*Discussion on Paper No. 6726**

The finite element method for analysis of elastic isotropic and orthotropic slabs

by

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and

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Mr R. J. Allwood (Reader in Structural Analysis, Department of Civil Engineering, Loughborough College of Technology) said that the Authors had derived a stiffness matrix for an element of a plate subjected to bending. Because the finite element method was so simple once the relevant stiffness matrix was available, and allowed the solution of a wide spectrum of problems, such as that solved and described by the Authors, it seemed important to Mr Allwood that the validity of the basic stiffness matrix should be beyond question. The following two points had occurred to him when he was comparing the proposed matrix with those published previously by Melosh.[12]

51. The function used to represent the lateral deflexion W (equation 11) was identical with that used by Melosh, except that Melosh placed the origin at the centre of the rectangular element whereas the Authors did not define their origin. Since their stiffness matrix was different from that of Melosh one could conclude that the origin was other than at the centre. One might therefore ask, what was the best position for the origin, particularly for the general quadrilateral element, so that stiffness matrices which would most closely approximate to the problem under consideration would be produced?

52. Secondly, Mr Allwood noted that the proposed stiffness matrix was applicable to the general quadrilateral (such as element A in Fig. 1 of the Paper) despite the violation of compatibility at the boundaries (§ 47). Clough[2] had shown that when using similar functions with triangular elements he obtained results which converged as the mesh size was reduced, but converged to an incorrect answer. The Authors had shown that for rectangular elements their stiffness matrix produced results which converged to the correct answer; one also needed to ask whether that behaviour held for the general quadrilateral as well.

Dr H. A. Slyper and Mr D. J. Dawe (Dept of Mechanical Engineering, University of Cardiff) wrote that the application of the finite elements procedure to plate bending problems had, as the Authors stated, received little attention in the past. They were to be congratulated on producing a paper which demonstrated the accuracy and range of applicability of the method.

54. The writers had for some time been investigating the application of the finite element method to plate vibration problems and had independently duplicated much of the work of the Authors. Both stiffness and inertia matrices had been derived and

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applied[13] for a uniform isotropic rectangular plate element using the same simple polynomial expression for lateral deflexion within the element as was chosen by the Authors. The strain energy approach as outlined by Melosh[4] was used for the derivation of the stiffness matrix in place of the method of virtual work employed by the Authors. The resulting stiffness matrix was of course the same as that quoted in the Paper (when reduced to the isotropic case). The contributors' primary interest was in the calculation of the eigenvalues and eigenvectors of vibrating plates of arbitrary shape. However, as a preliminary to this the stiffness matrix was applied to the solution of a number of static problems, some of which were considered by the Authors. In all cases where such problems had been repeated the numerical values obtained by the contributors confirmed those quoted in the Paper.

55. The expression for the lateral deflexion W is an obvious one, being the product of a third-order polynomial in X and Y, with the fifth- and sixth-order terms and the $X^2 Y^2$ term neglected. The expression is such that the lateral deflexion and both slopes θx and θy are continuous within the boundaries of an element. At the boundaries of neighbouring rectangular elements continuity of lateral deflexion is maintained across the lines separating such elements. However, such is not the case in general for the two slopes. In fact, it can easily be shown that continuity of the θy slope is maintained along the edges y=constant but not along the edges x=constant. Similar remarks apply to the θx slope.

56. The contributors consider that these slope discontinuities are the cause of an effect not shown by the Authors' results, *viz.* refinement of the finite element mesh does not automatically lead to a closer approximation to a required deflexion. This effect is illustrated by the results obtained for the central deflexion of an isotropic square plate (side L, Poisson's Ratio, $\nu=0{\cdot}3$), clamped around the boundaries, when subjected to a central point load F, as shown in Table 6.

TABLE 6: CENTRAL DEFLEXION OF ISOTROPIC SQUARE PLATE

Basis of Calculation	Central Deflexion
Finite Element 2×2	0·00592
4×4	0·00613
6×6	0·00591
8×8	0·00580
Theoretical Solution[5]	0·00560
Multiplier	FL^2/D

57. Ideally, the deflexion expression should be such that both the deflexion and the two slopes are continuous within an element and across element boundaries.

58. The use of a column vector of nodal forces consistent with a static distributed loading had been considered by the contributors. The consistent loading vector for a rectangular element under uniform loading had been evaluated using a similar approach to that outlined by the Authors. Solutions to deflexion problems of uniformly loaded plates were obtained using both the consistent loading vector and the lumped loading vector (in which the distributed loading over an element is regarded as being concentrated at the nodes). Under uniform loading the nodal forces for any internal point on the plate were identical in both approaches.

59. To indicate the improvement normally obtained by using the consistent loading vector, the calculated deflexion at a free corner of a square cantilever plate of side L, $\nu=0{\cdot}3$, under a uniform distributed load is given in Table 7. The percentage errors are based on the solution obtained by a point matching method[9] in which it is assumed that $\nu=\frac{1}{3}$.

