Formulae for continuous beams

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Mr Mawby has presented formulae for continuous beams based on the moment distribution method. His observation of the convergence of the geometric series obtained in a simple distribution is interesting. Further comment on the convergence process is perhaps in order. The equation $a(1+\lambda+\lambda^2+\ldots+\lambda^n)=a/(1-\lambda)$ as $n\to\infty$ is valid when $0\leq\lambda<1$. The λ value is a function of carry over and distribution factors which are, in turn, functions of the beam stiffnesses. When $\lambda=0$, the problem reduces to a series of simple beams connected to each other by hinges at the supports. If $\lambda\geq 1$, the series will diverge and the structure is unstable. In other words, a divergent series would mean the creation of energy by the system—an obvious impossibility. This criterion has been used to investigate the stability of structures subjected to both axial and transverse loads.[1]

If each of several moment distribution problems is represented in series form, the combination of these structures into one continuous structure can be represented as a more complex geometric series. By this approach, explicit expressions for moment influence lines for continuous beams with either a variable or a constant moment of inertia for any span have been compiled,[2] as well as moment expressions for prismatic beams with various arrangements of uniform loading. When the number of spans exceeds four or when frames are considered, explicit expressions for final moments or moment influence lines become impractical. Tacit evidence of this impracticality for larger structures is also present in the latter part of Mr Mawby's Note.

If matrix notation is used, the problem can be greatly simplified. It has been shown that the moment distribution procedure can be written as a matrix geometric progression.[3] The sum of an infinite number of terms of the series gives an explicit finite matrix expression. Generalized forces may also be treated without the need for calculating displacements.

The distribution procedure can therefore be viewed as the solution of a matrix geometric series or, in the limit, as the inversion of a matrix composed of carry over and distribution factors.

Mr Mawby

I wish to thank **Mr Mozingo** for his contribution and I am particularly pleased to read of the application of the sum to infinity of a geometrically progressing convergent series as the basis for further forms of analysis.

From the comments on the range of validity of λ it can be construed that when $\lambda=0$ the structure is statically determinate, and that when $\lambda\geq 1$ the product of two adjacent distribution factors must be ≥ 4, which is impossible if one remembers that distribution factors always lie between 1 and 0.

It must be pointed out that to solve the incompatibilities of the joints using matrix algebra or to produce influence lines[4] generally requires a computer, which was outside the scope of the Note. It was intended to show that the solution to multi-span problems can be successfully evolved by pencil and paper more readily than by other methods, even conventional moment distribution methods. Indeed, as often

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encountered, when all span stiffnesses are equal, solving the formulae is extremely simple. The reduced formulae for all k values equal are:

Three-span continuous beam

$$M_A = 4/15(X-\tfrac{1}{4}Y)-F_{AB} \quad (12)$$
$$M_B = 8/15(X-\tfrac{1}{4}Y)+F_{BA} \quad (13)$$
$$M_C = 8/15(Y-\tfrac{1}{4}X)-F_{CD} \quad (14)$$
$$M_D = 4/15(Y-\tfrac{1}{4}X)+F_{DC} \quad (15)$$

Two-span continuous beam

$$M_A = \tfrac{1}{4}X-F_{AB} \quad (16)$$
$$M_B = \tfrac{1}{2}X+F_{BA} \quad (17)$$
$$M_C = \tfrac{1}{4}X+F_{CB} \quad (18)$$

It is suggested that the degree of accuracy (obtained by hand) is higher than that obtained by the reiterative process and is sufficiently accurate for design considerations within the bounds of the elastic theory.

Profound agreement is felt for the statement that explicit expressions are impractical for more than four spans, and this is the reason for the table of sub-formulae (Table 2). This table has broken down what would otherwise be very unwieldy expressions for each joint. If these are solved numerically and added algebraically to the results of the general formulae, then exact answers will be obtained. It is felt that the derivation of these sub-formulae should be enlarged upon as follows.

