*Discussion on Paper No. 6853**

Collapse load of structures allowing for strain hardening

by

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Dr G. Augusti (Istituto di Scienza delle Costruzioni, University of Naples) wished to present some comments on the Paper.

30. He was afraid that there was a mistake on the left-hand side of the equations in §§ 15 and 25.

31. Take for instance the first equation. If, as it appeared from Fig. 5, $W = P/2$ was of type

$$W = W_0 + K\delta$$

the work on the left-hand side of the equation was *not* $W\delta$ but

$$\int_0^\delta (W_0 + K\delta)d\delta = W_0\delta + \tfrac{1}{2}K\delta^2$$

With this expression for the external work, the equation in § 16 became

$$W_0 + \tfrac{1}{2}K\delta = \frac{28}{3}\frac{M_0}{L} + \frac{272\alpha}{9L^2}\delta$$

whence

$$W_0 = \frac{28}{3}\frac{M_0}{L}$$

$$K = \frac{544\alpha}{9L^2}$$

i.e., the slope of the $W-\delta$ line was twice that derived in the Paper.

32. It was easy to see that, if the slope of the line 'including strain hardening' in Fig. 5 was doubled, a line was obtained parallel to the final line of the 'elasto-plastic analysis including strain hardening'. This was obvious as the horizontal shift between the two lines was due to the elasto-plastic deformations prior to the development of the $(n+1)$th hinge, which were neglected in the rigid-strain hardening hypothesis (Fig. 1).

33. Incidentally, the term 'virtual work equation' was incorrectly used in the Paper. In fact, the sets of displacements considered were small but not infinitesimal (i.e. not 'virtual') nor were the forces and stresses kept constant while they took place.

34. It followed that equations in the Paper did not represent the 'virtual' work principle; they indicated rather (apart from the correction on the L.H.S. discussed above) the equality between the 'actual' work of the external loads and of the internal stresses during the kinematic deformation, under the rigid-strain hardening assumption. The two approaches coincided only if loads and stresses were constant during the deformation, as they were according to the simple plastic theory.

35. Moreover, while the writer appreciated the Author's efforts to outline a simple procedure to take account of strain hardening in plastic collapse without any claim to accuracy, he remained doubtful of the significance of the approach suggested, because 'collapse' was identified with the attainment of 'a multiple of the elastic deflexion' (§ 5) and at the same time the elasto-plastic deformations prior to the

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development of the complete mechanism were neglected, though they might by themselves give a 'multiple of the elastic deflexion'.

36. In addition, the relative rotations θ in the plastic hinges in the rigid-strain hardening hypothesis might be quite different from the actual ones, which included the effects of the elasto-plastic deformations, and the plastic moments $M = M_0 + \alpha\theta$ might be substantially affected by this difference.

37. In these respects, the procedure used by Mr Sawko in his earlier papers [1,2] was much more reliable. Although this earlier procedure involved some step-by-step calculations, the simplification set forth in this last Paper did not seem, on the whole, to be an improvement.

38. Finally, the writer would like to know whether the Author had considered how his procedures could be extended to derive 'statically admissible' stress fields, in order to bound the collapse load from below when the collapse mechanism was not easily recognizable.

Dr M. Gregory (Civil Engineering Department, University of Tasmania) wrote that the present Paper represented a valuable addition to the Author's previous paper on the plastic behaviour of beams and grillages, but attention should be drawn, somewhat reluctantly, to the following point.

40. In his Paper, the Author confused the principle of virtual work, for a virtual displacement of a structure from its equilibrium position, with the principle of conservation of energy, where the work was computed over the whole of the displacement path of the structure. The difference might not affect his numerical examples to an appreciable degree, but the error was of a rather fundamental nature.

41. Consider a simple cantilever of 'rigid-strain hardening' material whose strength was given by $M = M_0 + \alpha\theta$. If the cantilever was of length l and carried an end load W, then the bending moment at the support was $M = Wl$, the rotation there was $\theta = (Wl - M_0)/\alpha$, and the deflexion under the load was $\delta = l\theta = l(Wl - M_0)/\alpha$, from first principles. The fact that the plastic hinge could no longer be localized at a point was neglected, as it was in the Author's examples. However, using conservation of energy, which in the Paper was unfortunately called virtual work, we had $(M_0 + \frac{1}{2}\alpha\theta)\theta = W\delta = Wl\theta$; hence $\delta = l\theta = l(Wl - M_0)/\frac{1}{2}\alpha$.

