving

$${}_{(1)}h_c = \sqrt[3]{(q^2/g)}.$$

The pre-suffix (1) indicated a first approximation. Since dh/dx had therefore an indeterminate form at the crest, de l'Hospital's rule had to be applied, giving ultimately

$${}_{(1)}\left(\frac{dh}{dx}\right)_c = -\sqrt{\left(\frac{{}_{(1)}h_c}{3r}\right)}.$$

The next higher derivates of h had been found in a similar fashion by successive differentiation and application of de l'Hospital's rule. For example

$${}_{(1)}\left(\frac{d^2h}{dx^2}\right)_c = \frac{4}{9r}+\frac{\lambda}{r}\sqrt{\left(\frac{{}_{(1)}h_c}{27r}\right)}$$

where λ/r^2 was the rate of change of curvature of the weir surface at the crest section. This would explain the perhaps mysterious appearance of the quantity λ in the final equations. To this same degree of accuracy ${}_{(1)}h_c = 2H/3$.

107. These first approximations had then been inserted in the more accurate equations (4a) and (6) and simplification carried out to a degree of accuracy compatible with the initial assumption that the ratios of H to r were small. This had finally yielded relationship (9) linking q and H.

108. This additional explanation would augment the brief description of the analytical technique given in § 14 of the Paper and answer the specific queries raised by Dr Engel in § 59 and § 60. In particular it should explain why α had no place in the final results.

109. The Author did not agree with Dr Engel's remarks in § 61 that the theoretical results would necessarily be applicable only to small-scale weirs. For round-crested weirs of large absolute size, surface tension and viscosity effects of the kind postulated would have a negligible effect at head ratios over 0·1, so that the coefficient of discharge would have the form $C_q = 0{\cdot}385\ (1+0{\cdot}272(H/r))$ valid roughly in the range $0{\cdot}1 \leqslant H/r \leqslant 0{\cdot}5$. As Dr Engel had himself admitted, this was in quite close agreement with Lauffer's results for such a weir and it was also in general accord with the Lausanne empirical equation. The Author would return later to the matter of which equation he recommended for practical use.

110. Eisner's results were altogether more surprising, showing up very marked scale effects at a value of H/r far beyond the range of the Author's theory. The actual value of H/r had not been stated but was probably of the order of 2. Strict comparison was difficult because no information had been given on how the coefficient of discharge had been defined or on the approach conditions.

111. It might well be that, as Dr Engel had suggested in § 74, some form of separation was occurring at this high value of H/r, so that the mechanism of flow was distinctly different from that assumed in the theory. This question of separation was discussed in greater detail later.

112. Many of Dr Engel's remarks had centred on the Reynolds and Froude numbers, presumably of the approach flow. It was the Author's belief that, in very high weirs or, in general, where the heads were small in comparison to the height of the crest above the invert of the approach channel, the Reynolds and Froude numbers of the approach flow played virtually no part in the weir behaviour. He believed that the zone of convergence was sufficient to transform any regime of flow upstream into the laminar boundary layer phenomenon considered in the Paper. There would, however, be a threshold beyond which these factors would begin to play an increasing role in the behaviour of the weir and consequently in the discharge equation.

113. The Author therefore agreed with Dr Engel that such a threshold would probably depend on a specific Reynolds number not necessarily related in any way to a 'critical' Reynolds number associated with transition.

114. Mr Montes had dwelt on the implications of the Paper with regard to the limiting size of model weir required for an acceptable reproduction of discharge coefficient. He had recalled the fact that, on occasions, higher and not lower coefficients had been observed on models as compared with prototype values and suggested that the explanation might lie in the fact that the surface roughness of the model could well be relatively greater than that of the prototype.

115. If a turbulent boundary layer could be sustained on the round-crested weir, the Author had shown[11] that higher coefficients of discharge would certainly result, but he had been quite unable to relate this to the matter of surface roughness or even to justify the existence of a turbulent layer unless some discontinuity existed on the upstream face of the weir crest.

116. It was perhaps worth recalling that many practical spill-weir profiles designed to match the lower nappe emanating from a sharp-crested weir did, in fact, have a point of discontinuity upstream of the crest. With real fluids this would produce a separation zone which could have a marked influence on the nature of the boundary layer on the crest. It might mean that the boundary layer, once disturbed in this fashion, would remain turbulent over the crest. On the other hand, the extent of the separation zone could be such that the profile of the 'equivalent' weir (a concept introduced in § 20) might be appreciably different from that of the actual weir, so that it would be no longer permissible to take the crest radius of curvature simply as r or even to assume that that critical section occurred sensibly at the physical crest of the weir. The precise mechanism of flow, depending as it did on boundary layer separation and consequently on fluid properties and absolute size, might therefore be quite different from model to prototype. This, in addition to surface roughness, might be an explanation for some of the anomalies which arose in comparing model and prototype results.

117. Mr Montes had assembled some other interesting experimental data for comparison. The fact that they tended to fit the 1 in. weir results rather than equation (14) might simply be due to the inapplicability of an a value of 0·272 to the cases quoted. The Author had shown in Table 1 that although a value of a of about 0·27 was satisfactory for a symmetrical parabolic weir, a value of 0·24 or less was appropriate for a cylindrical crest. He considered that no single value for a would be suitable for all round-crested weirs.

118. The Author felt that Mr Ackers had recast the theoretical results in a most interesting and revealing way by introducing the concept of 'effective head'. The Author had himself used such a device in bringing viscosity into the analysis via boundary layer theory but had eliminated it from his final results. Mr Ackers had shown quite clearly in § 86 and again in § 90 that its retention could be very helpful.

119. The Author was very pleased that Mr Ackers had made the point in § 88 arising from this that quite small errors in zero setting or in head readings could very easily mask the fluid property influences altogether.

120. In § 89 Mr Ackers had introduced a slightly different type of discharge equation with separate factors C_e and C_v to take account of geometrical and viscosity influences respectively (surface tension being assumed to be negligible). He had shown that the Author's results then bore a strong family resemblance to others relating to the smooth broad-crested weir and to weirs of triangular profile.

121. In the Paper, the Author had recommended the use of an equation of the form (16) for practical purposes. With Mr Ackers' contribution in mind, it might be more attractive to use a discharge equation of the form

$$q = C_e \bar{C} \sqrt{(2g)} H^{3/2}$$

where C_e was a quadratic function of (H/r) of the form

$$C_e = 0{\cdot}385\left[1+a\left(\frac{H}{r}\right)-b\left(\frac{H}{r}\right)^2\right]$$

taking account of the geometrical features, and $\bar{C}$, embracing viscosity and surface tension influences, might be of the form

$$\bar{C} = 1-c(\sigma/wr^2,\ \nu^{1/2}/g^{1/4}r^{3/4})r/H.$$

122. The Author, in answer to Dr Engel's question in § 64, felt that this form for C_e was simpler than that which would result from the use of Lauffer's multiplier (22) and yet could be fitted satisfactorily to any round-crested weir by suitable choice of the quantities a and b.

123. Mr Cooke was to be thanked for pointing out an inconsistency between equation (9) and (14). The correct index was $\frac{1}{2}$ as given in equation (9). The Author also agreed with the correction to the value of b in Table I made in § 95.

124. In summing up, the Author remarked that he was pleased at the interest the Paper had aroused and hoped that it would prompt exploration of some of the avenues of research still clearly open in the study of weirs of this and other types.

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