To arrive at the sub-formulae one must consider the correction moment, M' at the joint in question to be re-distributed. Bearing in mind the worked example in the Note, when joint D is allowed to rotate, re-distribution takes place across spans A–D and D–F. Investigating the three-span portion (Fig. 16), if $-\tfrac{1}{2}dM'$, $-\tfrac{1}{2}dM'$ and dM' are substituted for Y, F_{CD} and F_{DC} respectively in the general formulae (joint D remaining locked) then:

$$M'_A = M'.\tfrac{1}{4}dap = M'c'_A \quad (19)$$
$$M'_B = M'.\tfrac{1}{2}dap = M'c'_B \quad (20)$$
$$M'_C = M'.\tfrac{1}{2}d(1-b) = M'c'_C \quad (21)$$
$$M'_D = M'.d(1-\tfrac{1}{4}b) = M'c'_D \text{ (left)} \quad (22)$$

c'_D (left) $= c'_1$ generally, where c'_1 is the correction factor to the left of the unbalanced joint. Similarly, the re-distribution across spans D–F is as shown in Fig. 17. In this case, $-(1-d)M'$, $\tfrac{1}{2}(1-d)M'$ and $\tfrac{1}{2}(1-d)M'$ are substituted for the values of F_{AB}, F_{BA} and X respectively in the two-span general formulae, resulting in:

$$M'_D = M'.(1-d)(1-\tfrac{1}{2}r) = M'c'_D \text{ (right)} = M'c'_r \text{ generally} \quad (23)$$
$$M'_E = M'.\tfrac{1}{2}(1-d)(1-2r) = M'c'_E \quad (24)$$
$$M'_F = M'.-\tfrac{1}{2}(1-d)q = M'c'_F \quad (25)$$

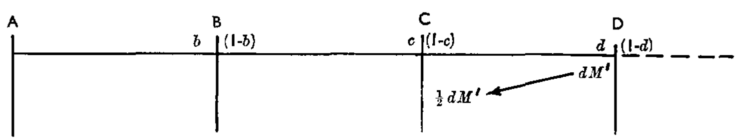

Fig. 16

Table 1. Conventional distribution of out of balance moment at joint D

	A	B		C		D		E		F	
		$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$		
						+50·00	+50·00				
				−18·75	+25·00 −6·25			+25·00 −8·33	−16·67		
		+3·75	−9·38 +5·63			−3·13 +3·65	−4·17 +3·65				
	+1·88			+2·82 −3·49	+1·83 −1·16			+1·83 −0·61	−1·22		
		+0·70	−1·75 +1·05			−0·58 +0·44	−0·30 +0·44				
	+0·35			+0·53 −0·56	+0·22 −0·19			+0·22 −0·07	−0·15		
		+0·11	−0·28 +0·17			−0·10 +0·07	−0·03 +0·06				
	+0·06			+0·08 −0·09	+0·04 −0·03			+0·03 −0·01	−0·02		
		+0·02	−0·05 +0·03			−0·02 +0·01	+0·01				
$\sum M$	+2·29	+4·58	−4·58	−19·46	+19·46	+50·34	+49·66	+18·06	−18·06	0	**A**
$\sum M \times 0{\cdot}9163$	+2·10*	+4·20*	−4·20	−17·83	+17·83*	+46·13*	+45·50*	+16·55*	−16·55	0	**B**
$\sum M$(D locked)	−82·55	+84·90	−84·90	+101·58	−101·58	+4·21	−95·84	+58·33	−58·33	0	**C**
Final moments (B+C)	−80·45	+89·10	−89·10	+83·75	−83·75	+50·34	−50·34	+74·88	−74·88	0	

*** Indicates values that may be obtained from Table 2 (Condition: Case 1 + Case 3).**

Table 2 (corrected version). Correction moments for four-, five- and six-span continuous beams ($m_x = \tau c'_x$) [*see Fig. 11*]

A | BC | D | EF | G

Condition	τ	c'_A	c'_B	c'_C	c'_D (left)	c'_D (right)	c'_E	c'_F	c'_G
Case 1+ Case 2 (six spans)	$\frac{M'}{c'_D\,(\text{left})+c'_D\,(\text{right})}$	$\frac{1}{4}d_1a_1p_1$ $(=\frac{1}{2}c'_B)$	$\frac{1}{2}d_1a_1p_1$ $(=2c'_A)$	$\frac{1}{2}d_1(1-b_1)$	$d_1(1-\frac{1}{4}b_1)$	$(1-d_1)(1-\frac{1}{4}a_2)$	$\frac{1}{2}(1-d_1)(1-a_2)$	$\frac{1}{2}(1-d_1)b_2q_2$ $(=2c'_G)$	$\frac{1}{4}(1-d_1)b_2q_2$ $(=\frac{1}{2}c'_F)$
Case 1+ Case 3 (five spans)	$\frac{M'}{c'_D\,(\text{left})+c'_D\,(\text{right})}$	$\frac{1}{4}d_1a_1p_1$ $(=\frac{1}{2}c'_B)$	$\frac{1}{2}d_1a_1p_1$ $(=2c'_A)$	$\frac{1}{2}d_1(1-b_1)$	$d_1(1-\frac{1}{4}b_1)$	$(1-d_1)(1-\frac{1}{2}r_3)$	$\frac{1}{2}(1-d_1)(1-2r_3)$	$-\frac{1}{2}(1-d_1)q_3$	