42. The error, which was not numerically as large as might at first appear, arose because conservation of energy applied over the whole displacement path gave some kind of average of W over that whole path from zero deflexion up to the actual deflexion; virtual work (equivalent to stationary potential energy in this case) should be applied only to a geometric perturbation of the equilibrium position; the two principles were equivalent if the loading remained constant as it acted through its displacement, but this was true only for the 'rigid-plastic' case. Fortunately, however, the larger value was generally in keeping with the facts (though this really depended on the value of α chosen), since deflexions were increased because the hinge could no longer be localized. (However, a careful treatment of the latter would require the specification of a moment–curvature relation not a moment–rotation relation.)

Professor M. R. Horne and Mr M. W. Chin (University of Manchester) wrote that the Author had presented a method for taking account of the beneficial effect of strain-hardening that was simple to apply.

44. The method had many similarities to a method previously presented by Professor Horne[6,7] in relation to frame stability effects, and now described in greater detail in a paper by Horne and Medland.[8] Both the methods used a relationship of the form $M = M_p(1 + A\phi)$ between the bending moment M at the plastic hinge and the rotation ϕ at that hinge. Whereas Professor Horne's method introduced A as a variable dependent in a simple way on the shape of the bending moment diagram, the Author's treatment made A constant. This latter approach ignored the well-established dependence of the strain-hardening characteristics of simply supported

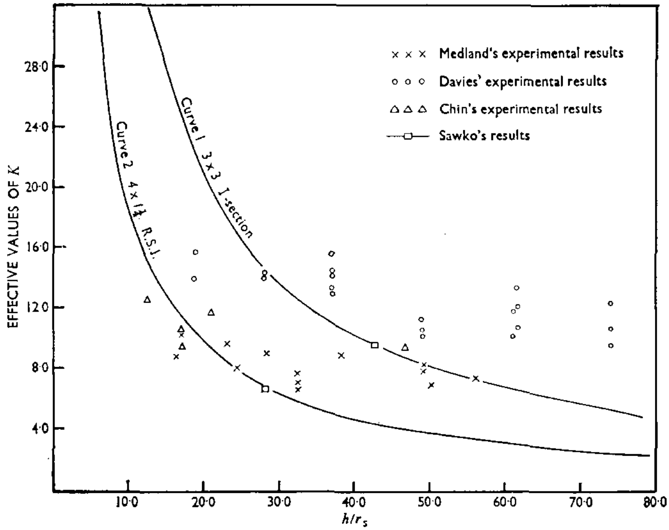

FIG. 7: STRAIN HARDENING FACTOR FROM VARIOUS TESTS

beams on the shape of the bending moment diagram.[9, 10, 11] If the Author's method were applied to the load-deflexion characteristic of a beam with a central region of uniform moment, it would be found to give results inconsistent with experiment and 'unsafe' in that the load predicted for a given deflexion would exceed the experimental load. An inspection of the available literature would show this to be the case.

45. The theory propounded by Professor Horne was derived from the 'rigid-plastic-rigid' approximation to material behaviour. The additional virtual work term due to an incremental hinge rotation $d\phi$ in a plastic work-hardening hinge with a rotation ϕ was expressed as

$$M_p\left(\frac{r_s}{h}\right)\left(\frac{E}{k\sigma_y}\right)\phi d\phi$$

where

M_p = full plastic moment,

$r_s = \frac{I}{S}$ = distance of strain centre from neutral axis,

I = second moment of area of section,

S = plastic modulus of section,

E = elastic modulus,

σ_y = yield stress,

k = ratio of strain at beginning of strain hardening to elastic strain at yield,

h = equivalent cantilever length.

46. The equivalent cantilever length for a plastic hinge spreading in one direction from a section of maximum bending moment was M_p/F where F was the shear force in the member at the hinge. Where a hinge was free to spread in two directions, $h = h_1 + h_2$ where h_1 and h_2 were the equivalent cantilever lengths on either side of the hinge. *Since in the Author's treatment the additional plastic work due to strain-hardening* was $\alpha\phi d\phi$, it followed that

$$\alpha = M_p\left(\frac{r_s}{h}\right)\left(\frac{E}{k\,\sigma_y}\right) = \left(\frac{r_s}{kh}\right)ES$$

and that α ought to vary with the characteristics of the bending moment diagram introduced by the quantity h. To the extent that rigid-plastic-rigid theory was an approximation, satisfactory correlation between test results for beams and for frames demanded a slight variation in the value assumed for the strain-hardening factor k for any given batch of material. The values of k as a function of h/r_s for a variety of test results obtained by Medland,[12] Davies,[13] and Chin were shown by the plotted points in Fig. 7. Most of the scatter was due to variation of material properties, the upper and lower boundary curves being of small slope.

