*Discussion on Paper No. 6629**

Stresses in circular pipes buried in open cuts

by

M. M. Hanna, B.Sc., Ph.D., A.M.I.C.E.

Mr A. L. Marshall (Lecturer in Civil Engineering, Sunderland Technical College) wrote that he accepted Dr Hanna's initial assumptions in the Paper which provided an interesting method of assessing the loads on a buried pipe, although it appeared to apply only to horizontal pipes.

53. In the case of the non-horizontal section of a syphon, as indicated in Fig. 1, or, say, a sloping culvert, the vertical load was not normal to the pipe axis. Hence, section AA as indicated in the figure would be elliptical for a circular pipe.

54. For very small slopes it was unlikely that the loads would be affected to any great extent, but in the case illustrated the slope might well be appreciable and the theory would not apply in its current form. It could, however, be modified, as illustrated in Fig. 12.

55. Consider an element or ring of the pipe, of unit length, as indicated by section A′–A′ in Fig. 12. The ring would then carry a vertical load caused by a prism of soil bent elliptically in plan. This load could be divided into two components, parallel and normal to the pipe axis.

56. The parallel component would depend on the friction between the soil and the pipe, i.e. dependent on the soil properties. The normal component would act in plane A′–A′ and hence was analagous to the load conditions considered in the Paper.

57. Consider an element on the circumference of the ring, width δx as indicated, perpendicular to the line of the pipe in plan.

58. Then the weight on the element of pipe surface

$$= \gamma \times \text{volume of soil prism above it}$$
$$= \gamma\,[(H+R\cos\alpha)\delta x - y\,.\,\delta x]\cos\alpha$$

(The horizontal thickness of the prism, in the direction of the pipe was 1 cos α and hence its plan area was $\delta x \cos \alpha$.)

59. The total weight on the ring was then

$$= \textstyle\sum_0^{2R}[(H+R\cos\alpha)\delta x - y\,.\,\delta x]\gamma\cos\alpha$$
$$= \gamma\cos\alpha[\textstyle\sum_0^{2R}(H+R\cos\alpha)\delta x - \sum_0^{2R} y.\delta x]$$
$$= \gamma\cos\alpha[(H+R\cos\alpha)2R - \pi R^2/2.\cos\alpha]$$

since $\sum_0^{2R} y\,.\,\delta x$ = the area of the projection on the vertical plane of the top half of the pipe.

60. Hence the normal force, W', on the ring, due to the weight of soil, was given by (this weight $\times \cos\alpha$) or $W' = \gamma\cos^2\alpha[(H+R\cos\alpha)2R - \pi R^2/2.\cos\alpha]$. This was similar in form to the weight W used in the Paper.

61. Assuming, as in the Paper, that the displacement of the pipe, normal to the axis, was proportional to the load, equation (1) would become

$$X_A + X_B = W'/\pi.$$

---

* *Proc. Instn civ. Engrs*, vol. 26, December 1963, pp. 587–598.

62. (II) *The second condition of equilibrium.* The same procedure as in the Paper could be followed but in that case

$$w = \gamma[H+R(1-\cos\theta)\cos\alpha]\cos^2\alpha$$

and equation (2) became

$$M_A - M_B + (X_A - X_B)R + \gamma R^2 \cos^2\alpha[\tfrac{1}{2}(H + R\cos\alpha) - R\cos\alpha/3] = 0 \quad . \quad . \quad (2b)$$

and equation (2a) was

$$M_B + X_B R = M_A + X_A R + \gamma R^2 \cos^2\alpha/6[3H + R\cos\alpha] \quad . \quad . \quad (2c)$$

63. (III) *The expression for the bending moment at any section between points a and c*

Again $$w=\gamma[H+R(1-\cos\theta)\cos\alpha]\cos^2\alpha$$

Hence equation (3) became

$$\begin{aligned} M_\phi = {} & M_A + X_A R(1-\cos\phi) - \tfrac{1}{2}\gamma R^2\cos^2\alpha(H+R\cos\alpha)\sin^2\phi \\ & + \gamma R^3\cos^3\alpha\sin\phi(\tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\phi) \\ & + \tfrac{1}{3}\gamma R^3\cos^3\alpha(\cos^3\phi - 1) \end{aligned} \quad . \quad (3b)$$

and $$\begin{aligned} M_C = {} & M_A + X_A R - \gamma R^2\cos^2\alpha/6[3H+5R\cos\alpha] \\ & + \pi/4\gamma . R^3\cos^3\alpha \end{aligned}$$

and equation (3a) was

$$M_C = M_B + X_B R - W'/2 . R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