A | B | C | DE | F

Condition	τ	c'_A	c'_B	c'_C (left)	c'_C (right)	c'_D	c'_E	c'_F
Case 4+ Case 2 (five spans)	$\frac{M'}{c'_C\,(\text{left})+c'_C\,(\text{right})}$	$-\frac{1}{2}c_4r_4$ $(=\frac{1}{2}c'_B)$	$-c_4r_4$ $(=2c'_A)$	$c_4(1-\frac{1}{2}q_4)$	$(1-c_4)(1-\frac{1}{4}a_2)$	$\frac{1}{2}(1-c_4)(1-a_2)$	$\frac{1}{2}(1-c_4)b_2q_2$ $(=2c'_F)$	$\frac{1}{4}(1-c_4)b_2q_2$ $(=\frac{1}{2}c'_E)$
Case 4+ Case 3 (four spans)	$\frac{M'}{c'_C\,(\text{left})+c'_C\,(\text{right})}$	$-\frac{1}{2}c_4r_4$ $(=\frac{1}{2}c'_B)$	$-c_4r_4$ $(=2c'_A)$	$c_4(1-\frac{1}{2}q_4)$	$(1-c_4)(1-\frac{1}{2}r_3)$	$\frac{1}{2}(1-c_4)(1-2r_3)$	$-\frac{1}{2}(1-c_4)q_3$	

Constants:

$$d_1 = \frac{k_3}{k_3+k_4}$$

$$c_4 = \frac{k_2}{k_2+k_3}$$

a, b, p, q and r are from the general formulae (see summary of equations). The suffixes refer to the Case Conditions only.

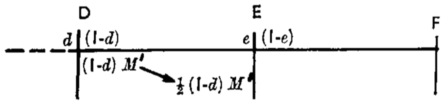

Fig. 17

For equilibrium at joint D, apparently, if the moment to the left of the joint when locked $= M_{Ll}$ and to the right $= -M_{Lr}$, then possibly

$$M_{Ll} + M'c'_1 = -M_{Lr} + M'c'_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

But $M' = M_{Lr} - M_{Ll}$, which when substituted in equation (26) results in $M_{Ll} = -M_{Lr}$. It is known that this is not the case, so obviously a second correction moment, say M'' is involved, and if this is re-distributed, a third becomes apparent, and so on. It follows that the sum to infinity of re-distributed correction moments to the right of the joint is given by

$$M'c'_r \frac{1}{1-\lambda'}$$

and to the left of the joint by

$$M'c'_l \frac{1}{1-\lambda'} \quad \text{where} \quad \lambda' = [1 - c'_r - c'_l]$$

Hence for equilibrium, at the out of balance joint:

$$M_{Ll} + \left(\frac{M'}{c'_l + c'_r}\right) c'_l = -M_{Lr} + \left(\frac{M'}{c'_l + c'_r}\right) c'_r \quad . \quad . \quad . \quad . \quad (27)$$

The function $[M'/(c'_l + c'_r)]$ is shown as τ in Table 2. As a general expression for any joint, X, the final moment, $M_X = M_{LX} + m_X$ where M_{LX} is the value obtained from the general formulae and $m_X = c'_X[M'/(c'_l + c'_r)] =$ the correction moment required for equilibrium obtained from Table 2. Table 2 is presented again, and the Author wishes to correct some errors that have come to light since the previous publication. It is also felt that τ is more simply represented as the product of the case correction moment and the reciprocal sum of the correction factors either side of the out-of-balance joint. Table 1 shows how Table 2 may be applied.

References

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4. Lawrence C. A method of deriving influence lines for a symmetrical arrangement of spans of uniform section. *Struct. Engr*, 1966, **44** (9), 313–318.