47. It was instructive to make a comparison between the above results and the Author's proposals by considering the constant α value suggested by the latter as defining the k value for various h/r_s. From the Author's test on a simply supported 3 in. × 3 in. at 8·5 lb/ft beam (Fig. 2), $a=96$, $E=13\,000$ ton/sq. in. (assumed), and $S=2{\cdot}98$ in.^{3}, so that from the last equation,

$$\frac{kh}{r_s} = \frac{ES}{\alpha} = 403$$

The resulting values of k for varying h/r_s were given by curve 1 in Fig. 7. Similarly, for a 4 in. × $1\frac{3}{4}$ in. at 5 lb/ft beam (Fig. 5 of reference 1), the Author had a variation of k with h/r_s given by curve 2 in Fig. 7. It was evident that the sort of variation implied in the Author's treatment was in disagreement with the results obtained from other quite extensive investigations.

48. The Author derived the equilibrium condition for his frames by applying the virtual work equation, so that the external work due to an incremental hinge rotation $d\phi$ was $(M_0+\alpha\phi)d\phi$. He integrated this to give $M_0\theta+\frac{1}{2}\alpha\theta^2$ (§ 12), and then equated the total internal work $\sum(M_0\theta+\frac{1}{2}\alpha\theta^2)$ to $\sum W\delta$ where the W were the *final* values of the loads. This was incorrect since the values of W did not remain constant during plastic deformation. The correct use of the virtual work equation was to limit attention to infinitely small increments of internal and external work, so that one should have

$$\sum W d\delta = \sum(M_0+\alpha\theta)d\theta.$$

Since $\frac{d\delta}{\delta} = \frac{d\theta}{\theta}$, it followed that

$$\sum W\delta = \sum(M_0\theta+\alpha\theta^2).$$

The use by the Author of the term $\alpha/2$ in place of α on the R.H.S. was therefore incorrect.

Mr P. H. Gray (Messrs G. Maunsell and Partners) considered that the Author's approach to the inclusion of the effect of strain hardening on the rigid-plastic failure load of a frame structure was refreshing in its simplicity. However, the plastic theory in its simple form was limited in application to those structures in which instability, either of individual members or of the whole structure, was not possible. The Author's extended theory must be subjected to similar restrictions. The effects of elastic changes of geometry and of instability were excluded.

50. In a recent research project undertaken at Cambridge University,[14] the behaviour of two series of multi-storey plane frames had been investigated, both experimentally and theoretically. It was intended to publish this work in the near future and it would therefore not be described in detail.

51. Each series consisted of frames of four and six storeys in height. One series, approximately one quarter full size, was constructed of 3-in. × $1\frac{1}{2}$-in. rolled steel joists. The other series, approximately one twentieth full size, was constructed of $\frac{3}{8}$-in. dia. × 20 gauge aluminium alloy tubes. Two alloys were used, H 15 W and H 30 WP, the former showing a higher degree of strain hardening than the latter. Each frame was loaded proportionally to failure. Vertical loads were applied at beam quarter points and horizontal loads at beam/stanchion joints.

52. The theoretical behaviour was computed on EDSAC II, the University digital computer, using a programme written originally by Livesley[15] for elastic-plastic frame analysis. Strain hardening at a hinge was introduced by relating the plastic moment to the hinge rotation, in a manner similar to that used by the Author. The reduction of plastic moment caused by axial load was also included. Table 1 gave failure load factors for the frames, both experimental and theoretical. The latter were calculated separately with material assumed to be rigid-plastic, elastic-plastic, and elastic-strain hardening.

53. Although agreement was not particularly good between the experimental failure load factors and those computed on the basis of elastic-strain hardening material, in no case did the inclusion of strain hardening in the calculations raise the elastic-plastic failure load factors even close to the rigid-plastic values. Indeed, for many of the aluminium frames, instability completely masked strain hardening.

54. These results were for frames with not particularly slender geometry, although the behaviour of the lower storeys was typical of that expected in multi-storey frames. The difference between experimental and theoretical failure load factors was considered to be attributable in part to the limitations of the test rig and computer programme, and in part to the underestimation of strain hardening in the theoretical investigations.

55. The dependence on frame geometry and loading of both the strengthening due to strain hardening and the weakening due to instability were clearly demonstrated.

56. It was hoped to extend the investigation into the effect of strain hardening on the behaviour of the steel frames, using an alternative method of analysis. Eventually, it was conceivable that general rules would be developed by which to calculate the effects on frame behaviour of both strain hardening and instability. Their use and value would depend, however, on better uniformity and predictability in material properties.

