load and also reveal the amendments which should be made in the assumed mode of collapse.

The methods described in this Paper have been illustrated by numerical examples which have been kept simple in order to clarify the exposition. These methods are, however, capable of dealing with much more complicated problems. As in the case of elastic analysis by moment distribution, the advantages of plastic moment distribution over other methods of design and analysis are more apparent the more extensive the frame under consideration.

The work described in this Paper was carried out at the Engineering Laboratory, Cambridge University, under the direction of Professor J. F. Baker, Head of the Department of Engineering. It forms part of a general investigation into the behaviour of steel structures in the plastic range being carried out at Cambridge with the assistance of the British Welding Research Association and the Department of Scientific and Industrial Research.

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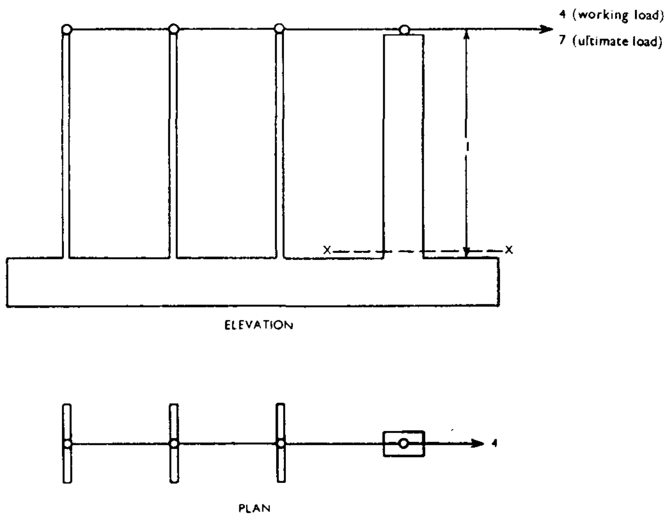
The Paper is accompanied by 12 sheets of diagrams, from which the Figures in the text have been prepared.

Discussion

Professor A. L. L. Baker observed that the plastic-hinge theory facilitated a true determination of the real strength of a structure and of the real value of the factor of safety; it avoided a number of unreal assumptions which had to be made in the elastic theory; it simplified calculations and secured economy.

Turning to the Paper itself, he could not yet entirely agree to the abandonment of strain compatibility and a consideration of approximate deformations, deflexions, and stresses under working load. As a simple illustration he sketched *Figs 20*, which showed the elements of a structural

Figs 20



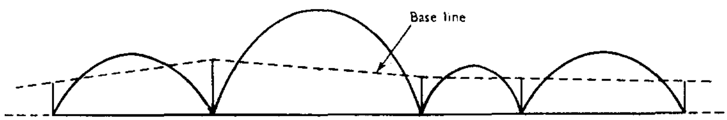
M_u	1·75	1·75	1·75	1·75
I	1	1	1	3
M_e	0·67	0·67	0·67	2

M_u Denotes ultimate moment of resistance
M_e ,, bending moment for elastic conditions
I ,, second moment of area

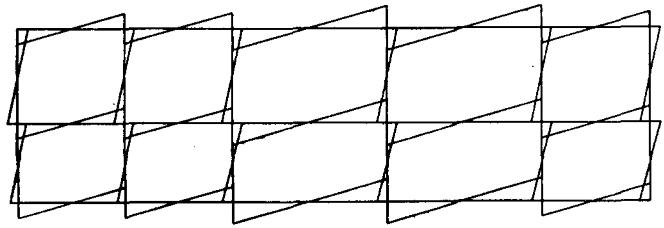
system consisting of four columns. Each column had a load factor of 1·75 and a rectangular cross-section, the fourth one having the longer side at right angles to that of the other three. He would assume that a force of 4 units was applied along the tops of the columns and that the lever arm was unity, whilst the second moments of area were in the ratio 1 : 1 : 1 : 3, so that the elastic moment (M_E) distribution would be $\frac{2}{3} : \frac{2}{3} : \frac{2}{3} : 2$. It would be seen that if the load factor was 1·75 under working load the right-hand column would fail, or at least it would yield plastically, and if the force were repeatedly reversed in direction then gradually there might be failure at the section XX. That was an extreme case, but it was a pointer to the kind of thing that might happen in a complex framework unless the distribution of bending moments and stresses for the elastic case were examined. It was only necessary to be approximate for the purpose.

Reinforced concrete engineers had usually been brought up on or converted to the use of Müller-Breslau's general elastic equations and, adapting those equations for the plastic-hinge case, they could check with the aid of simple tabulation whether hinge rotations were excessive, and also by a simple method of trial and adjustment, derive an approximate elastic

Figs 21



Continuous beam



Points of contraflexure centre in columns and inside beams

solution, and so find out fairly quickly whether there was a risk of the sort of occurrence which Professor Baker had described.

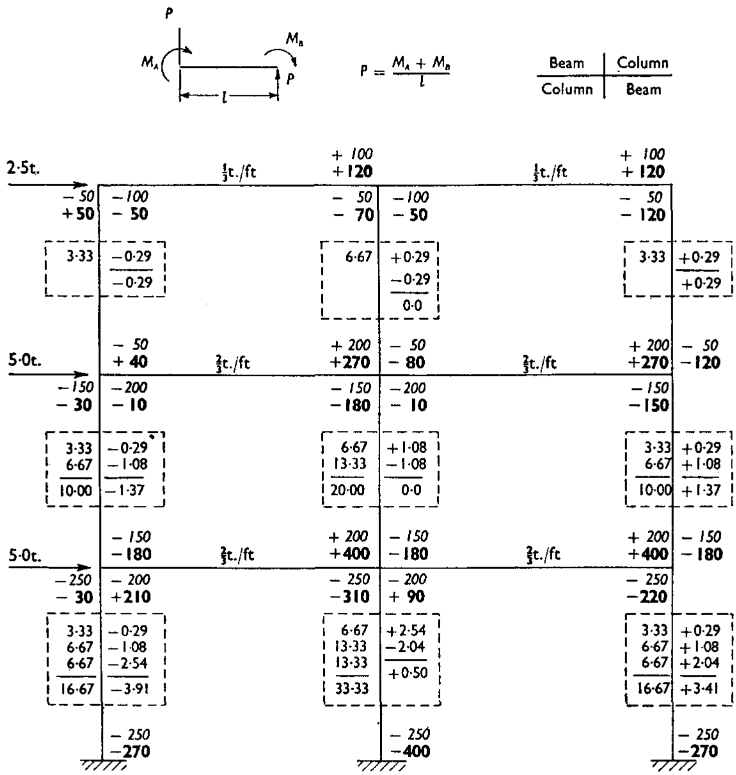
Structural steel was different from reinforced concrete and investigating a framework only at the final stage of failure as a mechanism might be satisfactory, but he was not content. For instance, in the case of a building frame, it seemed to him that once the step had been taken of calling the ultimate load the load which was applied when the structure became a mechanism, there was no arguable case against designing, for instance, a continuous multi-span beam by simply drawing a base-line through the bending-moment diagram for freely supported beams to give the best distribution of moments for economy in design and providing sections accordingly (see *Figs 21*), and, if the beam were one of a framework subject to sway forces, drawing-in the sway bending moments on a basis of central points of contraflexure in the columns and the inside beams (see *Figs 21*). With the correct or greater plastic hinge values at the appropriate sections that structure would resist the required or greater ultimate load, or was there a fallacy involved ?

