*Discussion on Paper No. 6698**

Analysis of irregular building frames

by

F. P. Wiesinger, Arch.-Engr, M.S., Ph.D.

and

Seng-Lip Lee, B.S.C.E., M.S.E., Ph.D.

Mr D. G. Alcock (Manager, Computing Service Sales (South), I.C.T. Ltd), said that Professor Hardy Cross[1] introduced a systematic method for the implicit solution of equilibrium equations at the joints of a rigid frame. His method still had two great advantages: it was easy to remember and it was self checking. However, there were originally two serious limitations: the method did not lend itself readily to the solution of frames or grids requiring consideration of axial or torsional deformation, and there was no simple and systematic extension of the method to cope with the effect of joint translation or sway.

38. For over thirty years engineers had been using 'Moment Distribution' where ever it could be applied and in many cases where it could not, and all too often problems which had defied the method had been twisted to fit it, or been left unsolved. Had the Authors' Paper been presented thirty years ago it would have been of real value to the profession as a practical tool. Today it must be considered of academic interest only.

39. For a method to be practical it must be economic. Mr Alcock said he could only guess at the man-hours employed in the analysis of the frames in Examples 1 and 2 by the Authors' method, and guess again at the cost of analysing the frame shown in Fig. 1, even under a single condition of loading. But the cost of these analyses, he said, must surely be many times greater than those quoted in Table 6.

TABLE 6: COST OF ANALYSIS BY COMPUTER

Fig. no.	Number of joints	Charge for analysis by Computer Bureau	Cost of each additional condition of loading
6	7	£3 19 8	£1 6 7
9	8	£4 10 9	£1 10 3
1	20	£10 17 4	£3 12 5

40. In his opinion the only economic method of straightforward structural analysis of frames, grids, three-dimensional structures, etc, was by computer, using the many standard programmes[22] available. There were Computer Bureaux throughout Britain which offered a service to civil and structural engineers, and it would seem bad economics to neglect them in favour of outdated manual methods.

* *Proc. Instn civ. Engrs,* vol. 28, pp. 295–312, July 1964.

41. The coding necessary to solve the two small frames in question is shown in Tables 7 and 8, and Tables 9 and 10 show the output provided by the computer. A brief explanation of the coding is given outside the heavy lines in Fig. 10 and a full description of the code, invented by Dr J. Ludley, is given elsewhere.[22] Table 6 shows the charges made by a company which offers a computing service to civil and structural engineers. The method of solution was introduced by Dr R. K. Livesley.[23]

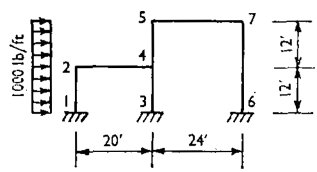
42. In order to complete the data Mr Alcock had had to work backwards from given information to obtain values usually available at the start, namely the second moments of area of the members. He had assumed realistic values for their cross-sectional areas although it would have been possible to enter very large values and thus neglect the effect of axial deformation as in the Authors' analysis. A value of the elastic modulus was also assumed in order to show sensible values of deflexions and rotations.[24]

43. He concluded that there was very little point in presenting new methods of structural analysis unless they could provide some new information, or provide standard information more cheaply than a Computer Bureau.

TABLE 7: CODING OF A 7-JOINT FRAME FOR COMPUTER ANALYSIS[20]

13 000	Elastic modulus (ton/sq. in)
9	Number of restraints
7	Number of joints
6	Number of members

1–1 1–2 1–3 3–1 3–2 3–3 6–1 6–2 6–3	Joint 1 restrained { Horizontally—1; Vertically —2; In rotation —3



0(2	0 12	X and Y co-ordinates of the joints with the origin arbitrarily taken at joint number 1.
20(3	0 12 24	
44(2	0 24	Bracket and number, e.g. 44(2 indicates repetition.

1–2 3–4 4–5 6–7	5(4	48(4	List of connected joints defining the members. List of cross-sectional areas.
2–4 5–7	4·6 4·8	20 24	List of second moments of area.