TABLE 7: DEFLEXION AT FREE CORNER OF A UNIFORMLY LOADED SQUARE CANTILEVER PLATE

Basis of Calculation	Lumped Loading		Consistent Loading	
	Deflexion	Error (%)	Deflexion	Error (%)
Finite element 1 × 1	0·170	+34·9	0·131	+4·0
2 × 2	0·137	+8·7	0·127	+0·8
3 × 3	0·131	+4·0	0·127	+0·8
Theoretical Solution[9]	0·126		0·126	
Multiplier	qL^4/D		qL^4/D	

60. It is evident in this case that for a small number of elements considerable improvement is obtained using the consistent loading vector but that this improvement becomes less marked when the number of elements is increased. Whilst this statement is true in general, there are exceptions in which it appears that use of the consistent loading vector leads to an inferior answer. Such a case is that of the central deflexion of a square plate (side L, $\nu=0{\cdot}3$) uniformly loaded and simply supported on all edges. The results are given in Table 8. The percentage errors are based on the solution quoted by Timoshenko.[5]

61. It would be interesting to see comparative deflexion values for a plate loaded under hydrostatic pressure. The advantage of using the consistent loading vector would possibly be more strongly emphasized in this case.

62. Results obtained by the contributors when investigating vibrational problems showed the finite element method to be a powerful tool for the calculation of the natural frequencies of plates.[13] The frequencies obtained might lie on either side of the true frequencies and it was impossible to estimate whether a particular frequency value was high or low. This contrasted unfavourably with the use of the well-known energy methods, in which the calculated frequencies were upper bounds of the true values. However, the finite element technique had much more general application and could be used in cases where other methods (such as the Rayleigh and Rayleigh–Ritz) became inaccurate. Investigations into the vibration of flat and curved plates of non-rectangular plan form and non-uniform thickness were being carried out by the contributors.[13,14] These required the development and use of stiffness and inertia matrices for elements of variable thickness or non-rectangular plan form. The basic procedure, however, was unaltered.

TABLE 8: CENTRAL DEFLEXION OF A SIMPLY SUPPORTED UNIFORMLY LOADED SQUARE PLATE

Basis of Calculation	Lumped Loading		Consistent Loading	
	Deflexion	Error (%)	Deflexion	Error (%)
Finite Element 2 × 2	0·00342	−15·8	0·00505	+24·4
4 × 4	0·00394	−3·0	0·00433	+6·7
6 × 6	0·00401	−1·2	0·00418	+3·0
8 × 8	0·00403	−0·7	0·00413	+1·7
Theoretical Solution[5]	0·00406		0·00406	
Multiplier	qL^4/D		qL^4/D	

Mr P. E. West (Senior Scientific Officer, British Railways Board Research Department, Derby) suggested that problems associated with rectangular slabs were, in general, merely special cases of skew slab problems. Moreover, particularly in the field of bridge analysis, skew slabs were much more common than rectangular.

64. Finite difference methods had been developed for the solution of skew slabs having relatively simple support conditions, but the methods were not suitable for the more complex ones, which were usually those most likely to occur in practice. The method of finite elements, based on the work of Argyris[15], avoided the problem of edge support by considering the edge as a series of nodes, and by imposing support conditions only at these nodes.

65. The basic procedure to be adopted in the finite element method was outlined by the Authors. However, the polynomial expansion (1) they suggested for a typical element was, for a skew element (Fig. 8), liable to error. A more suitable expansion could be derived as follows:

Let

$$\omega = A_0+A_1x+A_2y+A_3x^2+A_4xy+A_5y^2+A_6x^3+A_7x^2y+A_8xy^2 +A_9y^3+A_{10}(x-y\tan\theta)^3y+A_{11}(x-y\tan\theta)y^3 +A_{12}(x-y\tan\theta)^2y^2 \quad (33)$$

By imposing the condition that this must satisfy the biharmonic equation for an unloaded slab, it follows that:

$$\omega = A_0+A_1x+A_2y+A_3x^2+A_4xy+A_5y^2+A_6x^3+A_7x^2y+A_8xy^2 +A_9y^3+A_{10}f(x,y)+A_{11}g(x,y) \quad (34)$$

where

$$f(x,y) = x^3y-\frac{6c^3}{b(b^2+3c^2)}x^2y^2+\frac{3c^2(c^2-b^2)}{b^2(b^2+3c^2)}xy^3+\frac{2c^3}{b(b^2+3c^2)}y^4$$

$$g(x,y) = \frac{3bc}{b^2+3c^2}x^2y^2+\frac{b^2-3c^2}{b^2+3c^2}xy^3-\frac{bc}{b^2+3c^2}y^4.$$

If $c=0$, i.e. if the element is rectangular, then (34) reduces to the Authors' equation (11).