64. Substituting W' for W in equation (4) gave

$$M_\phi = M_B + X_B R(1+\cos\phi) - W'R/\pi . (\pi-\phi)\sin\phi$$

yielding again $$M_C = M_B + X_B R - W'/2 . R \quad . \quad . \quad . \quad . \quad (4a)$$

65. (Va) *The tangents at A and B remain horizontal*

Equation (7) became

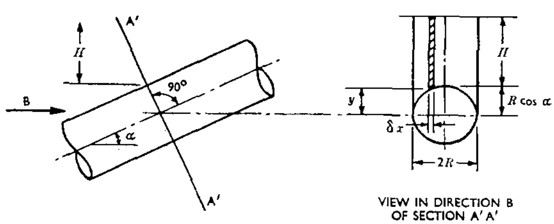
$$M_A + M_B + \frac{W'R}{\pi}\left(1-\frac{4}{\pi}\right) + \gamma R^3\cos^3\alpha\left(\frac{16}{9\pi}-\frac{7}{12}\right) - \tfrac{1}{4}\gamma H R^2\cos^2\alpha = 0 \qquad (7a)$$

66. Equation (8) became

$$\frac{2M_A}{R} = (X_B - X_A) - \frac{W'}{\pi}\left(1-\frac{4}{\pi}\right) - \frac{\gamma R\cos^2\alpha H}{4} - \gamma R^2\cos^3\alpha\left(\frac{16}{9\pi}-\frac{5}{12}\right) \qquad (8a)$$

and Equation (9) became

$$\frac{2M_B}{R} = (X_A - X_B) - \frac{W'}{\pi}\left(1-\frac{4}{\pi}\right) + \tfrac{3}{4}\gamma H R\cos^2\alpha - \gamma R^2\cos^3\alpha\left(\frac{16}{9\pi}-\frac{9}{12}\right) \qquad (9a)$$



α = angle of slope of pipe

y, H and R cos α are measured vertically

FIG. 12

67. (Vb) *The horizontal displacement of B relative to A is zero*
Equation (10) became

$$X_A = \frac{W'}{12\pi} - \frac{\gamma R^2 \cos^3 \alpha}{48} \quad \ldots \ldots \ldots \quad (10a)$$

68. Hence equation (11) became

$$M_A = 0{\cdot}2272\gamma HR^2 \cos^2 \alpha + 0{\cdot}0218\gamma R^3 \cos^3 \alpha \quad \ldots \quad (11a)$$
$$M_B = 0{\cdot}1967\gamma HR^2 \cos^2 \alpha + 0{\cdot}0330\gamma R^3 \cos^3 \alpha$$

equation (12)

$$\left.\begin{aligned} X_A &= 0{\cdot}0530\gamma HR \cos^2 \alpha - 0{\cdot}0095\gamma R^2 \cos^3 \alpha \\ X_B &= 0{\cdot}5836\gamma HR \cos^2 \alpha + 0{\cdot}1461\gamma R^2 \cos^3 \alpha \end{aligned}\right\} \quad \ldots \quad (12a)$$

and equation (13)

$$M_C = -[0{\cdot}2197\gamma HR^2 \cos^2 \alpha + 0{\cdot}0356\gamma R^3 \cos^3 \alpha] \quad \ldots \quad (13a)$$

69. It could therefore be seen that considerable modification of the loads was required for any other than very shallow slopes. It was also obvious that the effect of the second part in the equations was even further diminished.

70. It should be noted that the revised equations were a more general form of the originals, since, when $\alpha = 0$, the originals were obtained.

Dr Ing. S. D'Agostino (University of Naples) wrote that in the study of the stresses in circular pipes subjected to vertical loads only, Dr Hanna assumed that the reaction of soil was proportional to the radial displacement at every point, and that the cross-section of the pipe did not deform, so that the total displacement was vertical and the same at every point. Whilst the first hypothesis was in accord with the widely accepted theory of beams on elastic foundation, the deformability of the pipe appeared to make the second hypothesis less applicable. It would seem advisable to allow for the deformation of the pipe, so that the reaction was $p = Ky$, where the radial displacement, y, was different across the section, and K was the modulus of the foundation; that did not appear in Hanna's treatment.

