A study of unsteady flow in branched channels

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Mr J. W. Kamphuis, Queen's University, Kingston, Canada

The Author describes the solution of a junction problem in unsteady flow in a clear and logical manner but neglects to tell his readers that this system is only useful if one knows the boundary conditions at the other ends of one's three or four branches. In other words, using this method one can only treat situations which contain just a single junction and these are indeed rare cases in nature. The more usual situation occurs, for instance, on the St Lawrence River,[5] where a number of tributaries join the main river and where midstream islands are present.

Consider only one midstream island as in Fig. 4.

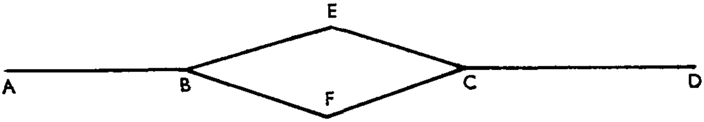

Fig. 4. Situation for one midstream island

The situation at B and C (individually) could be described by means of Fig. 2 and equations (9) through (16). However, the solutions at B and C are dependent upon each other. If the branches BEC and BFC are reasonably long and complicated so that they cannot be represented by one distance step, then a time consuming iteration must be performed between the solutions at B and C. This is still a rather simple problem, but it could easily be complicated by tributaries entering, for instance, at E and F, in which case computer time required would be very great. This would lead one to believe that the method of characteristics is very difficult to apply to a branching system, involving very large computer storage and time requirements.

Dr M. B. Abbott, International Courses in Hydraulic and Sanitary Engineering, Delft

The Note contributed a valuable formulation of the junction problem, but perhaps some words of caution should be appended. In effect, the formulation assumes that flows remain nearly horizontal and energy conserving at the junction. However, A. N. Jolliffe[6] has given the following counter example. Consider the steady state situation whereby flows in uniform rectangular channels 1 and 2 join to form flow in a similar channel 3. Then, under the given assumptions, the net input energy flux E_i is given by

$$E_i = \rho Q_1\left(\frac{u_1^2}{2}+gd_1\right)+\rho Q_2\left(\frac{u_2^2}{2}+gd_2\right)$$

and the output energy flux E_0 is given by

$$E_0 = \rho Q_3\left(\frac{u_3^2}{2}+gd_3\right)$$

so that

$$E_i-E_0 = \frac{\rho}{2}\{Q_1(u_1^2-u_3^2)+Q_2(u_2^2-u_3^2)\} \quad . \quad . \quad . \quad . \quad (21)$$

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In the event that the junction is flow wise symmetric, we shall have $Q_1=Q_2$ and $u_1=u_2$ so that the energy defect is zero if $u_1=u_2=u_3$, i.e. if there is no expansion or contraction of flow at the junction.

In practice, if there are such sudden expansions or contractions, flows that are not nearly horizontal will intervene, together with energy losses, so that the Riemann invariant relations cannot be extended to the point where $d_1=d_2=d_3=d$. A procedure for simulating this intervention computationally has been described by Jolliffe, while another has been used by J. Grubert (private communication) for computations of certain Indian rivers, using energy variables in the Riemann invariant relations.

In natural watercourses it is in fact observed that either $u_1 \approx u_2 \approx u_3$ (junctions of major streams) or $Q_2 \ll Q_3$ (junction of small stream with larger) so that the error involved by retaining nearly-horizontal flow assumptions may be quite small. Thus, computation of irrigation networks by Karunaratne (private communication) using implicit numerical techniques,[7] which relied upon the same assumptions, showed negligible gain or loss in computed energy. The Author's method was thus probably sound enough for most practical purposes.

Dr T. R. E. Chidley, University of Aston

The Author presents a very good description of the method of characteristics as applied to an open channel flow system comprising branching channels. I do not agree, however, that the fixed time step method of solution is worthwhile. Most of the advantages of using characteristics are lost if one has a fixed time step and in a highly complicated situation such as the River Ganges delta upon which I am currently working, it becomes almost impossible to apply without very much care in choice of timestep. For this River Ganges flow simulation I have elected to use a finite difference method, in common with all of the other major unsteady flow simulations to date.[8,9]

The system used is to reduce the flow network to a set of nodes where water is stored and a set of channels which connect the nodes. Mass transactions take place at the nodes while energy transactions in the channels (Fig. 5).

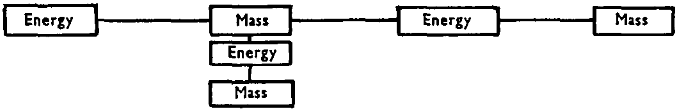

Fig. 5. Mass transactions take place at the nodes while energy transactions take place in the channels

One of the advantages of this formulation is that almost any type of rules for mass and energy transfer can be applied to the sub-systems, e.g. pollutant or saline water can be routed through the system as a series of mass transfers.

It should be made clear that equation (4) refers to a set of rectangular channels of constant width. For a channel of general shape equation (4) becomes

$$\left(g.J.A+\frac{dQ}{dt}\right)-\frac{dA}{dt}\cdot\left(\frac{2.Q}{A}-\frac{dx}{dt}\right)=0 \quad . \quad . \quad . \quad . \quad (22)$$

where J is a term which embraces friction, bed slope and channel shape parameters. When one applies the Author's method to branching channels one comes to the problem that while there is only one depth at a junction, there is more than one area.

However, this can be overcome, but the solution is more complicated. Mr Ellis's approximation may be justifiable if

$$\sqrt{g.\frac{A}{B}} = \sqrt{g.d}$$

and A_1, A_2 etc. are substituted for B_1, B_2 etc. in equation (16).