66. The stiffness matrix derived from (34) had been applied to a simply supported skew slab; the results exhibited the same order of accuracy, by comparison with a finite difference solution[16] as those obtained by the Authors.

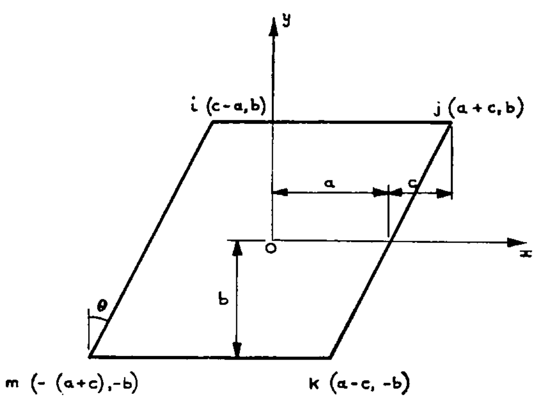

FIG. 8: SKEW ELEMENT

67. To illustrate the versatility of the finite element method, a solution was also obtained for a skew slab supported only at its corners; current work was concerned with the more general problem of a skew slab on discrete supports.

Mr B. Irons (Senior Stress Engineer, Rolls Royce Ltd, Derby) wrote that with reference to § 5 of the Paper, the Rolls-Royce group had recently discovered what appeared to be entirely satisfactory shape functions for the triangular element of a plate in bending. The following two conditions were taken as sufficient to ensure that a solution converged towards the truth as the mesh was finely sub-divided.

(a) If the nodal slopes and displacements were consistent with a state of constant stress or curvature, then the chosen shape functions should implement this state.[17] In particular, rigid body motions must be correctly represented.[12]

(b) The shape functions should guarantee conformity of deflexions between elements[12] and of slope across boundaries between elements.[18]

69. Professor Zienkiewicz had suggested in private discussion that nodal deflexions could be ignored in describing the bending distortions of a triangle. The normal displacements at the nodes defined a rigid body movement, and the deformations were adequately defined by the slopes at the nodes relative to this rigid body movement. *Following this suggestion, it may be stated that if a triangle with nodes* 1, 2 and 3 is considered, a vector rotation θ^*_{x1} or θ^*_{y1} applied at node 1 (the asterisk signifies that θ^*_{x1} or θ^*_{y1} and the resulting normal deformation w^*_{x1} or w^*_{y1} are relative to the rigid body displacement)

$$\begin{aligned} w^*_{x1} &= \{\phi_{12}(y_1-y_2)+\phi_{13}(y_1-y_3)\}\theta^*_{x1} \\ w^*_{y1} &= \{\phi_{12}(x_2-x_1)+\phi_{13}(x_3-x_1)\}\theta^*_{y1} \end{aligned} \quad . \quad . \quad . \quad . \quad (35)$$

With the other four shape functions obtained by cycling the indices, the displacements based on (35) satisfy both the requirements for convergence, if the ϕ are as follows:

$$\phi_{12} = L_1^2L_2+\tfrac{1}{2}L_1L_2L_3-\tfrac{1}{2}\epsilon_1+\tfrac{1}{2}\epsilon_2-\tfrac{3}{2}k_3\epsilon_3 \quad . \quad . \quad . \quad . \quad (36)$$

L in this expression = the so-called area co-ordinates; the three equations

$$\left.\begin{aligned} L_1x_1+L_2x_2+L_3x_3 &= x \\ L_1y_1+L_2y_2+L_3y_3 &= y \\ L_1+L_2+L_3 &= 1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

relate $[L_1, L_2, L_3]$ to a point $[x, y]$ in the triangle.
The constant multiplier k_3 in (36) depends on element geometry:

$$k_3 = \frac{(x_1-x_2)(x_1+x_2-2x_3)+(y_1-y_2)(y_1+y_2-2y_3)}{(x_1-x_2)^2+(y_1-y_2)^2} \quad . \quad . \quad (38)$$

To calculate ϕ_{13}, indices 2 and 3 are interchanged in (36) and (38).

70. The contributor stated that he believed the functions ϵ in (36) were an innovation:

$$\epsilon_1 = \frac{L_1L_2^2L_3^2}{(L_1+L_2)(L_1+L_3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

for example, and ϵ_2 and ϵ_3 follow in cyclic order. Their characteristic was that the slope normal to one edge varied parabolically along that edge, while the slopes across the other two were zero, and the deflexions were zero along all the edges. In (36) the ϵ were applied with the objective of making the normal slope vary linearly along all three edges, to ensure that the slope conformity requirement of (a) was satisfied. (To achieve this the conditions leading to the proof given in ref. 19—already broken in ref. 20—have to be transgressed and for this reason ϵ_1 for example must have singularities at nodes 2 and 3 of such a type that second derivatives do not exist at *those points. The ϵ are in the nature of corrections, and should not be present in* large amounts: Fig. 9 showed that these singularities need not frighten an engineer. Indeed, it could be argued that in the neighbourhood of a node point the several adjacent elements have different curvatures, so that no unique curvature attaches to