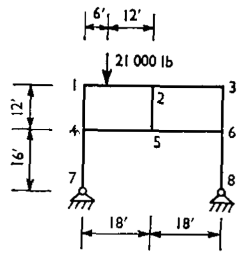
L 1000 H 1–2 4–5	Line load of 1000 lb/ft acting horizontally on members 1–2 and 4–5.
Z Z	End of loading condition and problem.

TABLE 8: CODING OF AN 8-JOINT FRAME FOR COMPUTER ANALYSIS[21]

13 000	Elastic modulus (ton/sq. in)
4	Number of restraints
8	Number of joints
9	Number of members

7–1 7–2 8–1 8–2	See Table 7 for further explanation.

0	28(3
18	
36	
0	16(3
18	
36	
0	0(2
36	



1–2	8(2	180(2
2–3		
4–5	7·5(2	144(2
5–6		
7–4	8·5	192
4–1	7	120
5–2	6	72
6–3	7	120
8–6	8·5	192

P–21 000 V 1–2 (0·333 Z	Point load of 21 000 lb acting vertically downwards on member 1–2 at the 'third point'.
Z	End of loading condition.

TABLE 9: COMPUTER OUTPUT. 7-JOINT FRAME

JT.	X-DIRN	Y-DIRN	ROTN
1	+0·0000	+0·0000	+0.0000
2	+3·1433	+0·0030	−2·0318
3	+0·0000	+0·0000	+0·0000
4	+3·1495	−0·0012	−2·9157
5	+6·8671	+0·0006	−1·4207
6	+0·0000	+0·0000	+0·0000
7	+6·8616	−0·0036	−2·1463

MEMBER		AXL 1	SHR 1	MOM 1	AXL 2	SHR 2	MOM 2
1	1→ 2	−1·34	+6·89	+37·98	+1·34	−1·53	+12·59
2	3→ 4	+0·53	+2·63	+26·33	−0·53	−2·63	+5·27
3	4→ 5	−0·80	+4·17	+8·92	+0·80	+1·19	+9·00
4	6→ 7	+0·80	+1·18	+18·06	−0·80	−1·18	+10·31
5	2→ 4	−1·53	−1·34	−12·59	+1·53	+1·34	−14·19
6	5→ 7	+1·18	−0·80	−9·00	−1·18	+0·80	−10·31

TABLE 10: COMPUTER OUTPUT. 8-JOINT FRAME

JT.	X-DIRN	Y-DIRN	ROTN
1	+0·0637	−0·0247	−0·6242
2	+0·0595	−0·4667	+0·2332
3	+0·0571	−0·0035	+0·0254
4	−0·0656	−0·0136	−0·0044
5	−0·0613	−0·4632	−0·0747
6	−0·0589	−0·0027	+0·0666
7	+0·0000	+0·0000	+0·0534
8	+0·0000	+0·0000	+0·0128

MEMBER		AXL 1	SHR 1	MOM 1	AXL 2	SHR 2	MOM 2
1	1→ 2	+2·03	+7·00	+17·75	−2·03	+2·37	−4·24
2	2→ 3	+1·14	−0·51	−2·74	−1·14	+0·51	−6·49
3	4→ 5	−1·95	+0·81	+7·82	+1·95	−0·81	+6·80
4	5→ 6	−1·07	−1·05	−10·44	+1·07	+1·05	−8·39
5	7→ 4	+7·82	−0·08	+0·00	−7·82	+0·08	−1·25
6	4→ 1	+7·01	−2·03	−6·56	−7·01	+2·03	−17·75
7	5→ 2	+1·86	+0·88	+3·63	−1·86	−0·88	+6·97
8	6→ 3	+0·51	+1·14	+7·23	−0·51	−1·14	+6·49
9	8→ 6	+1·56	+0·07	+0·00	−1·56	−0·07	+1·16

Deflexions (*in.*). Rotations (*rad.* × 100)
Forces (*tons*). Moments (*tons-ft*)

Professor R. G. Robertson (Department of Civil Engineering, University of Cape Town), said that the Authors were to be congratulated on their new method, which had advantages for all frames subjected to sway and not only on irregular frames. He had found difficulty in appreciating some of the descriptions, and gave his remarks for clarifying his interpretation of the method.