He thought that for a great many frames used commonly in practice, once the step had been taken of ignoring strain compatibility it was possible to make such simple assumptions and design accordingly without using moment distribution. Reinforced concrete engineers in Continental countries were very much in favour of using ultimate load theory, and he thought it must come in the future.

Mr K. G. Eickhoff referred to the Author's statement (p. 69) that "Allowance for the reduction of full plastic moments due to axial loads may readily be made during the design process. . . ." It had been well known for some years that the full plastic moment which a member could stand was dependent upon the axial thrust which it carried. It could easily be shown that it was possible to draw a curve relating the fully plastic moment to the end thrust; the moment would be maximum when the end thrust was zero, and zero when the end thrust was equal to the yield stress multiplied by the known area of cross-section.

In order that due allowance might be made in the design process, it was necessary to form some estimate of the load in the stanchions, and that could be done by adding another column of figures to the Author's method. For comparison, Mr Eickhoff had chosen the frame given in *Fig. 11* and had converted the loads into tons. The additional columns of figures were shown within the broken-line enclosures in *Figs 22.* Considering the top storey, where the beams were uniformly loaded and treated initially as encastré, the reactions at each end of a beam were the same, and those reactions were put down in columns on the left of the stanchions. Those could be carried down to the lower stanchions, and similarly for the next storey down and the next. Although starting, for convenience, with a beam with equal end moments, those moments were altered during the distribution process, and the result would be a beam with an additional shear in it equal to the sum of the bending moments at either end, divided by the length of the beam. It was therefore possible from that relationship to determine the additional reaction at each end of the beam corresponding to the alterations in end moments made during the distribution process, and it was convenient to place those additional reactions in the columns to the right of the stanchions. The moments at the ends of a top-storey beam were originally 100 tons-inches and -100 tons-inches, giving zero sum, and after the distribution process they were $+120$ tons-inches and -50 tons-inches, giving a total of $+70$ tons-inches. If that was divided by the length of the beam a figure of 0·29 tons was obtained. That could be done for the other beams and carried down to the other storeys. By adding the contents of the two columns on either side of the stanchions the final load in the stanchions was determined. The process was conveniently left until the distribution process was finished, and the final result would show whether the estimated fully plastic moment of the section would be adequate when allowance was made for the axial load. In the case considered, taking the central stanchion, the axial load reduced the full plastic moment by 10 per cent, so that it was worth while making the correction.

Figs 22



Mr N. S. Boulton remarked that the case for plastic moment distribution had been well substantiated in the Paper. The Author had compared his new moment-distribution method with that used for elastic structures and had pointed out that the two methods were not in any way identical. In addition to the points of difference which the Author had mentioned, the plastic method, unlike the elastic method, involved a finite number of steps—quite a small number, judging by the examples given in the Paper—and also difficulties arising from slow convergence, which sometimes arose in the elastic method, were absent in the plastic method. On the other hand, whereas the elastic method was almost automatic, as the Author had pointed out in his introductory remarks, the new method appeared to require considerable thought and skill throughout the process, since it allowed, as he had shown, a considerable degree of choice.

The great merit of the Author's method was that it was a direct design process in which convenient practical sections could be assigned to the members within a possible range of full plastic moments which was determined in the process. As used for calculating the collapse load of frames of members of given dimensions it seemed less simple, as the Author had mentioned in his introduction, than the method of Neal and Symonds, in which modes of collapse were combined until the least load factor was obtained. However, Mr Boulton felt that a valuable feature of the Author's method was that it could be used to indicate the probable collapse mode before the final solution was obtained. Moreover, the moment distribution Table could be at once used to verify that the full plastic moments were nowhere exceeded, instead of having to calculate the moments at points where plastic hinges did not occur, in order to verify that the final mode of collapse was correct.

In his " Conclusions " the Author had stated that " there will usually be only a secondary advantage in seeking the frame which has the theoretical minimum weight," since the loss of economy was usually small in structures designed by the procedure advocated in the Paper. The saving of even a few per cent in weight, however, might be worth while in a large structure, and it was therefore desirable to have a simple guiding principle to minimize the total weight during the moment distribution process. The Author evidently had such a method, although for reasons of space he did not describe it. It would be of interest if he would indicate the principles of it in his reply to the discussion.

The method described by Dr Foulkes in a recent Paper [9] seemed to be suitable for use with moment distribution, and, as Dr Foulkes could not be present, Mr Boulton would like to mention the essential steps of the method. Having found the full plastic moments of members to support the given loads by the Author's (Dr Horne's) method, those moments were varied two at a time so as to increase the load factor as judged from the virtual work equation while keeping the total weight constant. The collapse mode and the corresponding load factor for those modified moments was then found by the procedure described on p. 70. The full plastic moments were then reduced in a constant ratio so as to restore the load factor to unity. Other modifications to the full plastic moments were made in the same way until the minimum weight was obtained. Would the Author recommend that method of minimizing the weight by moment distribution, or did he know of a better method ?

Dr B. G. Neal referred to a case which he thought could be treated only with some difficulty by the Author's method. Basically, that

[9] J. Foulkes, "Minimum Weight Design and the Theory of Plastic Collapse." *Quart. J. Appl. Maths*, vol. 10, p. 353 (Jan. 1953).

method was to write down against the frame a set of bending moments which satisfied immediately all the equilibrium requirements except those at the joints. The next step was to balance the joints while preserving the other equilibrium conditions which were already satisfied, and finally further adjustments were made to satisfy all the equilibrium requirements, until a suitable design was found.