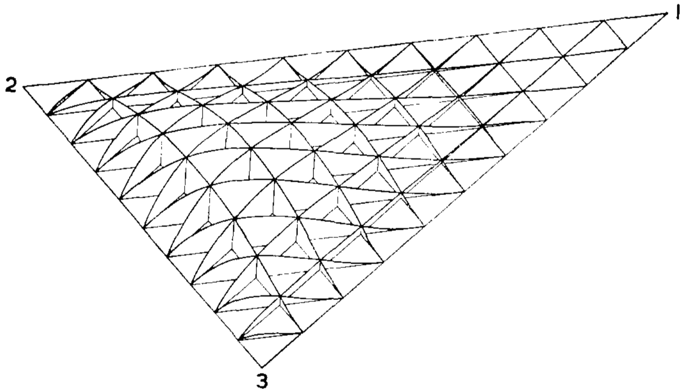

FIG. 9: ILLUSTRATION OF ϵ FROM EQUATION (39) SHOWING NATURE OF SINGULARITIES AT POINTS 2 AND 3

the node point itself. This feature must be nearly universal in finite element methods, and it did not appear to prevent their converging).

Mr C. I. Mathew (Principal, Government Polytechnic, Perintalmanna, Kerala State, India) wrote that the possibility of solving a slab considering it as a grid was put forward in 1952 by Ewell, Okubo and Abrams.[21] Since then efforts had been made to approximate a plate to a grid and *vice versa*, but in most of these only continuous members between supports had been considered. In the finite element method presented, the members were continuous only between the fictitious nodes.

72. The method of influence coefficients in the solution of a grid which assumed three actions, two concentrated moments and a force, had been dealt with by P. B. Morice.[22] The problem of whether a square grid could be approximated to a plate had been investigated by the contributor.[23] Using the influence coefficients method, square grids were solved with 3, 5, 7, and 11 members each way simply supported, with torsion prevented at the supports. The grids were solved with the joints considered both as pinned and rigid with $\frac{GC}{EI}=0{\cdot}8$. The same grids were then solved as plates, using Navier's solution, and the values of bending moments and deflexions were compared. Only symmetrical loads were considered. For a uniformly distributed load the values agreed very well. But for a central point load, even though the deflexions showed fair correlation, the bending moments did not show the required agreement. This could be attributed to the error in successive differentiation of the deflexion function. This indicated that the agreement in deflexions need not necessarily be followed by agreement in bending moments.

73. The general theory presented in the Paper considered the element as a quadrilateral. It was also pointed out that elements of other shapes could be considered. In recent years attempts had been made to approximate a diagrid to a plate, but without much success. K. C. Ray[24] presented a method which was not satisfactory. Hendry and Jaeger,[25] modifying the Navier solution, devised a method to solve diagrids. This method, even though it showed fair agreement in deflexions, did not show the necessary agreement in bending moments. Thus even rotation of square elements had failed to give the required results. It was doubtful whether a quadrilateral or any other element would be useful in the problem.

74. With the advent of the electronic computer, solution of a matrix had become comparatively easy. The usefulness of the method presented was not in doubt. For problems where the thickness of plate varied and where there were holes in the plates, this method might provide the nearest approach to the correct values. But it was the contributor's opinion that the shape of the element in relation to the support conditions played a decisive part in achieving the required result.

75. The contributor had presented a solution of a triangular grid.[26] The bending moments at the same node in different directions would be considerably different and any approach might have to deal with this aspect of the problem.

76. The investigations into a square grid with eleven members each way led to the formation of a 45×45 matrix. A method had to be evolved to check the influence coefficients. Thus forming a large matrix also had its disadvantages. If a method could be devised to direct the computer to form the matrices on feeding in the data of the element/structure, it might simplify the problem considerably.

Professor A. Hrennikoff and **Professor S. S. Tezcan** (Department of Civil Engineering, University of British Columbia) wrote that they found the method of analysis of plates described in this Paper very interesting and wished to compare it with the approach developed by one of them.[27, 28]

78. This consisted of replacing the plate subjected to plane stress or transverse bending by a framework of cells of repeating pattern made up of elastic bars endowed with extensional stiffness in case of plane stress and flexural stiffness in plate bending. The types of cells and the member stiffnesses were determined from the conditions of equal deformabilities of the plate prototype and the framework model under the action of uniform stresses. With infinitesimal size of cell the framework imitated the plate exactly.

79. The early solutions of the framework were performed by relaxation and were very laborious, but a high speed digital computer had been used successfully and the method had been developed further and presented in matrix terms.[29–31] Several types of cells had been proposed, but only the square and rectangular shapes had actually been used in solutions.