Another interesting complication in using this method arises when there is an element of overbank storage in one or more of the channels. For a system as in Fig. 6, where it is assumed that there are no energy losses attributable to the flood plain and that virtually all of the water enters the reach through the main channel, the equation of continuity becomes

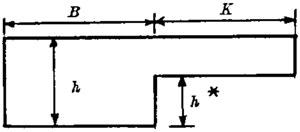

Fig. 6. Overbank storage in one or more of the channels

$$\frac{\partial Q}{\partial x}+(B+K)\cdot\frac{\partial h}{\partial t} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

The new characteristic equations are approximately equal to

$$\frac{dx}{dt} = u.\left(1+\frac{K}{2B}\right) \pm \sqrt{gd-\frac{K}{B}\left(gd+\frac{u^2k}{4B}\right)} \quad . \; . \; . \; . \quad (24)$$

$$d.\left(g.J+\frac{du}{dt}\right)-\left(1+\frac{K}{B}\right)\cdot\frac{d^2}{dt}\left(u-\frac{dx}{dt}\right) = 0 \quad . \; . \; . \; . \quad (25)$$

Thus, even for rectangular channels the equations become highly non-linear and the Author's simple equations become more difficult to handle. These are further complicated by the fact that K only exists when h exceeds the threshold of flooding h^*.

Mr Ellis

The Author wishes to thank those who contributed to the discussion. In presenting the Note describing the solution of a junction problem it was not intended to present a complete solution of an unsteady flow network, but merely to examine one element or module of a program for solution, i.e. the problem of a junction. In a complete program in which the physical network would be described by a set of nodes (the nodes being at boundaries, junctions or simply in the middle of a reach) procedures exist for the solution of each of these different problems. Thus, at the nodes which **Mr Kamphuis** describes as boundaries, the solutions are produced by the use of equations appropriate to the configuration of the problem at the node. The junction procedure may be used as often as necessary within any program and is therefore not restricted to one confluence. The Author has successfully used this routine in the simulation of flow around an island situated in a tidal channel in which there are two junctions as Mr Kamphuis rightly says.

The solutions at these junctions are independent of each other for each increment of time provided that the spacing between them, $DX \geqslant (u \pm (gd)^{1/2}).DT$, and no iteration is required. The problem of a tributary entering into one of the branches would be tackled in the same way provided that the spacing between the nodes obeys the above criterion and the storage requirements and computer time are in no way excessive.

The Author is indebted to **Dr Abbott** for his comments regarding the limitations of application of the junction equations derived in the Note. He confirms my impressions that for the majority of cases encountered the simple equations presented are adequate and it is for these that the method is intended.

I cannot accept **Dr Chidley's** statement that the advantages of using characteristics are lost with a fixed mesh technique. Indeed, where geometrical changes are large along the length of a channel as with fiordic type inlets, the fixed mesh technique offers a more accurate representation of the geometry of the channel. The only occasion when a naturally occurring mesh can be advantageous is when the possibility exists of bore formation.

Dr Chidley is wrong in his statement that equations of the form (4a) and (4b) in the Note are restricted to rectangular channels of constant width. Consider the continuity equation,

$$\frac{\partial Q}{\partial x}+B\frac{\partial h}{\partial t} = 0$$

Expanding this gives,

$$uB\frac{\partial d}{\partial x}+ud\frac{\partial B}{\partial x}+Bd\frac{\partial u}{\partial x}+B\frac{\partial h}{\partial t} = 0$$

If over a small increment of time it is assumed that the top width of the channel does not change then we have,

$$u\frac{\partial d}{\partial x}+d\frac{\partial u}{\partial x}+\frac{\partial d}{\partial t} = -\frac{ud}{B}\cdot\frac{dB}{dx}$$

On transforming the above equation and the corresponding dynamic equation into characteristic form, equations (3), (4a) and (4b) result. If, on completion of each integration process an interpolation into stored geometrical data of top width and mean depth is made, the variations of channels properties with depth may be accurately followed. Thus, equations (3), (4a) and (4b) are applicable to channels of arbitrary and variable cross section and there is no need to resort to the complicated form of equation (22) and modifications to the junction equations as proposed by Dr Chidley are unnecessary.

Dr Chidley also raises the matter of storage and proposes equation (23) which gives characteristic equations (24) and (25). With regard to equation (24) it is seen that the rate of propagation of a wave in a channel is directly proportional to the breadth of storage K. This is physically unreasonable, for one would no more expect the rate of propagation of a wave to be any more directly influenced by the breadth of storage K than by, say, the diameter of a pipe withdrawing water from the channel for a power station cooling system. Thus equations (24) and (25), while they may appear mathematically correct, are certainly of doubtful physical significance.

The Author has studied the problem of storage on sand flats associated with estuaries. The continuity equation may be written as,

$$\frac{\partial Q}{\partial x}+B\frac{\partial h}{\partial t}-Q^* = 0$$

where Q^* is the water entering storage/unit length of the channel. Whether the fluid being drawn off is for industrial usage or merely entering storage is not considered at this time. Corresponding modifications are made to the dynamic equation. After the equations are transformed into characteristic form the storage Q^* may be related to the geometrical configuration of the flood plain. The result is a simple integral added to the right-hand side of equations (4a) and (4b) with no alteration in the wave transmission properties of the channel given by equation (3), surely a much more realistic state of affairs. These integrals simply disappear when the storage breadth becomes zero. The Author has used, with a high degree of success, equations (3), (4a) and (4b) in a study of the Clyde Estuary,[10] in which the ship channel is accompanied by extensive sand flats over much of its length. The use of equation (24) would certainly account for the difficulties encountered by Dr Chidley in attempting to assign a suitable time increment to his model of the River Ganges.

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