TABLE 1

Frame data			Failure load factors			
Material	Number of storeys	Ratio of horizontal to vertical load per storey	Experimental	Theoretical		
				Rigid-plastic	Elastic-plastic	Elastic-strain hardening
Steel	4	0	52·50	50·12	44·63	44·63
,,	4	1:48	48·00	48·88	44·12	45·88
,,	4	1:12	34·00	37·88	31·62	32·88
,,	6	1:24	42·50	44·12	36·38	37·12
,,	6	1:12	32·50	34·62	29·12	29·38
Alloy	4	1:48	4·88	7·27	3·92	3·92
H 15 W	4	1:12	3·47	5·55	2·95	2·95
,,	6	1:48	3·31	6·70	2·89	2·89
,,	6	1:12	2·24	4·17	1·89	1·92
Alloy	4	1:48	4·31	5·83	3·33	3·33
H 30 WP	4	1:12	2·88	4·52	2·55	2·64
,,	6	1:48	2·81	5·33	2·53	2·53
,,	6	1:12	1·83	3·33	1·64	1·67

The Author wished to thank all the writers for their interesting contributions. He was particularly indebted to Dr Augusti and Professor Horne and to Mr M. W. Chin for pointing out the mistake in equations in §§ 15 and 25. Dr Augusti was quite correct in stating that the slope of the line 'including strain hardening' in Fig. 5 was doubled.

58. While writing the Paper, the Author was faced with the difficulty of finding a suitable name for the equations used to derive collapse load with strain hardening effect, and the term 'virtual work' was used because of the similarity between the present equations and the conventional procedure where moment at a plastic hinge remains constant. He agreed with Dr M. Gregory that 'the principle of conservation of energy' might have been a more accurate expression to use. Unfortunately, it would have masked the similarity between the Author's extensions and 'virtual work' equation.

59. The problem of the constant value for the strain hardening factor α was dealt with in the discussion to the Author's original paper[1] and the Author would only like to emphasize one or two important points. In the discussion, he had suggested that conservative values of α could be selected from beam tests. Testing of full size sections had been going on during the past eighteen months at Leeds University, and was not yet completed. Tests had been performed on a whole range of sections up to 29 in. deep, and in some cases with different spans for the same section size. Two items were of interest: (*i*) the theoretical formula for strain hardening suggested by Professor M. R. Horne and Mr M. W. Chin did not seem to agree exactly with test results, presumably because of the effect of shearing forces and strain concentrations at plastic hinges; (*ii*) it had been extremely difficult to test beams with a small 'cantilever length' because of the web buckling.

60. These points seemed to reinforce the validity of some approximate treatment, and the Author believed that the one suggested in the Paper was reasonable. He was in full agreement with Mr P. H. Gray that instability was a very important effect and one which would, at least in the case of multi-storey frames, govern the collapse mode. Strain hardening should thus be examined not only in a purely theoretical manner, but also in the light of practical aspects of design.

61. The Author was also grateful to Mr Gray for emphasizing the simplicity of the proposed approach. The full elasto-plastic analysis of structures with strain hardening, instability and large deflexion effects all being included, was at present being studied at Leeds, but it was quite obvious that even small frames required a large amount of computer time on fast machines because of the multiple stage solution required for convergence at the formation of each plastic hinge. Thus plastic theory, which was originally put forward as a simple competitor to elastic design, was now becoming too sophisticated and long even for the fastest computer available. The Author's approach represented the move in the other direction and a return to simplicity. The proposed method could be used in any design office. Clearly all the problems had not yet been solved, but the improvement in the agreement between experimental and theoretical values even with the simple proposed approach (*i*) could not be disputed.

62. The Author believed that the suggested equations would give some idea of the *measure* of the increase in strength due to strain hardening of structures where instability was not a critical factor. As Dr Augusti had pointed out, the slope of the load-deflexion line would be parallel to the one obtained in the full theoretical treatment for which the complete elasto-plastic analysis had to be performed. This 'measure' of strain hardening could be used to give some idea of whether the increase in strength was important, and whether any benefit would be derived by performing the complete step by step analysis.

63. Dr Augusti had also referred to the difficulty of deriving the statically admissible stress fields to determine the lower bound of collapse loads for plastic analysis. In this problem, again, electronic computers offered promising results.

64. An approach similar to that used by Livesley[16] for minimum weight design using linear programming, offered considerable scope and was being investigated. Here again, considerable advantage was gained by assuming that plasticity was *confined* to plastic hinges at ends of members, and by neglecting the spread of plastic zones. A cyclic solution successively improved estimates of collapse loads, and thus converged to the exact solution or at least approached it. A very high speed computer with a large 'memory' was essential, but even then multi-bay multi-storey frames with large numbers of joints would present difficulties, and some simplified approach must be sought.

65. The Author would once again like to acknowledge the financial aid of the SRC which was enabling the present investigation to continue.

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