80. Knowing the member stiffnesses, the three deformations of each corner were related to the resultant corner stress by means of a 12 by 12 stiffness matrix. The general matrix of the framework was composed of the cell matrices, and its product with the column matrix of the joint deformations was equal to the external loads acting at the nodal points. The matrix treatment of the problem was then analogous to the one proposed by the Authors, the contributor's 'cell' being similar to the Authors' 'element', even though the two methods were quite different in their approach. The stiffness matrix of a single unit was formed in the framework method more directly than in the Authors' method, where it was obtained by combining four separate matrices. The Authors' method was more versatile, permitting different shapes of elements, even though their irregularity might bring in considerable mathematical complications. The 'cell' possessed the same deformability (under uniform stress) as the prototype. This was also true of the 'element' insofar as it satisfied the differential equation of the plate. With a decrease in the size of cell the framework solution converged to the exact. This was also true of the Authors' solution. The relative speed of the two convergencies was not certain, although the following circumstances might have some bearing on it. In the Authors' method preservation of continuity between the elements was effected only at the nodes and was ignored elsewhere. In the contributors' approach the model structure did not exist physically between the nodes and consequently the continuity need not be considered.

81. The accuracy of the contributors' method was found to compare favourably with that of Melosh,[4] and Marcus.[10]

82. The framework method had been applied to problems reported in Tables 1 and 3 and the results were given in Tables 9 and 10, including torsion moments and

shears. Comparison with the values found by the Authors' method and the exact solution would be desirable. The deflexions and bending moments in these tables were very close to the ones found by the Authors. The framework method was equally applicable to cylindrical shells[27] whose bars combined the properties of the plane stress and flexural cells.

TABLE 9: SQUARE PLATE WITH CLAMPED EDGES UNDER UNIFORM LOAD, 8×8 FRAMEWORK. $\mu=0{\cdot}3$

Joints (See Fig. 10)	V_x	V_y	M_x	M_y	M_{xy}	w
Multiplier	$qL10^{-2}$	$qL10^{-2}$	$qL^2 10^{-2}$	$qL^2 10^{-2}$	$qL^2 10^{-2}$	$qL^4 10^{-3}/D$
1	38·527	0	−5·010	−1·684	0	0
2	27·508	0	−0·886	0·040	0	0·275
3	15·384	0	1·212	1·329	0	0·725
4	6·799	0	2·142	2·130	0	1·122
5	0	0	2·400	2·400	0	1·257
6	35·203	8·740	−4·525	−1.525	0	0
7	24·896	1·308	−0·760	0·061	−0·341	0·247
8	13·602	−3·202	1·102	1·213	−0·389	0·673
9	5·902	−5·902	1·910	1·911	−0·236	1·003
11	24·305	15·656	−3·110	−1·044	0	0
12	16·807	0·901	−0·452	0·062	−0·584	0·169
13	8·205	−8·204	−0·746	0·745	−0·650	0·455
15	3·217	−3·143	−1·031	−0·252	0	0
16	3·014	−2·994	−0·177	−0·177	−0·541	0·063
18	0	0	0	0	0	0

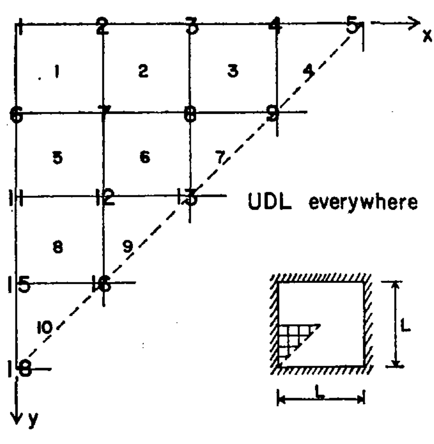

FIG. 10.

TABLE 10: SQUARE PLATE SUPPORTED AT FOUR CORNERS UNDER UNIFORM LOAD, 10×10 FRAMEWORK. $\mu = 0{\cdot}3$

Points	M_x	M_y	w
Multiplier	$qL^2 10^{-2}$	$qL^2 10^{-2}$	$qL^4 10^{-3}/D$
1	0	15·658	17·872
2	11·033	11·033	25·453

The Authors in replying to the discussion wrote that they experienced a certain measure of difficulty. The reason for this was the progressing nature of research and the large time lag between the writing of the Paper, its printing, and the date of their reply (March 1963 to December 1965). Developments achieved by others and by the Authors in the intervening period had produced more refined and versatile element characteristics. In particular the success achieved in devising a suitable stiffness matrix for an arbitrary, triangular-shaped element opened the door to a very much wider area of application of the finite element procedure in the field of plate study and its practical applications to foundation slabs, bridge decks and, not surprisingly, shell structures. Although some of these developments were being concurrently published elsewhere[32] some of the results and salient details would be given in this discussion to 'round off' the present problem.