In the structures dealt with in the Paper, which were all of rectangular form, the equilibrium requirements could be stated very simply, as follows :—

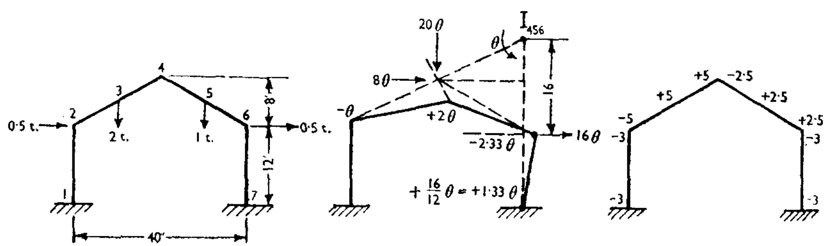
1. Equilibrium of each beam.
2. Sidesway equilibrium of each storey.
3. Rotational equilibrium of each joint.

For a pitched roof portal, however, the formulation of the equilibrium requirements was a matter of some difficulty, as he proposed to illustrate by a particular example.

Fig. 23 (*a*) showed a simple portal problem. The first step was to

Figs 23

(*a*) (*b*) (*c*)



determine the correct number of equilibrium requirements, and Dr Neal did that by observing that if the bending moments at the seven numbered points in *Fig.* 23 (*a*) were given it was possible to draw the bending-moment diagram for the entire frame. A frame of that type had three redundancies, so that there had to be four equilibrium relations between the seven unknown bending moments. Three of those relations could be written down without difficulty, as follows :

$$2M_3 - M_2 - M_4 = 20 \quad . \quad . \quad . \quad . \quad (1)$$

$$2M_5 - M_4 - M_6 = 10 \quad . \quad . \quad . \quad . \quad (2)$$

$$M_2 - M_1 + M_7 - M_6 = 12 \quad . \quad . \quad . \quad . \quad (3)$$

The fourth equation was most conveniently written down as a connexion between the moments of M_2, M_4, M_6, and M_7. Dr Neal thought that the easiest way of deriving that relation was by the application of the principle of virtual work. Considering a mechanism with hinges (not plastic hinges) at the cross-sections 2, 4, 6, and 7, an equilibrium equation could be derived by equating the virtual work done by the loads to the virtual work absorbed by the hinges. The kinematics of the mechanism was best solved by noticing that the apex 4 was constrained to rotate about the end 2 of the rafter member, whereas the top of the stanchion at 6 was constrained to move horizontally. The instantaneous centre of rotation of the rafter member 456 was therefore at I_{456}, as was shown in *Fig. 23* (*b*). If the rafter member 456 was given a small rotation θ about its instantaneous centre, then the top of the right-hand stanchion would move 16θ feet horizontally, and therefore the rotation at the hinge at the bottom of that stanchion would be $16\theta/12 = 1{\cdot}33\theta$. By a similar argument it could be seen that the hinge rotation at cross-section 2 was of magnitude θ. Then, since the left-hand rafter rotated clockwise through an angle θ, and the right-hand rafter rotated anti-clockwise through the same angle, the rotation of the hinge at the apex must be 2θ. The vertical downward movement of apex 4 was seen to be 20θ feet.

It was then possible to write down the virtual work expression for that mechanism :

$$-\theta M_2 + 2\theta M_4 - 2{\cdot}33\theta M_6 + 1{\cdot}33\theta M_7$$
$$= (10\theta \times 2) + (10\theta \times 1) + (16\theta \times 0{\cdot}5)$$

which, after rearrangement yielded the fourth equilibrium equation :

$$-M_2 + 2M_4 - 2{\cdot}33M_6 + 1{\cdot}33M_7 = 38 \quad . \quad . \quad . \quad (4)$$

If the procedure outlined in the Paper was to be followed, it would be necessary to write in along the left-hand rafter three moments of magnitude 5, at the ends and at the centre, that value being obtained from equation (1) by making M_3, M_2, and M_4 equal in magnitude. The left-hand rafter would be entered similarly with moments of magnitude 2·5 obtained from equation (2), and finally moments of magnitude 3 would be written against the tops and bottoms of the stanchions, the value 3 being obtained from equation (3), the equation of side-sway equilibrium. Reference to *Fig. 23* (*c*) showed that a complete set of moments had been obtained, but the fourth equilibrium equation was not yet satisfied. Thus the method explained in the Paper was not directly applicable to that type of problem.

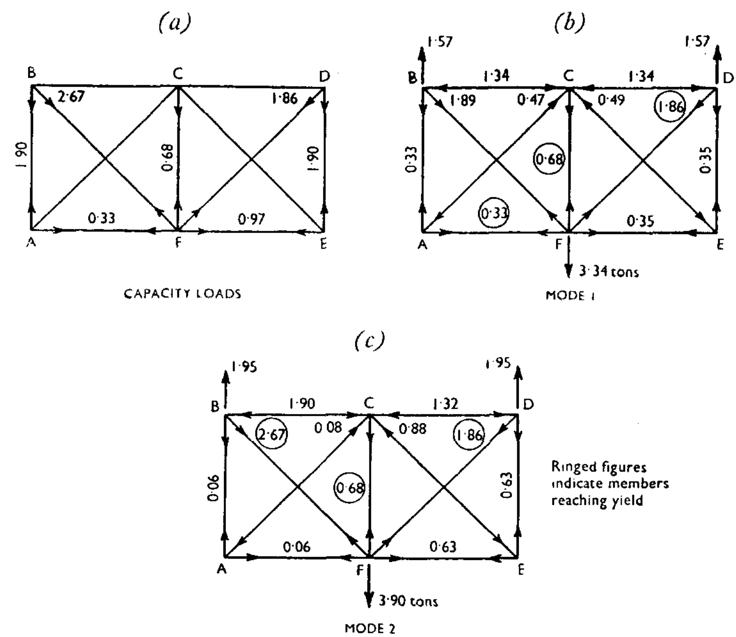
In conclusion, Dr Neal made the general point that since it was possible to derive the equilibrium requirements through an essentially kinematic process, as he had done in the example which he had just given, it must follow that there was an extremely close connexion between the statical approach outlined in the Paper and the kinematic approach which had been explained elsewhere,[6] and in a sense it was misleading to think of those approaches as fundamentally different. He thought that they were fundamentally very much the same.

