Structural design for minimum cost using the method of geometric programming

A. B. TEMPLEMAN

Mr A. S. T. Chee, Bowaters United Kingdom Paper Company Limited

In example 2, optimum section modulus Z^* is first found by expanding equation (7a) for the second term of equation (23) giving $Z^*=764\,000$ mm^3. When an expansion of equation (7b) is performed for the second term of equation (24):

$$c_{12}x_1^{a_{121}}x_2^{a_{122}}x_3^{a_{123}} = \frac{w_{12}}{w_{10}}$$

i.e:

$$\left(\frac{4\cdot69\,WBA^3}{EK_2}\right)n^{*-1}Z^{*-4/3} = \frac{4}{5}$$

giving a value of $Z^*=750\,100$ mm^3.

43. Although these results are of the same order, there is still almost 2% discrepancy. I would like Dr Templeman to remark on this.

44. The rounding process suggested by the Paper seems to have given a final optimum design of $y_0^*=$ £3519 which is a good 3% lower than the initial optimum cost of £3626. This seems a bit illogical as I would assume that any value of y_0^* corresponding to the initial optimum values of the variables t, Z and n would be the absolute minimum attainable for the given set of constants. In other words, the combination of $t=4\cdot76$ mm, $n=92$ and $Z=842\,000$ mm^3 should not be more economical than that of $t^*=4\cdot2$ mm, $n^*=103\cdot1$ and $Z=764\,000$ mm^3.

45. From equation (27), $y_0^* \propto k$ with $k=[K_1^8/K_2^4]^{1/11}$. Using the assumed values of $K_1=0\cdot78$ and $K_2=1\cdot95$, $k=0\cdot655$ then:

for 14 × 6¾ UB @ 51 kg/m $K_1=0\cdot75$ and $K_2=1\cdot92$, $k=0\cdot642$
for 15 × 6 UB @ 52 kg/m $K_1=0\cdot75$ and $K_2=2\cdot02$, $k=0\cdot625$

46. The above shows that having carried out the rounding operation necessary, the final optimum design should be derived using the actual values of K_1 and K_2 in order to arrive at the true optimum result.

47. Finally, I would suggest that the unit for live load W, being a force, should be in N/m^2. This would obviate the need to introduce $g=9\cdot81$ m/s^2 to convert kg to newtons, especially when the elastic modulus E is expressed in MN/m^2.

Professor F. Sawko, University of Liverpool

The method of geometric programming is a process of satisfying the minimum cost criteria subject to certain restraints, which in the field of structural design are either external (e.g. headroom limitations, maximum sizes of members, etc.), or strength criteria (maximum working stresses, instability, etc.). It thus immediately follows that a structure which satisfies all restraints (i.e. stress levels, minimum depth, etc.) is not necessarily the cheapest, and in the field of structural design 'the first stab' at a section rarely satisfies minimum cost criterion. The method presented by the Author is a powerful tool for satisfying the cost criterion in addition to all structural criteria simultaneously. The success of the method when applied to solution of design problems is dependent on an accurate supply of cost figures, and as I attempted an

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optimization study of bridge design in 1964, would urge contractors to make cost figures more readily available.[5]

49. It is important to interpret the Paper in its true context, i.e. as a *technique* for arriving at minimum cost design.

50. In § 12 the Author suggests that the problem of the tower '. . . could be solved in a number of ways . . .' and engineers should not be afraid to experiment with different techniques. I have re-analysed the tower problem using a procedure which all engineers will find straightforward; a simple process of minimizing the cost function. The Author's comments on this approach would be most welcome.

51. Starting with the Author's equation (8):

Cost of tower $$y_0 = K_w \pi D h + K_R \frac{\pi D^2}{4}$$

Volume $$V = \frac{\pi D^2 h}{4}$$

The inequality sign is not necessary, since the minimum volume and required volume are the same. From this equation, h can be expressed in terms of V and eliminated from the cost equation. Thus:

$$y_0 = K_w \pi D \times \frac{4V}{\pi D^2} + K_R \frac{\pi D^2}{4}$$

$$= K_w \times 4V \times \frac{1}{D} + K_R \frac{\pi}{4} D^2$$

For minimum cost $\partial y_0 / \partial D = 0$. Therefore:

$$-K_w 4V \frac{1}{D^2} + K_R \frac{\pi}{4} \times 2D = 0$$

Hence:

$$D = 2\left(\frac{K_w V}{K_R \pi}\right)^{1/3}$$

the Author's equation (20). Substituting to equation for volume:

$$h = \frac{4V}{\pi} \times \frac{1}{4} \times \left(\frac{K_R \pi}{K_w V}\right)^{2/3} = \left(\left[\frac{K_R}{K_w}\right]^2 \frac{V}{\pi}\right)^{1/3}$$

i.e. the Author's equation (21).