72. Secondly, the Author assumed the soil reaction at the bottom to be normal to the pipe surface (i.e., no friction between soil and pipe), but considered the weight of the soil above the pipe to act vertically, thus introducing a tangential component of the load at every point, i.e., a frictional effect between soil and pipe surface. To avoid this contradiction, the contributor suggested that it was better to assume the action of the soil above also to be radial.

73. From these hypotheses, Voellmy[7] had derived and applied the influence lines of displacements and stresses for radial loads on the pipe. A similar treatment had been developed by Hetényi.[8] These influence lines allowed the weight of the soil and the lateral pressure (Spangler diagrams) to be studied separately. The purpose of the current discussion was to present along these lines the formulae for the bending moment, limited to the effect of the weight of the soil.

74. In accordance with Hetényi,[8] let

$$\eta = \sqrt{\left[\frac{R^4 K}{EI} + 1\right]}; \quad \alpha = \sqrt{\frac{\eta - 1}{2}}; \quad \beta = \sqrt{\frac{\eta + 1}{2}}$$

$$A = \frac{\alpha \cosh \alpha\pi \sin \beta\pi + \beta \sinh \alpha\pi \cos \beta\pi,}{\eta\,(\sinh^2 \alpha\pi + \sin^2 \beta\pi)}$$

$$B = \frac{\alpha \sinh \alpha\pi \cos \beta\pi - \beta \cosh \alpha\pi \sin \beta\pi}{\eta\,(\sinh^2 \alpha\pi + \sin^2 \beta\pi)},$$

where K = modulus of foundation

and EI = bending rigidity of pipe.

75. Hetény's formula for the moment in S′ due to a concentrated force P (Fig. 14) was

$$M = -\frac{PR}{2}\left(\frac{1}{\pi\eta^2} + A \sinh \alpha\phi \sin \beta\phi + B \cosh \alpha\phi \cos \beta\phi\right) \quad . \quad . \quad (14)$$

where ψ = angle defining the load element, as in Fig. 13.
ϕ and θ = angles relating the load section to the section where the bending moment is calculated (Fig. 14).

76. In Fig. 15

$$dP = \frac{dW}{\cos \psi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where dW = weight of an elementary prism of soil, above the pipe,
and dP = radial action on the pipe.

77. Thus the calculations took into account a fictitious system of horizontal forces. The weight dW was (Fig. 13)

$$dW = \gamma HR \cos \psi d\psi + \gamma R^2(1 - \cos \psi)\cos \psi . d\psi ; \quad . \quad . \quad . \quad (16)$$

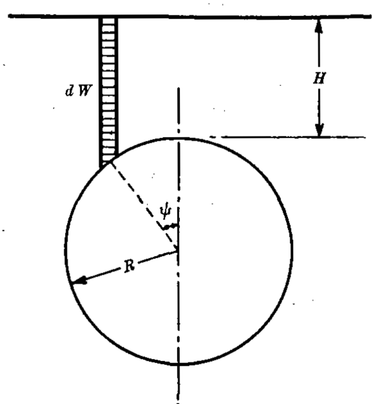


Fig. 13

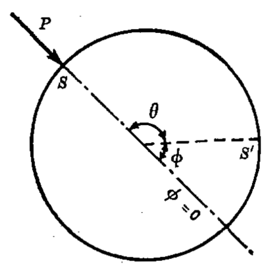


Fig. 14

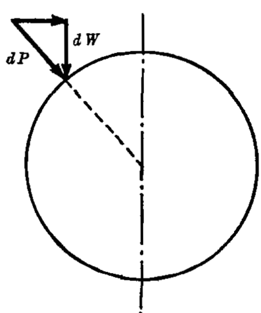


Fig. 15

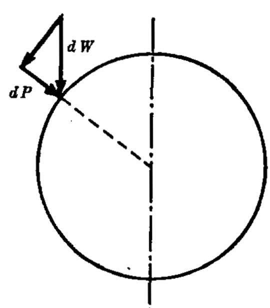


Fig. 16

the relations between θ, ϕ, and ψ were

$$\psi = \theta+\phi-\pi, \quad \cos\psi = -\cos(\theta+\phi), \quad d\psi = d\theta; \quad . \quad . \quad . \quad (17)$$

and rearrangement of equation (15), introducing equations (16) and (17) yielded

$$dP = \gamma HRd\theta+\gamma R^2[1+\cos(\theta+\phi)]d\theta \quad . \quad . \quad . \quad . \quad . \quad (18)$$