Dr R. W. Steed said that many practising engineers welcomed the underlying philosophy of the plastic theory, but ideally that philosophy should embrace all types of structure and not just building frames. For some time he had been interested in the application of the plastic theory to triangulated structures.

On p. 52 there were given three requirements of a bending-moment distribution representing a state of collapse. When considering triangulated structures those requirements might be re-written thus :

(1) The equilibrium condition (the force distribution to be in equilibrium with the applied loads).
(2) The collapse mechanism condition (existence of sufficient plastic members for either the whole structure or part of it to become a mechanism).
(3) The yield condition (the capacity load of any individual member should nowhere be exceeded).

Figs 24



It would be noticed that in the first requirement he had replaced " bending-moment distribution " by " force distribution." In the second, " plastic hinges " had been replaced by " plastic members." In the third requirement, " the full plastic moment " had been replaced by " the capacity load of any individual member."

In connexion with research on the application of plastic theory to such structures he had encountered the following problem. Considering the triangulated pin-jointed truss shown in *Fig. 24* (*a*) the capacity loads of the tension members were as indicated. The load capacities of the compression members were large in comparison with the load capacities of the tension members and were omitted from the Figure. In *Fig. 24* (*b*) a likely mode of failure was shown which he had called Mode 1. The mode of failure was that the member AF, the member CF, and the member DF had reached yield, and all three requirements which he had given earlier had been fulfilled. The frame was in equilibrium and if the mode of failure was correct, three members had reached yield. Nowhere had the load capacity of any individual member been exceeded and upon the application of further load large deformations of the truss would develop. Was 3·34 tons to be taken as the value of the collapse load for that structure ?

Another mode of failure, Mode 2, was shown in *Fig. 24* (*c*), and there the three members to yield were BF, CF, and DF. That gave a collapse load of 3·9 tons, and the three requirements had also been met. Which was the true collapse load, 3·34 tons or 3·9 tons ? If the two modes were realized, then according to the proposition that a structure would support the maximum load it possibly could, the collapse load was 3·9 tons. That was subsequently found to be correct by testing a small-scale pin-jointed truss to destruction. There was a danger, however, without an experimental test or an elasto-plastic analysis, that only the theoretical Mode 1 would be discovered ; it satisfied the three requirements, and so, incorrectly, 3·34 tons might be given as the collapse load for that truss.

Had the Author any experience of the application of plastic theory to triangulated structures, and was there another requirement, in addition to the three mentioned above, needed for that type of structure ?

The Chairman said that the Author had specified three requirements *for a bending-moment distribution representing a state of collapse.* It did not seem obvious to the Chairman that those conditions could always be satisfied : the Author had pointed out that it might be difficult to satisfy them simultaneously, and had said that they were more readily satisfied in pairs. The Chairman was not sure what was meant by that ; presumably they had to be satisfied simultaneously, and the Author's statement required clarification.

On p. 53 the Author had referred to a structure which might be subjected to two or more different loading combinations and the Chairman found difficulty in understanding the treatment. *Figs 9* showed two different load systems, and the final design had to be a compromise between the requirements of those. Was it not possible that part of the structure could collapse under load system M_1 and part under system M_2 ? How was that dealt with ? The same problem was presented in the elastic design of a redundant structure for two different load systems. It was possible to design directly for one load system, but he did not think that it was possible to do so for two different load systems.

In the example given in the Paper there were two different load systems, but the points of application of the loads were the same in both (with an extra side load in one). What happened if a load, such as a crane, travelled across the structure ? Was it possible to deal with that problem ?

The Chairman also asked how, if yield had developed at one hinge, one could be certain of not going beyond the horizontal portion of the stress/strain curve there and so entering part of the curve for which the analysis was incorrect, while the other hinges were developing.

Professor A. L. L. Baker, who evidently found the subject very easy and straightforward, had stated that the Author's method would give economical design. The Chairman could not accept that as a general result, because it depended on the relative values taken for the load factor in "collapse" design and the factor of safety in elastic design. Having designed upon a certain load factor, was there any guarantee that, some portion of the structure would not yield under normal working loads or deflect so much as to cause serious troubles? There was no need for the Chairman to detail the obvious dangers of a too flexible structure in certain types of building.

It was obvious, as the Author had said, that there were an infinite number of solutions to the problem he had tackled, but what criterion would make it possible to choose the best ? Would it be necessary, having designed by the plastic method, to adopt a criterion involving the working conditions of the structure ? It was not sufficient to select the structure of the smallest weight, since that would be the one in which trouble would most likely occur under working conditions.

Professor J. F. Baker said that he would like to ask for more information regarding Professor A. L. L. Baker's remark about Continental activity in the field of plastic theory. It was to be hoped that Professor A. L. L. Baker was not starting a Continental hare, in addition to the American hare, by lending support to the suggestion that other countries were always ahead of Britain. No such impression had been gained by the exponents of plastic design, who went to the Continent from time to time to lecture on the subject. There was too great a tendency for national self-derogation in such matters. That was no longer amusing now that it was so important to attract students of technology to Britain.

Mr R. S. Jenkins remarked that there was one "aside" of the Author's with which he could not agree, although it did not affect the Paper itself. The Author had stated at the end of the Paper "As in the case of elastic analysis by moment distribution, the advantages of plastic moment distribution over other methods of design and analysis are more apparent the more extensive the frame under consideration." Mr Jenkins had had a fair amount of experience in the old-fashioned sort of analysis, and he thought that moment distribution became very tedious in complicated statically-indeterminate structures, and there were very much better methods. He had been going to say that since the plastic case was quite different, and distribution did not spread all over the structure, the

Author might be right in what he said with regard to plastic design, but, after listening to Dr Neal, Mr Jenkins began to think that that statement might not apply to plastic theory either, and that in complicated structures there might be better ways of doing it.

Mr Jenkins thought that the Author was being a little less precise than usual when he spoke of " elastic analysis " ; Mr Jenkins thought a better term would be " linear analysis," because one could have non-linear elasticity.

The premises of linear theory were that the material obeyed Hooke's law and that the displacements were small. Fortunately, structural materials, although elastic, were not like rubber, and their strains within the working-stress range were such small ratios that that small-displacement premise answered for most structures, with some well-known exceptions. When considering plastic hinges, however, it was obviously impossible to accept that small-displacement premise without investigation. In many cases it made no difference, but he asked the Author if that was the qualification referred to on p. 69, where he had stated that he had not dealt with moments introduced by sway deflexion. Had the Author made any investigations on frames of the type to which he referred, where it might be that the structure could not support the load based on small displacements, since other moments were introduced because the displacement became large ? That was but one matter among the many about which information was required before the plastic hinge theory for structural steel could be established for general use in practice.