84. Before proceeding with outlining of this new work, which incidentally touched on some of the points raised in the discussion (particularly those of Mr Irons), the Authors would like to reply, *seriatim*, to the various questions raised.

85. *Convergence:* If the displacement function chosen for the element was such that continuity of both displacements and slopes was satisfied, then, as could easily be shown,[33] the finite element solution of the type proposed must converge to the correct answer and at any stage would result in an 'upper bound' to the total strain energy of the system (i.e. in general would underestimate deflexions).

86. The Authors contended that it was possible to achieve convergence (though not bounds) to the correct solution by displacement functions which did not completely satisfy the continuity conditions (except naturally at nodes) provided these functions were of such a type that they could portray, when appropriate nodal displacements were given, a state of constant strain within the element. This proposition was discussed in more detail elsewhere[32] but the results presented would bear out the point.

87. Dr Allwood, Dr Slyper, and Mr Dawe all referred to this point and mentioned the function derived by Melosh.[12] This was identical to the function presented in the Paper, which was submitted before Melosh's publication but unfortunately printed late. In common with the function used here it violated slope continuity across element junctions while maintaining (for rectangular elements) the continuity of displacement there. (This point appeared to have been misused by Melosh in his publication, where he claimed complete continuity.) Dr Allwood mentioned in this connexion the matter of the origin of the co-ordinate system. In the Paper this was arbitrarily taken at one of the nodes, but clearly its position was immaterial providing the orientation of the axes was preserved. A mere shift of origin would always result in the same stiffness matrix.

88. Reference was made by the above Contributors to examples which apparently proved that convergence to incorrect solutions might occur. Clough[34] apparently demonstrated this point in some solutions using triangular plate elements and simple polynomial functions. The Authors investigated similar functions during the course of research leading to the present paper and found that numerical difficulties due to

singular nature of the *C* matrix (similar to the one defined by equation (13)) prevented a reasonable solution being reached. The results presented by Clough in this context were therefore subject to some doubt.

89. Dr Slyper and Mr Dawe made the same point (that reduction of element size does not necessarily lead to convergence) and quoted results of their Table 6. Although convergence was not monotonic it certainly appeared that once again the results in fact converged to the correct ones.

90. *Confirmation of results.* The Authors were delighted that the work carried out by Dr Slyper and Mr Dawe confirmed both the general algebra and numerical values obtained. The derivations of the stiffness matrix via strain energy consideration would obviously, in a linear case, lead to the same results as the use of virtual work. The latter approach had however some generality, being adaptable to non-linear problems.

91. The many possible pitfalls in computer programming, data preparation, etc. made the numerical verification obtained extremely valuable.

92. *Consistent loads.* Although the derivation of 'consistent' load matrix for both static and dynamic cases was presented in the Paper, the numerical examples therein used a physical load lumping process which, while necessarily convergent with decreasing size of sub-division, introduced additional errors.

93. The results given by Dr Slyper and Mr Dawe in Table 7 which showed the remarkable improvement in accuracy achieved using 'consistent' load distributions were of great interest. The slight deterioration of accuracy in the example given in Table 8 was therefore a little surprising. Clearly some compensation of errors must have occurred here. The Authors had used the 'consistent' type mass matrix in vibration problems. Here its use appeared imperative—and results of a study of a vibration of a cantilever plate divided into 12 elements only showed that frequencies of the 13th mode could be assessed with an accuracy of the order of some 3% with its use.[35] With lumped mass matrices such accuracy was limited to the lowest frequencies.

94. *Analogue models.* Much attention had been given in the literature to problems of representing plate continua by grid-work systems and not surprisingly, to the reverse of this process. The Paper was concerned with *true plates* and if at some stage the methods presented were used for solving actual gridwork problems by engineers, the onus of proof of the equivalence of the plate constants used and the actual system would lie on the user of such a representation and need not be discussed.

95. The first problem, however, was that of relevance. If a gridwork system with similar nodes and identical stiffness to that obtained by the finite element procedure could be found then, obviously, the formulation of the combined equations and their solutions would follow identical paths in both approaches and give identical answers. The difficulty lay in determining the characteristics of such an equivalent grid system. Professor Hrennikoff and Professor Tezcan had arrived at such representation by ensuring equal deformabilities of the analogues and the continuum elements. The finite element procedure allowed the establishment of precisely the same requirement without recourse to the intermediate analogue 'model'. The new standardized simple procedure of assuming suitable displacement functions was found by the Authors to be both physically and mathematically more meaningful, and avoided some of the pitfalls so often encountered with the grid type representations at such points as re-entrant corners, etc.

96. The remarks of Professor Mathew bore out some of the difficulties encountered in such analogues. Professor Mathew mentioned an additional difficulty encountered in the formation of the combined matrix. In the finite element analysis the whole procedure from the moment of insertion of the nodal co-ordinates, plate characteristics, and loads as data was completely automatic. The results could in fact be obtained, once the programme was written, by relatively unskilled personnel.