45. Referring to §§ 1, 16, 17, and 24, he considered that the term 'stiffness' should be reserved for the moment created at a joint by a unit rotation of that joint, and that the moments created at other joints were better described as 'carry over' or possibly 'distributed moments' as the latter term also applied to the moments created at the ends of the members meeting at the joint rotated.

46. The operation of the method appeared to be similar to that devised by himself for frames with inclined members, and the Author's reservation in § 34 might be waived. Professor Robertson said that his interpretation was given below and made this point clearer.

47. Clamps were provided as necessary to restrain the rotation or translation of the joints. Loading caused couples or forces to be applied to these clamps.

48. The couples or forces on the clamps were successively relaxed by allowing rotation or translation at each clamp in turn.

49. The effect of each relaxation was found by creating a unit rotation or a unit translation at each clamped joint in turn, while all clamps were present. The moments created by this operation were the 'carry over ' or 'distributed' moments which, divided by the stiffness of the joint operated on, gave the 'carry over' or 'distribution' factors. Thus each joint may be balanced in turn, as in the usual process, but using multiple carry over moments, which were found at every joint.

50. Professor Robertson went on to give some further remarks on the text and some errata.

(*a*) The fundamental values of T_i and Q_{ij} might have been given, for more ready reference, as follows.

For a fixed end J:

$$T_i = k_i(1+c_i)\frac{E.I}{L^2} \quad \text{and} \quad Q_{ij} = (k_i+k_j+2c_ik_i)\frac{E.I}{L^3}$$

For a hinged end J:

$$T_i = k_i(1-c_ic_j)\frac{E.I}{L^2} \quad \text{and} \quad Q_{ij} = k_i(1-c_ic_j)\frac{E.I}{L^3}$$

(*b*) Equation (4) might more clearly be written $Q_{\text{group}} = \dfrac{1}{\sum \frac{1}{Q}}$

(*c*) Equation (5d) should be $U_i = \dfrac{T_i}{\sum_Q}$ not $\dfrac{T_i}{Q_{1,2\ldots}}$ which is not clear.

(*d*) The last line in Table 1 should read 16·889. Table 1 should show the *U* moment values as these were requird for subsequent work.

(*e*) The sixth column of Table 5 should be headed E_3 E_7 E_4 not G_3 G_7 G_4.

(*f*) In Fig. 9 the *T* values for members 1 and 2 should be 3·33 not 33·3.

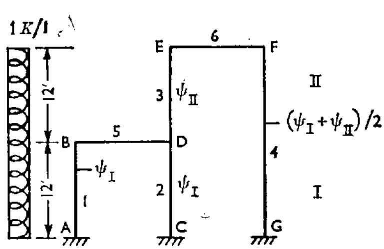
51. Professor Robertson said he had made the above remarks in view of the importance of the advance in techniques, which he considered had been made by the Paper.

Professor Masao Naruoka (Nagoya University, Japan) said that the Authors had contributed something to the analysis of irregular building frames by the moment distribution method. The moment distribution method proposed by H. Cross[1] was excellent, but was conveniently applied to rectangular rigid frames (regular building frames), and in the case the Authors had dealt with it was better to use the

TABLE 11

	ϕ_B	ϕ_D	ϕ_E	ϕ_F	ψ_I	ψ_{II}	=
at B . . .	12	5			1		−12
at D . . .	5	20	3		2	3	12
at E . . .		3	18	6		3	−12
at F . . .			6	20	2	2	0
Storey I . .	1	2		2	8/3	2/3	−72
Storey II . .		3	3	2	2/3	8/3	−48

$\phi = 2EK\theta, \quad \psi = -6EKR$



fundamental slope deflexion equation directly than to use the moment distribution method. If the slope deflexion equation was applied to the equilibrium condition at rigid joints and at each story, the equations in Table 11would be obtained. These equations were symmetrical to the main diagonal, and therefore the errors would be easily checked.

53. The simultaneous equations would be easily solved by computer. Therefore, to use the slope deflexion equation seemed best for such a complicated rigid frame.

54. Professor Naruoka said that he had some doubt whether it was necessary to use the moment distribution method to solve irregular building frames by the complicated steps shown in Table 1, 2 and 3.