Mr G. P. Manning referred to the fact that the Author had opened his introductory remarks with an apology for introducing yet one more method of structural analysis. Mr Manning could assure him that no such apology was necessary, because a variation of the Author's method had actually been in use amongst French engineers 40 years ago.

He thought that the Author should have made it quite clear in the Introduction that the Paper was intended primarily for steel. That was not in fact made clear until the last page, and much of what was said in the Paper would not apply to many structures in reinforced concrete, although the beginning of the Paper was couched in quite general terms. Mr Manning also gathered that it was intended primarily to apply to structures where the members were basically plain steel joists which were necessarily of uniform section from one end to the other, and where the practical criterion was the maximum bending moment at any point. It did not apply exactly to any member where the moment of resistance might be varied, such as a plated joist, where the total weight of the member was a combined function of the maximum bending moment and the total area of the bending-moment envelope.

Various attempts had been made to compare different methods of analysis. Mr Manning's view was that there was no one method of analysis which was best for all problems in all materials. Some were of more universal application than others; some were very attractive in theory, in the lecture room, but proved to be very disappointing in the drawing-office. He did not know how many statically-indeterminate structures he had designed in his career, but it must run into hundreds. People asked him what methods he had used. He had tried most of them, including variations of the one now under discussion, which he did not much like. Various speakers had mentioned the factor of safety. Speaking as a mere observer, he would say that it was an imaginary number inversely proportional to the number of competitive tenders.

How did he now design the various statically-indeterminate structures? Everyone that evening seemed to be becoming more and more mathematical, but he used what might be termed a method of "exproximation." He first calculated, by his own method, the theoretical moments throughout the frame; he then proceeded to check a few approximate bending moments, $WL^2/16$, $WL^2/12$, and so on, and finished by putting the sections in by eye. There were therefore three distinct stages in his design, each one getting less and less mathematical. He could recommend that method, which had stood him in good stead for many years. The Author's method of assuming hinges and manipulating the bending moments was quite good within its own field, but, like the Chairman, Mr Manning felt that with a complicated frame one might get so lost that one might overlook some possible method of failure. He found that in all engineering problems the difficulty was not so much in solving any one specific problem, but in being able to state exactly what the problem was which had to be solved. It was quite possible in a complicated frame to overlook one of the problems which should be solved, and therefore to produce a frame which had not even a theoretical factor of safety.

Mr A. L. Moss-Morris referred to Mr Manning's opinion that possibly true economy did not come from the type of design which assumed that the members had a uniform plastic moment along their length, and said that that was particularly evident in continuous beams, which were subjected to a uniformly-distributed load. In that case, the moments in the span were approximately constant over a considerable length, while at the supports there was a rapid change in moment. By the method of design outlined in the Paper it was not necessary to know the modulus or area of section to be used, and those were actually determined from the design. He felt that if the support moments were generally made larger than the mid-span moments, a considerable saving could be effected. Since beams were normally spliced at their supports, their was no great difficulty in increasing the plastic moment of resistance of those short lengths above that of the rest of the beam. He was engaged on work similar to that of the Author but with reinforced concrete as the material, and even in that case, where the amount of ductility available was considerably less, the required amount of redistribution of moment could be obtained without excessive deformations.

Several speakers had asked if it was possible to be sure that at the design loads the deflexions of the structure would not be excessive. He thought that a simple criterion for that could readily be found. If all but one of the plastic hinges were placed at the joints between two members, and only that hinge required to transform the structure into a mechanism was put into the span of the members, the deflexions in that case must always be less than they would be for a structure made from simply-supported members, or for a framed structure with pin joints. He thought that that was fairly obvious, but, if not, it could be derived from the virtual-work equations.

Would the material be able to withstand the required strains, if, for instance, one plastic hinge was formed at a very low load and the other plastic hinges at considerably higher loads ? He thought that the technique for obviating that, at least in beam members, was to make the span-moment/support-moment ratio not less than that for the corresponding encastré beam.

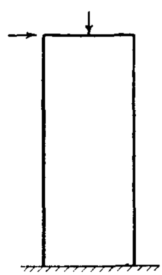
The Author, in reply, said that the chairman, Professor A. L. L. Baker, and Mr Moss-Morris had all raised the question of stresses and deformations at working loads. It might of course be necessary to consider such matters, just as in structures designed to a safe stress it was sometimes necessary to modify the design in order to avoid excessive deflexions. The first requirement of any design process—whether elastic or plastic—was that it should set forth some simple criterion of strength or suitability which might be expected to lead in general to a rational economical design. Long experience of orthodox design methods had enabled one to state with some confidence when it was necessary to calculate deflexions under working loads. That experience was not yet generally available for structures designed by the plastic theory, and it might at present be necessary to calculate deflexions at working loads more frequently than in structures designed to an elastic criterion. Since structures designed by the plastic theory used lighter sections, deflexions might be expected to be greater than in similar structures designed by elastic theory. Against that, however, had to beset the high premium placed in plastic theory design on the maximum use of rigid joints, and it was doubtful whether there would be any appreciable net increase in deflexions at working loads.

Mr Moss-Morris had put forward an argument relating to the deflexions of beams with which he was in entire agreement. It would be shown [10] that in multi-storey multi-bay frames with rigid joints and distributed beam loads the maximum bending moments in the beams occurred always at the ends. Plasticity would therefore first occur at the ends of the beams, the rest of the beams remaining elastic practically until the collapse load was reached. There was in fact reason to expect that the deflexions of such beams would be less—not more—than the deflexions of corresponding beams designed, according to orthodox methods, as simply supported.

[10] References 10 to 15 are given on p. 98.