78. When the load was distributed between ψ_1 and ψ_2, the moment M was obtained by integrating equation (14) in $\theta_1 = \psi_1-\phi$, and $\theta_2 = \psi_2-\phi$. The introduction of equation (18) gave

$$M\phi = -\frac{\gamma R^2}{2}\int_{\theta_1}^{\theta_2}\left[\frac{1}{\pi\eta^2}+A\sinh\alpha(\pi-\theta)\sin\beta(\pi-\theta)\right.$$

$$\left.+B\cosh\alpha(\pi-\theta)\cos\beta(\pi-\theta)\right]\{H+R[1+\cos(\theta+\phi)]\}d\theta, \quad . \quad (19)$$

79. The integral could be solved by means of the formulae for $\int e^{ax}\sin\beta x dx$ etc.

80. The value of dP could otherwise be assumed as in Fig. 16.

$$dP = dW\cos\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In that case the tangential component was neglected, but a shear force dT must be introduced for reasons of equilibrium (Fig. 17), and therefore the effective value of dW became uncertain. Assuming, for the sake of simplicity, that dW was still given by equation (16), the expressions for dP and M_θ became

$$dP = \gamma HR\cos^2(\theta+\phi)d\theta+\gamma R^2[1+\cos(\theta+\phi)]\cos^2(\theta+\phi)d\theta, \quad . \quad . \quad (21)$$

$$M_\phi = \frac{\gamma R^2}{2}\int_{\theta_1}^{\theta_2}\left[\frac{1}{\pi\eta^2}+A\sinh\alpha(\pi-\theta)\sin\beta(\pi-\theta)\right.$$

$$\left.+B\cosh\alpha(\pi-\theta)\cos\beta(\pi-\theta)\right]\{H\cos^2(\theta+\phi)$$

$$+R[1+\cos(\theta+\phi)]\cos^2(\theta+\phi)\}d\theta \quad . \quad . \quad . \quad (22)$$

81. This discussion only indicated an approach which seemed adequate for tackling the problem. It should be noted that the bending moments for lateral loads were obtainable from integrals similar to (19) and (22).

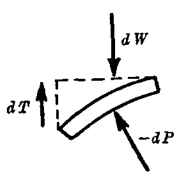


FIG. 17

82. It must be stressed that in the above procedure the characteristics of the soil and the deformability of the pipe were taken into account. The most recent trends with buried pipes appeared to follow these lines; it should be borne in mind that pipes of corrugated sheets, having a radius, R, of 1·50 m and 4·2 mm thick,[9] could be buried 20 m deep in non-cohesive soil, taking advantage of the redistribution of soil reaction due to the deformability of pipes.

Mr D. G. Alcock (I.C.T. Ltd) wrote that the Author had made a valuable practical contribution in presenting his equations (11), (12), and (13).

84. The equation relating the radial reaction of the soil to the vertically applied load was particularly ingenious as it was both realistic and mathematically simple.

85. The casual reader of the Paper might wonder at the inclusion of part M in Fig. 11 on the grounds that the effect of this load was very small, whereas the effects of axial deformation, shear deformation and shift of the neutral axis (caused by the curvature of the member) had all been neglected.