The Authors, in reply, said that they appreciated the discussions and were, in general, in agreement with them. The use of computers is certainly more convenient than manual calculation. In this regard the Authors agreed with Mr Alcock. However, computers were readily and economically available only in highly industrialized countries at the present time. Furthermore, it was doubtful that machines would ever eliminate the requirement that engineers understand and be able to analyse structures.

56. Even though some details of nomenclature were probably matters of personal preference, the errata and helpful comments provided by Professor Robertson were valuable additions to the Paper. It should be mentioned in connexion with Professor Robertson's introductory remarks that the Authors discussed 'ordinary' frames in a recent paper.[25]

57. The Authors were unable to verify Professor Naruoka's Table 11. Applying the slope–deflexion equation to Example 1 (Fig. 6), as proposed in the discussion, led, by inspection, to

$$\begin{array}{c} \\ B \\ D \\ E \\ F \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} A1 & B1 & B5 & C2 & D2 & D5 & D3 & E3 & E6 & F6 & F4 & G4 & 1 & 2 \end{array} \\ \begin{bmatrix} 192 & 384 & 96 & 0 & 0 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 576 & 0 \\ 0 & 0 & 48 & 192 & 384 & 96 & 384 & 192 & 0 & 0 & 0 & 0 & 576 & 576 \\ 0 & 0 & 0 & 0 & 0 & 0 & 192 & 384 & 96 & 48 & 0 & 0 & 0 & 576 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 96 & 192 & 96 & 144 & 144 \\ 576 & 576 & 0 & 576 & 576 & 0 & 0 & 0 & 0 & 0 & 144 & 144 & 2452 & 144 \\ 0 & 0 & 0 & 0 & 0 & 0 & 576 & 576 & 0 & 0 & 144 & 144 & 144 & 1296 \end{bmatrix} \end{array} \begin{bmatrix} \theta_{A1} \\ \theta_{B1} \\ \theta_{B5} \\ \theta_{C2} \\ \theta_{D2} \\ \theta_{D5} \\ \theta_{D3} \\ \theta_{E3} \\ \theta_{E6} \\ \theta_{F6} \\ \theta_{F4} \\ \theta_{G4} \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \\ -12 \\ 0 \\ -216 \\ -72 \end{bmatrix} \qquad (8)$$

in which the first four equations represented the equilibrium condition of the joints and the remaining two the sway equilibrium conditions. The contributions to the stiffness influence coefficients of each end of the members were listed in separate columns for convenience of later calculations.

Setting

$$\begin{aligned} \theta_B &= \theta_{B1} = \theta_{B5} \\ \theta_D &= \theta_{D2} = \theta_{D3} = \theta_{D5} \\ \theta_E &= \theta_{E3} = \theta_{E6} \\ \theta_{A1} &= \theta_{C2} = \theta_{F6} = 0 \end{aligned} \qquad (9)$$

and solving yielded

$$\begin{bmatrix} \theta_B \\ \theta_D \\ \theta_E \\ \theta_F \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0{\cdot}172 \\ 0{\cdot}246 \\ 0{\cdot}120 \\ 0{\cdot}181 \\ -0{\cdot}184 \\ -0{\cdot}218 \end{bmatrix} \qquad (10)$$

Post-multiplying the transpose of this matrix by the 6 × 12 matrix obtained by omitting the last two columns of the matrix in equation (8), and adding the fixed end moments, yielded the final moments:

$$[-85{\cdot}2 \;\; -28{\cdot}3 \;\; 28{\cdot}3 \;\; -59{\cdot}0 \;\; -11{\cdot}8 \;\; 31{\cdot}8 \;\; -20{\cdot}2 \;\; -20{\cdot}2 \;\; 20{\cdot}2 \;\; 23{\cdot}1 \;\; -23{\cdot}1]$$

The above computations were carried out by an electronic computer, without special programmes by using only standard matrix routines. Nevertheless they could be easily and rapidly performed longhand, even for large and complicated frames, as shown in Table 12, except for the solution of the simultaneous equations. The latter