Professor A. L. L. Baker had given an instructive example in which yield stress would be reached, and the full plastic moment developed, at working loads for a structure which had been designed by the plastic theory to a load factor of 1·75. Experience had shown that it was not usual for plastic designs to exceed the yield stress at working loads unless the structure contained, adjacent to each other, members of highly unequal stiffness. An example in which yield stress would most likely be exceeded at working loads was afforded by the tall portal frame shown in *Fig. 25.* In that example, in which yield stress would be exceeded at the feet of the stanchions, the high stresses at working loads would not pass unnoticed because of the obvious necessity of calculating the sway deflexion of the frame. In such a structure, the limiting deflexion, not the ultimate strength, would necessarily be the real design criterion. In those parts

Fig. 25



EXAMPLE OF FRAME IN WHICH YIELD STRESS MIGHT BE REACHED AT WORKING LOAD

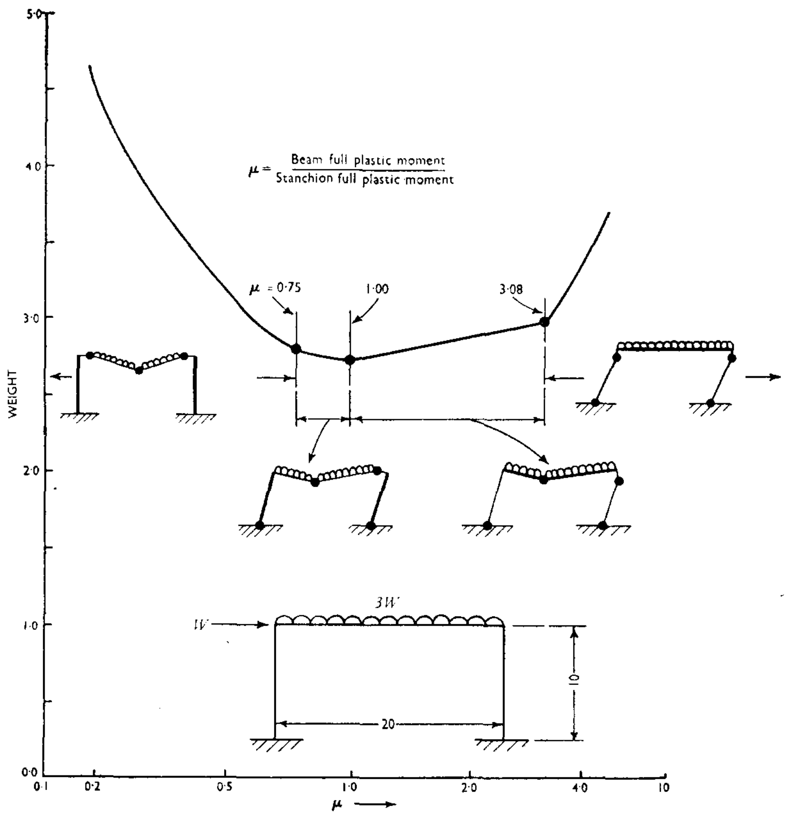
of a structure where deflexions were not critical, some degree of plastic deformation at working loads might not be objectionable, since the structure would behave elastically after the first application of the maximum load. When the loads were of an alternating character, special considerations arose, but that subject was beyond the scope of the discussion.

The question of excessive hinge rotations was of great importance in ultimate-strength theories for reinforced-concrete structures, but experience had so far shown that hinge rotations sufficient to enable the loads to reach the theoretical collapse values could take place in structures of mild steel.

Professor A. L. L. Baker appeared to suspect the design process of simply " drawing in " base-lines for continuous beams and multi-storey frames. According to the plastic theory that was quite a legitimate procedure, the only objection being that, for mild-steel structures composed of uniform members, the design would not in general represent a state of collapse at the factored loads, and would not therefore be " economical." The moment-distribution process described in the Paper was proposed as a logical means of deriving economical structures composed of prismatic members, but it was not suggested that it was the only such method.

Professor A. L. L. Baker, Mr Manning, and Mr Moss-Morris had all raised the question of members of varying sections. Such members were not so readily obtained in mild steel as in reinforced concrete, but even for

Figs 26



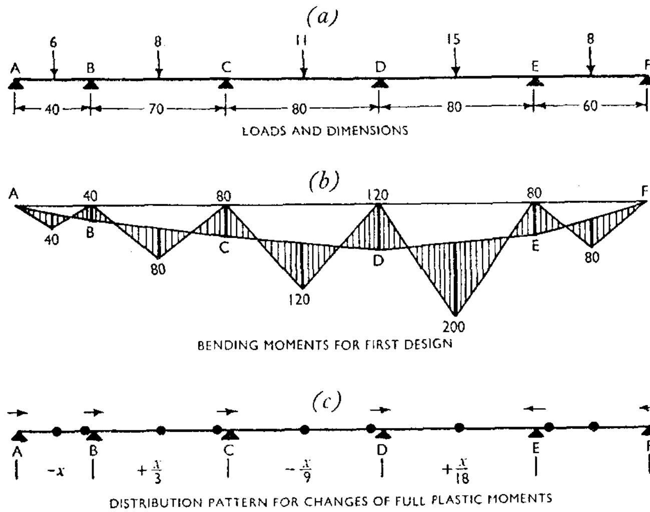
Weight Comparisons for Fixed-Base Portal Frame

steel it was a matter well worth investigation. The Author had himself considered the matter in some detail in another paper,[11] and had found that, as Mr Moss-Morris had stated, considerable economy could be achieved by increasing the moment of resistance of the beams locally at the ends.

The Author said that Mr Eickhoff had made a very useful contribution in his suggested method of recording the axial loads in the stanchions. It was certainly convenient to have a means of recording on the same work-sheet all the essential information required in design.

Mr Boulton had raised the question of obtaining the minimum-weight design. *Figs 26* showed weight comparisons for a rectangular fixed-base portal frame subjected to a distributed beam load of 3 W and a side load of W. It had been assumed in the calculations that the weight per unit length of a member was proportional to $M^{0 \cdot 6}$ where M was the full plastic moment. The diagram showed the weight plotted vertically against μ horizontally, where μ was the ratio of the full plastic moment of the beam

Figs 27



Example of Minimum Weight Design by Moment Distribution

to that of the stanchions. Within the limits $\mu = 0{\cdot}75$ to $\mu = 3{\cdot}08$, there was very little variation in total weight, and the moment-distribution process would always lead to a design within those limits. That was a typical example illustrating the point made by the Author in the Paper that there was comparatively little advantage in seeking the absolute-minimum-weight frame, since the small potential saving in weight would, in practice, be overshadowed by other important considerations.