86. The contributor investigated these effects by computing deformations, f_{ik}, from

$$f_{ik} = \frac{1}{AE}\left[\int\frac{MiMkds}{Ry_0}+\int NiNkds+\frac{4(1+\nu)}{3}\int ViVkds + \int\frac{MiNkds}{R}+\int\frac{NiMkds}{R}\right] \quad . \quad . \quad (23)$$

where the integrals were taken over the whole ring and, in addition to the Author's notation,

A = area of section
N = axial force on section
V = shear on section
ν = Poisson's ratio
y_0 = shift of neutral axis from centroid.[10]

87. Calculation of y_0 required use of the wall thickness, t. R is the *mean* radius.

$$y_0 = -RZ/(1+Z)$$

where

$$Z = -\frac{1}{A}\int_{-t/2}^{+t/2}\frac{ydA}{R+y} = \frac{1}{3}\left(\frac{t}{2R}\right)^2+\frac{1}{5}\left(\frac{t}{2R}\right)^4+\frac{1}{7}\left(\frac{t}{2R}\right)^6+\ldots \quad . \quad . \quad (24)$$

88. The analysis showed the remarkable fact that the bending moments were independent of Poisson's ratio, and the axial thrusts were independent of Poisson's ratio and curvature.

89. The five equations took the form:

$$M_A = (-0{\cdot}2272-p)\gamma HR^2+(-0{\cdot}0218-q)\gamma R^3$$
$$M_B = (-0{\cdot}1967-p)\gamma HR^2+(-0{\cdot}0330-q)\gamma R^3$$
$$M_C = (+0{\cdot}2197-p)\gamma HR^2+(+0{\cdot}0356-q)\gamma R^3$$
$$X_A = -0{\cdot}0530\gamma HR+0{\cdot}0095\gamma R^2 \quad \text{Compression}$$
$$X_B = -0{\cdot}5836\gamma HR-0{\cdot}1461\gamma R^2 \quad \text{Compression}$$

where

$$p = 0{\cdot}6553Z/(1+Z)$$
$$q = 0{\cdot}1248Z/(1+Z)$$

90. In order to conform to the sign convention normally used in the theory of curved members, the Author's sign convention was reversed. When the ratio of radius, R, to wall thickness, t, was greater than 33, the moment equations reduced to the Author's equations correct to all 4 decimal places in the coefficients. Thus the

TABLE 1

R/t	p	q
33	0·0001	0·0000
10	0·0005	0·0001
5	0·0022	0·0004
2	0·0139	0·0026
1·5	0·0250	0·0048
1·0	0·0588	0·0112
0·6	0·1954	0·0372

Author was fully justified in neglecting all but flexural deformations and in retaining the small load shown as part M in Fig. 11.

91. *It was interesting to see that the curvature reduced the magnitude of the* moment at C and increased the magnitudes at A and B.

92. A small programme was written for the Sirius computer to evaluate p and q for various ratios of R/t, some of which are shown in Table 1. When values of M and X are used to compute stresses the curvature should be considered.

93. The programme is being rewritten to calculate principal stresses where internal pressure, external hydrostatic pressure, and differential temperature gradient through the wall are all acting. This programme should be useful for the study of thick-walled pressure vessels subject to the above disturbances, as well as for buried pipes and culverts.

The Author in reply thanked the contributors not only for the interest they showed in commenting on the basic assumptions of the Paper, but for the mathematical work and programming for computers that gave interesting additions and applications for equations (11), (12), and (13).

95. Mr Marshall applied the general theory given in the Paper to the case of inclined pipes. It was natural that equations (11a), (12a), and (13a) given by Mr Marshall took the same form as those obtained in the Paper with the first terms multiplied by $\cos^2 \alpha$ and the second term multiplied by $\cos^3 \alpha$.

96. The slope of the inclined parts of the culverts as shown in Fig. 1 of the Paper is usually between 1:8 and 1:12, which is of little effect on the values of the bending moments. If the angle α is more than say 10° the effect on the inclination should not be ignored.

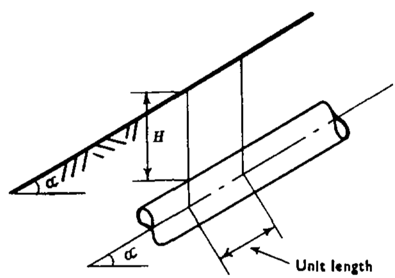


FIG. 18

97. For sloping pipes two cases are to be considered:

(a) The slope of the earth embankment is parallel to the pipe axis, for which Mr Marshall's analysis is applicable (Fig. 18). The component of the loads perpendicular to the pipe axis should give a case similar to that given in the Paper.