TABLE 12

	A	B		C	D			E		F		G	Check	
	A1	B1	B5	C2	D2	D5	D3	E3	E6	F6	F4	G4	1	2
	−12	12					−12	12					216	72
θ_B × row B	33	66	16			8							99	
θ_D × row D			12	47	94	24	94	47					141	141
θ_E × row E							23	46	11	6				69
θ_F × row F									9	17	35	17	26	26
ψ_1 × row 1	−106	−106		−106	−106						−27	−27	−451	−27
ψ_2 × row 2							−125	−125			−31	−31	−31	−281
M	−85	−28	28	−59	−12	32	−20	−20	20	23	−23	−41	0	0

could be accomplished without a computer by successive approximations. (Frames with not too many unknown rotations and translations could be analysed long hand without using successive approximations, but this was of no great interest, because for such frames Kleinlogel type ready-made solutions and even tabulations were widely available.)

58. The presence of the terms representing translation in equation (8) caused slow convergence. It was helpful to eliminate ψ_1 and ψ_2 by means of the last two of equations (8) resulting in

$$\begin{bmatrix} 55{\cdot}4 & 247{\cdot}4 & 96{\cdot}0 & -136{\cdot}6 & -136{\cdot}6 & 48{\cdot}0 & 15{\cdot}2 & 15{\cdot}2 & 0 & 0 & -30{\cdot}3 & -30{\cdot}3 \\ -121{\cdot}2 & -121{\cdot}2 & 48{\cdot}0 & 70{\cdot}8 & 262{\cdot}8 & 96{\cdot}0 & 141{\cdot}6 & -50{\cdot}4 & 0 & 0 & -90{\cdot}9 & -90{\cdot}9 \\ 15{\cdot}2 & 15{\cdot}2 & 0 & 15{\cdot}2 & 15{\cdot}2 & 0 & -65{\cdot}6 & 126{\cdot}4 & 96{\cdot}0 & 48{\cdot}0 & -60{\cdot}6 & -60{\cdot}6 \\ -30{\cdot}3 & -30{\cdot}3 & 0 & -30{\cdot}3 & -30{\cdot}3 & 0 & -60{\cdot}6 & -60{\cdot}6 & 48{\cdot}0 & 96{\cdot}0 & 169{\cdot}3 & 73{\cdot}3 \end{bmatrix} \begin{bmatrix} \theta_{A1} \\ \theta_{B1} \\ \theta_{B5} \\ \theta_{C2} \\ \theta_{D2} \\ \theta_{D5} \\ \theta_{D3} \\ \theta_{E3} \\ \theta_{E6} \\ \theta_{F6} \\ \theta_{F4} \\ \theta_{G4} \end{bmatrix} = \begin{bmatrix} 37{\cdot}4 \\ 88{\cdot}0 \\ 14{\cdot}6 \\ 19{\cdot}0 \end{bmatrix} \quad (11)$$

which was better suited for a solution by successive approximations.

59. This was exactly what the method as presented in the Paper accomplished. Lines 5, 8, 12, and 15 of Table 2 were equations (11). Indeed, Table 2 was nothing but the total amount of numerical work necessary to carry out the above mentioned elimination of ψ_1 and ψ_2.

60. Table 3 represented the total amount of numerical work of a successive approximation solution and back-substitution.

61. The Authors felt that their method was not only efficient for manual calculation, but had the added advantage of dealing with the frame itself rather than with algebraic manipulation of equations.

REFERENCES

22. INTERNATIONAL COMPUTERS AND TABULATORS. The analysis of plane structural frames. *I.C.T. List CS 354*, London 1963, and The analysis of structural grids. *I.C.T. List CS 357*, London 1963.

23. LIVESLEY R. K. Analysis of rigid frames by an electronic digital computer. *Engineering, Lond.* 1953, **176** (Aug.), 230–3.
24. SAWKO F. Determination of influence lines and surfaces by electronic computer. *Struct. Engr*, 1964, **42** (no. 8, Aug.), 258.
25. WIESINGER F. P. and LEE S. L. Analysis of frames subjected to translation. *Publ. int. Ass. Bridge struct. Engng*, 1963, **23**, 381–90.