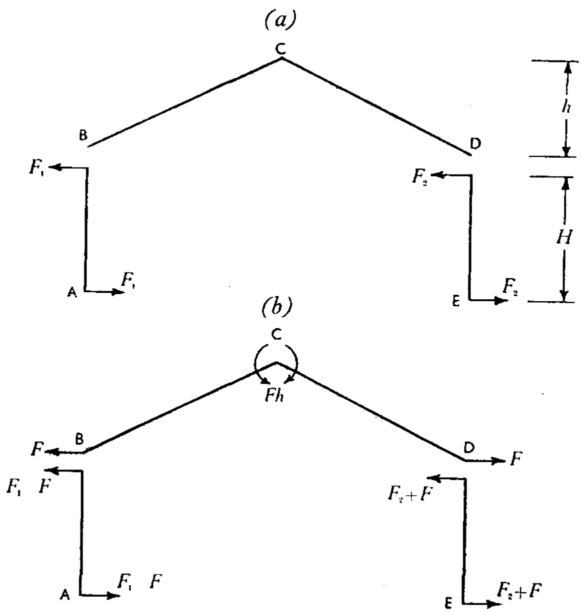
At the same time, the Author would agree that the problem of minimum-weight design deserved attention. Mr Boulton's suggestions for using moment distribution to obtain the minimum-weight frame were in general agreement with the Author's proposals. Those latter proposals could be introduced by considering a continuous beam loaded as shown in *Fig. 27* (*a*). Suppose that moment distribution (or some other method) resulted in the design shown in *Figs 27* (*b*) and (*c*), where *Fig. 27* (*b*) showed the bending-moment distribution and *Fig. 27* (*c*) the positions of plastic hinges at collapse. Wherever a hinge occurred at the end of a span, an arrow was drawn above the support in a direction away from the span in which the hinge occurred. Then if the full plastic moment of the first span AB were decreased by x, it could readily be shown that the full plastic moment of the second span BC would, as a result, have to be increased by at least $\frac{x}{3}$. It would then be found that the full plastic moment of span CD could be decreased by $\frac{1}{3} \times \frac{x}{3} = \frac{x}{9}$, and so on, the process being allowed to continue so long as the changes could travel in the direction of the arrow. The " carry-over factor " for change of section was therefore, in general, $-\frac{1}{3}$. The span DE had an arrow in the opposite direction at end E, and the change of section was not carried over the support at E. The carry-over factor to the last span concerned, DE, was $-\frac{1}{2}$ in place of $-\frac{1}{3}$. If changes of full plastic moments according to the resulting " pattern " reduced the total weight of the frame, then a value of x was chosen of sufficient magnitude to change the direction of one of the arrows. It was necessary to establish a number of rules in addition to those, particularly when dealing with multi-storey frames, but sufficient details had been given to make clear the basic principles of the process.

The Author said that, in connexion with the problem of minimum-weight design, some elegant fundamental work had been done by Mr J. D. Foulkes.[12] Much of that work had not yet been published, and had not been completed when the Author had written the Paper then under discussion. The work of Mr Foulkes would render his own treatment out of date, and the Author therefore thought that it was unnecessary to elaborate further on his own method of minimum-weight design.

The Author remarked that Dr B. G. Neal had given a clear exposition of the equilibrium conditions for a pitched-roof portal frame. Such structures were very difficult to deal with in elastic analysis, and it was not surprising that some difficulty also arose in calculating the plastic collapse loads. The Author was of the opinion that, for single-span pitched-roof frames, the graphical approach [1] was superior to any other. Dr Neal and Professor Symonds had shown that the method of combined mechanism could be applied successfully to multi-bay pitched-roof frames,[13] and there was also a method of applying the moment-distribution technique to that problem.

In pitched-roof portal frames, the distribution of the total horizontal shear force as between the stanchions directly affected the bending moments in the rafters. That complication did not arise in frames which contained vertical and horizontal members only. Thus if the shear forces in *Fig. 28* (*a*) were modified so that a shear force F were transferred from stanchion AB to stanchion DE, then a sagging bending moment Fh would be induced at the apex C of the intermediate rafters. At the same time, the sum of the end moments in stanchion AB would be decreased by FH, the sum of the end moments in stanchion DE being increased by the same amount. Provided no change occurred in the distribution of shear

Figs 28



Horizontal Shear Forces in a Pitched-Roof Portal Frame

forces between stanchions, any change in the bending moments in the two rafters could be treated as though both members constituted together a single horizontal beam, C being at mid-span. Hence any possible changes in the bending moments could be built up from the extended Distribution Table shown in Table 3. Lines (*a*), (*b*), and (*c*) were identical with the first three lines of Table 1, whilst line (*d*) showed how the sum of the top and bottom moments of each adjacent stanchion could be modified to bring about a change of shear-force distribution between the stanchions.

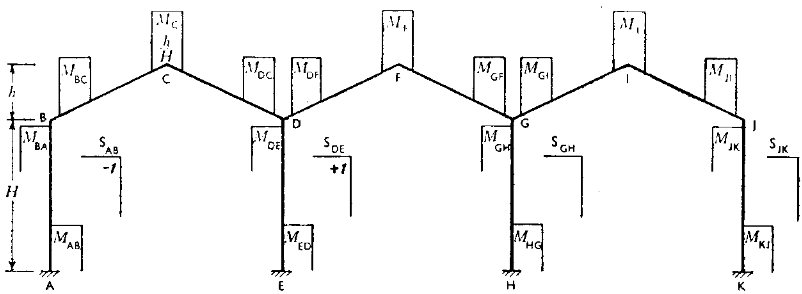
The bending moments which had to be recorded when dealing with a three-bay pitched-roof frame were shown in *Fig. 29*. The columns S_{AB}, S_{DE}, S_{GH}, and S_{JH} were individual sway columns for each stanchion,

TABLE 3.—DISTRIBUTION TABLE FOR PITCHED-ROOF FRAMES

	Left-hand sway	Left-hand end moment	Apex moment	Right-hand end moment	Right-hand sway
a	—	1	$\frac{1}{2}$	0	—
b	—	0	$-\frac{1}{2}$	1	—
c	—	1	0	1	—
d	−1	—	$\frac{h}{H}$	—	1

H denotes height to eaves.
h denotes height from eaves to apex.