For pipes with larger slopes, the component parallel to the pipe axis is appreciable. The distribution of the skin frictional forces is not necessarily uniform and is an interesting soil mechanics problem. From the stress analysis viewpoint, these parallel forces produce stresses along the pipe generator and principal stresses should be studied at points a, b, and c, Fig. 11.

(b) The earth embankment is horizontal while the pipe axis is sloping (Fig. 19). The height of the earth prism is variable and thus the stresses are different along the same generator of the pipe. An approximate solution may be obtained by taking the mean height of the earth prism under consideration.

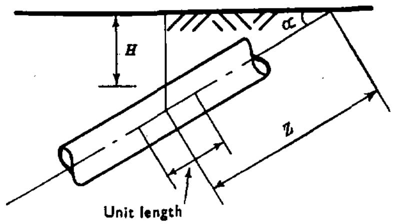


FIG. 19

98. Both Mr Marshall and Mr Alcock accepted all the initial assumptions of the Paper for the loading and reactions on the pipe. Dr Agostino agreed only to the assumption that the reaction of soil is proportional to the radial displacement at every point, but did not accept the assumption of equality of vertical displacement of pipe (as shown in Fig. 5) due to the deformability of the pipe.

99. As stated in many parts of the Paper and mainly § 25, the Paper gives the structural analysis for circular pipes under these assumptions and according to the loading shown in Fig. 7.

100. In the Author's opinion, the analysis given in the Paper is applicable and correct for all rigid pipes in which the deformations are small such as those constructed of plain or reinforced concrete or made of cast iron or burned clay. The design for such pipes may be based directly on equations (11), (12), and (13) since the sides of the rigid pipes will deform outwards very little and passive resistance will not develop until the pipe starts to fail.

101. On the other hand, the stability of flexible pipes constructed of thin-walled or corrugated sheets depends mainly on the passive earth resistance on the sides. Dr Agostino may be justified then in his remark.

102. The early work carried out by Voellmy[7] in 1936 gave the radial deflexions of the pipe when subjected to a radial line load and buried in an elastic medium that takes both tension and compression. Such an assumption may not be fully to the satisfaction of soil engineers. Moreover, the solved numerical example in Voellmy's paper involved such an amount of computation that its application was limited, contrary to the simple substitution in the equations of the present Paper.

103. The extensive and continuous work published by Iowa Engineering Experiment Station[1] gave the vertical and horizontal loads for flexible pipes as illustrated in Figs 4b and 6a respectively. Therefore, Dr Agostino's suggestion for converting the vertical earth loading to radial forces to take advantage of Voellmy's influence line is of little practical use.

104. It may be of interest to compare the results given in Voellmy's example for the case of rigid pipe (as given in Fig. 6 of his paper) with those obtained by applying equations (11) and (13) of the present Paper.

TABLE 2

B.M.	Voellmy	Hanna
M_a	-274 *kg cm/cm*	-261 *kg cm/cm*
M_c	$+286$,,	$+254$,,
M_b	-402 ,,	-228 ,,

105. It is interesting to note that the results do not differ greatly from each other although the assumptions and approaches are different.

106. Mr Alcock contributed valuable work by writing a programme for the Sirius computer to study the effect of axial and shear deformation as well as the effect of the curvature on the results of the analysis as given in the Paper. The values of p and q as given by Mr Alcock in Table 1 show that for $R/t \geqslant 5$, the bending moments M_a, M_b, and M_c differ by only about 1% of the values obtained from equations (11) and (13) of the Paper.

107. Mr Alcock's work leads to the conclusion that for all practical problems of design of buried pipes used in civil engineering where $R/t \geqslant 5$, the equations given in the Paper may be applied with sufficient accuracy.

References

7. VOELLMY A. Erddruck auf elastisch eingebettete Rohre. *Int. Assn for Bridge and Struct. Engng.* Publ. 4. Zurich 1936, pp. 591–611.
8. HETÉNYI M. *Beams on elastic foundation.* University of Michigan Press, Ann Arbor, 1958, p. 159 ff.
9. ARMCO-FINSIDER. *Condotte metalliche ondulate.* Pompei. 1959.
10. SEELY F. B. *and* SMITH J. O. *Advanced mechanics of materials.* John Wiley and Sons, New York, March 1959, Chapter 6.