97. *Alternative element shape and characteristics.* Mr West mentioned the possibility of using an improved specialized function for skew type elements. While the

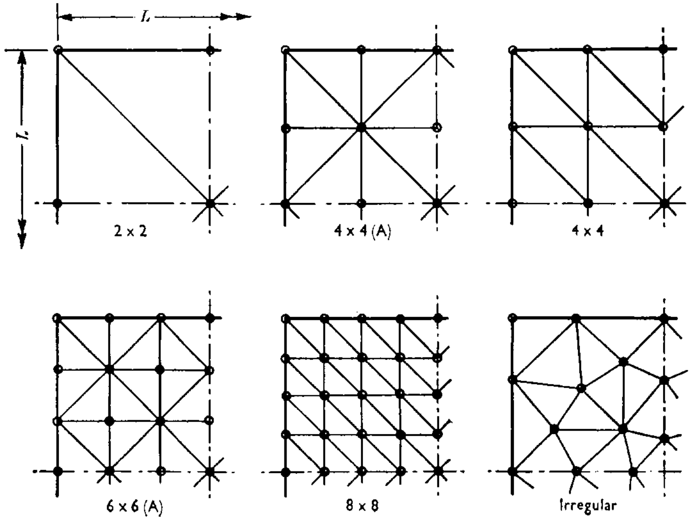

FIG. 11: SQUARE PLATE: ELEMENT DIVISIONS

simpler functions presented in the Paper might not be as accurate for non-rectangular elements the Authors felt that their efforts should be concentrated on finding the characteristics of an arbitrary triangular-shaped element. If a solution of this type could be obtained then a general programme suitable for skew bridges as well as for any arbitrarily shaped plates would be available, making the production of specialized programmes of limited applicability unnecessary. The recent work of the Authors indicated that such a point had in fact been reached and that a general programme based on triangular elements could now be considered as available for professional needs. However, if the procedure of deriving special stiffness matrices for skew element as suggested by Mr West were to be followed, then perhaps it would be most suitable to use inclined co-ordinates and the same form of the displacement function as specified in the Paper, which had the advantage of ensuring no discontinuity of displacement. This would, it was believed, result in expressions slightly different from those of Mr West.

98. Mr Irons had referred in his contribution to certain shape functions which possessed 'symmetry' with respect to the so-called area co-ordinates, for triangular shapes. These functions, developed jointly with the Authors were the basis of the new stiffnesses derived for such elements.

99. The functions given in equation (36) satisfied both the continuity and the 'constant stress' criteria but presented considerable difficulties in integration necessary to derive the element characteristics. In fact numerical integration was imperative in their use. To avoid these difficulties the element characteristics were derived using only the first part of the displacement functions of equation (36), i.e. functions of type

$$\phi_{12} = L_1L_2^2 + \tfrac{1}{2}L_1L_2L_3.$$

Such functions, being cubic along the element sides, satisfied there the continuity condition of deflexion but violated the normal slope compatability between adjoint elements. They did, however, still respect the constant strain criterion and therefore converged.

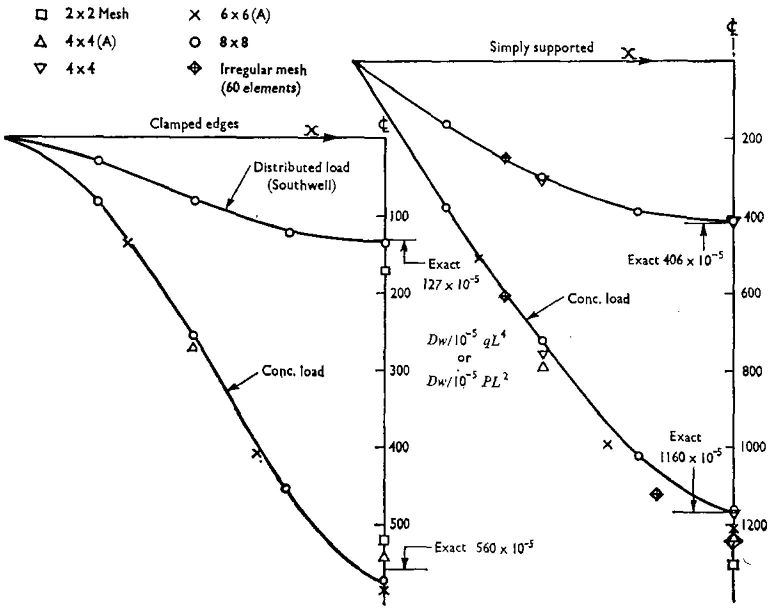

FIG. 12: SQUARE PLATE: DEFLEXIONS ON CENTRE-LINE

100. Figures 11 and 12 showed the results obtained by this general programme for square plates under various load conditions and with various triangular shape element divisions. The accuracy obtained was of a similar order to that given in the examples cited in the Paper. Now however *any* complex shape or non-homogeneity was tractable. In Fig. 13 some results for a problem of a circularly perforated slab were compared with an experimental (Moiré) solution. The slopes only were plotted in this figure as their contours were in fact the Moiré fringes.