Fig. 29



TREATMENT OF A THREE-BAY PITCHED-ROOF PORTAL FRAME BY MOMENT DISTRIBUTION

replacing the combined sway column for rectangular frames. The sum of the stanchion moments in any one stanchion had to agree with the moment in the corresponding sway column. The sum of the sway moments in all the columns S_{AB}, S_{DE}, S_{GH}, and S_{JK} had to remain constant, but the distribution between the columns could be altered by using the last line of Table 3, as shown for the first bay of the frame in *Fig. 29.*

Dr Steed had raised the question of triangulated structures, and although this was strictly outside the scope of the present discussion, the Author would do his best to answer the queries raised. Dr Steed had correctly interpreted the three conditions to be satisfied in a triangulated structure at collapse, provided an over-riding assumption, which he had not stated, was justified. That assumption was that any member should, at the ultimate load, be capable of offering a constant resistance during deformation. Such an approximation could not be justified for compression members in general The Author would therefore deprecate any easy assumption that the philosophy of the simple plastic theory as applied to building-frame structures could be extended to triangulated frames. At the same time, the general plastic theory did afford a means of studying the behaviour of triangulated mild-steel structures beyond the elastic limit, provided one resisted the temptation to jump to easy conclusions, and it should certainly be possible, for example, to throw light on the much discussed problem of secondary stresses.

Setting aside the query regarding the behaviour of compression members, the principles stated by Dr Steed would be found adequate to solve any problem without ambiguity. In the case of Dr Steed's example, Mode 1 (*Fig. 24* (*b*)) did not constitute a mechanism. If members CF and DF were assumed to extend, all other members except AF remaining of constant length, the mechanism shown in *Fig. 30* was obtained. According to that mechanism, the member AF was required to contract, whereas it had been assumed to be yielding in tension. Similar false modes of failure could be obtained for building frames if one did not take the precaution of drawing the collapse mechanisms, and the Author would recommend a similar procedure as being essential when dealing with triangulated structures.

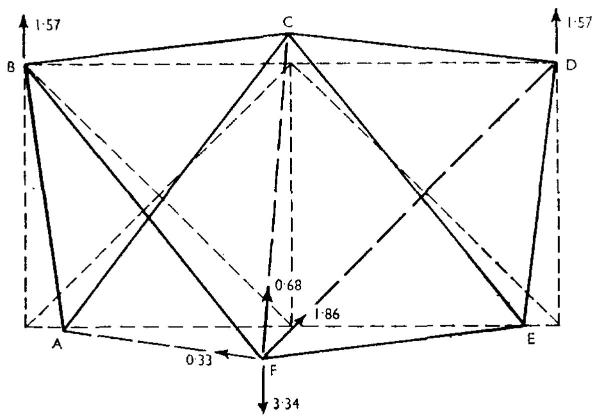
The Author felt that the Chairman had perhaps misunderstood the significance of the three conditions to be satisfied at the collapse load. Any one structure would collapse at a definite intensity of loading, and at collapse, the three conditions of equilibrium, mechanism, and yield would necessarily be satisfied. The difficulty of obtaining a bending-moment distribution satisfying the three conditions was only an analytical one, but had to be overcome in order to calculate the collapse load. A bending-moment distribution satisfying the equilibrium and yield conditions but not that of mechanism gave not the collapse load but a load of lower intensity. A bending-moment distribution which satisfied the equilibrium and mechanism condition but not that of yield would correspond to a load intensity greater than the value at collapse.

The Author had hoped that in presenting the example of a design for two diffeient systems of loads, he had demonstrated the versatility of the moment-distribution technique. In a rigid-pointed structure, under a system of loads A, one part X of the structure might be giving support to another part Y. Under a different system of loads B, under which part X was more severely loaded, part Y could be called upon to give extra support to part X. In elastic design, it was usually impossible to make more than accidental use of interaction of this nature, whereas it could be used to the full by using plastic design methods.

The Author was able to inform the Chairman that some study had been made of the effect of moving loads on structures designed by the plastic theory.[5] The question of the optimum design of members of varying cross-section under loading conditions of that nature had also received some attention.[14] With regard to the strain-hardening range, it was true that, at collapse some plastic hinges would, in actual structures, show stresses above the horizontal portion of the stress/strain curve. It had, however, been found that that effect was insufficient to modify significantly the distribution of bending moments, and some theoretical work [15] had shown that that result was to be expected.

Mr Jenkins had expressed doubts as to the superiority of moment-distribution methods in the elastic range. The Author would agree that multi-storey frames were tedious to analyse by moment distribution

Fig. 30



Deformation of Truss according to the Erroneous Model
(see *Fig. 24* (*b*))

because of the sway problem, but his own opinion held to the general superiority of moment distribution. Mr Jenkins was of course quite correct in his criticism of the term " elastic analysis " when reference should have been made to " linear analysis."

Mr Jenkins had raised the question of the effect of change of geometry at collapse on the conditions of equilibrium. In the type of structure under consideration, the only important effect of change of geometry was that additional stanchion moments could be introduced by sway deflexion, and the Author had in his treatment assumed that such moments could be neglected. There certainly would be some structures in which the effect would be considerable—particularly those structures in which deflexions at working loads were appreciable. The Author's general impression was that, in structures for which ultimate strength rather than deflexion at working loads was the criterion for design, the change of geometry at the point of collapse was insufficient to reduce the collapse load significantly below the value given by simple theory. That was, however, a problem which demanded further study, and in fact some as-yet-unpublished work had already been done at Cambridge on the subject.

The Author said that he had been somewhat at a loss to understand what Mr Manning was seeking in the three stages of his design method. Mr Manning had himself remarked that in engineering the difficulty was more often to state the problem which had to be solved than to solve it when it had been stated. The conception of ultimate strength could in general be more nearly related to what was required of a structure than the conception of safe stresses. It had been shown by the work of the Steel Structures Research Committee, and more recently by work at the Building Research Station, that calculated elastic stresses often bore but little relationship to stresses actually induced. Hence the devotion of laborious hours to elastic analysis might sometimes fulfil no useful purpose, and may merely serve to exemplify the lack of any clear perception of the nature of the problem facing the designer. The plastic theory, based as it was upon a thorough investigation, both experimental and theoretical, into the behaviour of structures right up to the ultimate load, did contain a clear perception of the problem which had to be solved. There was of course no pretence that the plastic theory itself—still less the particular technique suggested in the Paper—could take account of all the important factors which had to be considered in design. The elastic theory of structures had still to be regarded as an important tool to be used where necessary. There was also a great deal which could be learned only by experience, but there had to be a general conception of design which could be communicated. If there was to be any consistent advance in structural design, one could scarcely allow the individual to pick and choose his basic criteria of suitability, as Mr Manning would appear to advocate.

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14. M. R. Horne, Discussion, Final Report, 4th Congress, Intl Assn Br. & Struct. Engng, 1952, p. 119.
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