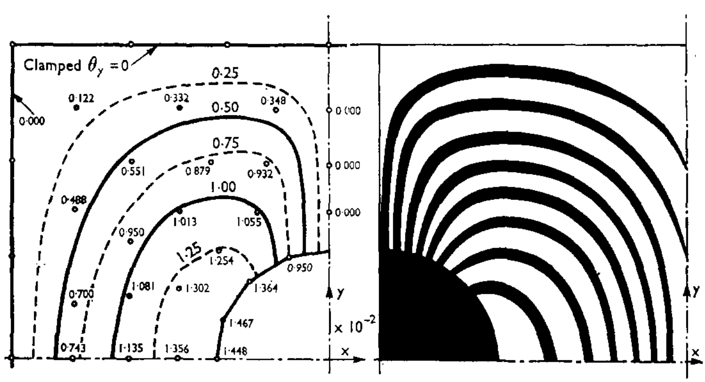

FIG. 13: SQUARE PLATE WITH HOLE: SLOPE CONTOURS $\theta y = +\partial w/\partial x$ AND SLOPE CONTOURS BY MOIRÉ EXPERIMENT. ONE FRINGE $= 0{\cdot}213 \times 10^{-2}$

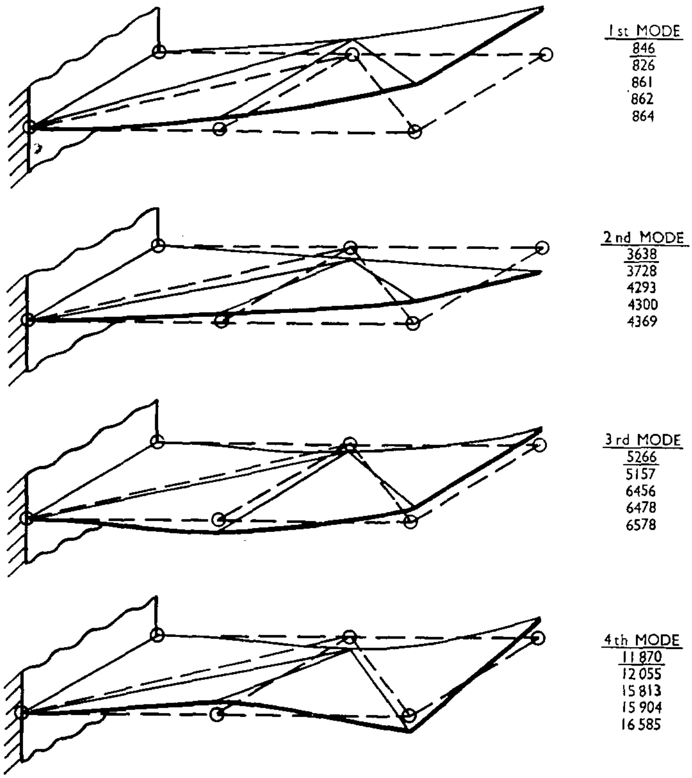

FIG. 14: VIBRATION OF A CANTILEVER PLATE DIVIDED INTO FOUR ELEMENTS: MODAL SHAPES. Values of frequencies are given in the following order: (1) exact; (2) non-conforming $\epsilon = 0$; (3) $\epsilon = \epsilon^a$; (4) $\epsilon = \epsilon^b$; (5) $\epsilon = \epsilon^c$ (3–5 conforming). Data: Steel $E = 30 \times 10^6$ p.s.i.; thickness of plate $= 0{\cdot}1$ in.; $\nu = 0{\cdot}3$

101. The accuracy attained was sufficient for all engineering purposes and clearly the solution was convergent. Problems of the type discussed by Mr West no longer presented any difficulties and skew slabs with or without supporting beams or columns could be simply analysed.[32]

102. The possible accuracy attainable with vibration problems was illustrated in Fig. 14. Here four triangular elements were used to obtain a solution for a vibrating cantilever plate. Various forms of the 'continuity-restoring' functions had been used here but the best results were once again obtained with the simplest non-continuous solution. Four frequencies were determined to an accuracy of $\pm 3\%$ even with this very crude model. The possibilities in studying vibration problems ranging from aircraft wings and turbine blocks to those of bridge decks, etc, were limitless.

103. Although the progress achieved since the appearance of the Paper had been considerable, the 'last word' in development of the finite element methods for even the limited plate problem had by no means been spoken. Certainly other displacement functions would be tried—possibly with even better results. So called 'equilibrium' formulations gave an alternative approach of promise[35] and would certainly

be further developed. It was hoped that the Paper served its purpose in stimulating interest in a new method of general applicability eminently suitable for present day computing machines.

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