52. It will be observed that there is always only one total cost function to be minimized, the differentiation of this function should always be possible. The Author might possibly throw some light on the limitations of this approach.

53. I have recently used a similar approach for optimization studies in the design of a reinforced concrete slab bridge. Several simplifying assumptions had to be made, but the treatment will, in my opinion, be of interest to engineers. For optimization purposes only the variable cost items have to be optimized. Thus, for a bridge of a fixed span, the surfacing, handrailing, footpaths, etc. are independent of deck thickness. The variable items were the depth of deck (which affects the dead load moment) and the area of main reinforcing steel. Transverse and top steel were also assumed as fixed cost items.

54. The following notation was used:

span of slab	L
overall depth of section	d
depth to steel	d_1, $c = d - d_1$
lever arm	$la = k \times d_1$
permissible tensile stress in steel	t_{st}
permissible compressive stress in concrete	ρ_{cb}
area of steel (tensile)	A_{st}
weight of concrete	ρ/unit volume
cost of concrete	C^*/unit volume
cost of steel	S^*/unit volume

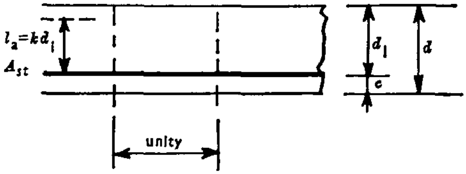

Fig. 5

55. Consider a strip of unit width.

$$\text{Total bending moment} = \underbrace{\text{Live load } mt + \text{finishes}}_{} + \text{slab moment}$$

$$= \text{Constant } mt \quad + \text{slab moment}$$

$$\text{Bending moment} = M_L + \frac{1}{8}L^2\rho d = M_L + \frac{1}{8}L^2\rho c + \frac{1}{8}L^2\rho d_1$$

$$= M + \frac{1}{8}L^2\rho d_1$$

It is here assumed that steel cover and diameter of tensile steel will be constant. Area of steel required:

$$A_{st} = \frac{\text{Bending moment}}{l_a \times t_{st}} = \left(\frac{M}{kd_1} + \frac{1}{8k}L^2\rho\right)\frac{1}{t_{st}}$$

assuming that the lever arm does not alter substantially with the area of steel. Hence:

$$\text{volume of concrete} = L(d_1 + c)$$

$$\text{volume of steel} = \left(\frac{M_L}{kd_1} + \frac{L^3\rho}{8k}\right)\frac{1}{t_{st}}$$

Therefore:

$$\text{total cost/unit width} = L(d_1 + c) \times C^* + \left(\frac{M_L}{kd_1} + \frac{L^3\rho}{8k}\right) \times \frac{S^*}{t_{st}}$$

For minimum cost:

$$\frac{\partial\,(\text{cost})}{\partial d_1} = 0$$

Hence:

$$LC^* + \frac{MLS^*}{t_{st}k}\left(-\frac{1}{d_1^2}\right) = 0 \quad \therefore\ t_{st}.k.C^*d_1^2 = MS^*$$

Therefore:

$$d_1 = \sqrt{\frac{MS^*}{kC^*t_{st}}}$$

56. It will be observed that this simple calculation produces the effective depth of slab immediately. This is of considerable benefit for the determination of headroom requirements, deck level and value of fill in embankments.

57. The calculation can be carried a stage further to obtain the total cost of variable quantities. Thus:

$$\text{Total cost/unit width} = LcC^* + LC^*\sqrt{\frac{MS^*}{kC^*t_{st}}} + \frac{L^3\rho S^*}{8kt_{st}} + \frac{MLS^*}{kt_{st}}\sqrt{\frac{kC^*t_{st}}{MS^*}}$$

$$= LcC^* + L\sqrt{\frac{MS^*C^*}{kt_{st}}} + \frac{L^3\rho S^*}{8kt_{st}} + L\sqrt{\frac{MS^*C^*}{kt_{st}}}$$

$$\text{Total cost} = 2L\sqrt{\frac{MS^*C^*}{kt_{st}}} + LcC^* + \frac{L^3\rho S^*}{8kt_{st}}$$

58. Before this calculation was carried out, a slab depth of $d=21$ in. ($d_1=17{\cdot}75$ in.) was assumed, for dead load calculations and analysis. Optimization calculations gave a figure for $d_1=13{\cdot}2$ in. The reduction of cost of the deck itself was $14{\cdot}5\%$ and the consequent saving in embankment cost due to the 4·5 in. reduction in construction depth was quite considerable.

59. It is important to emphasize that the method is a gross over-simplification, but clearly any degree of refinement could be introduced. Research is required into the practical consequences of optimization studies and I am fortunate to have on my staff the Author, who is not only pursuing this topic himself, but also acting as a stimulus to his colleagues.

Dr L. C. Schmidt, University of Melbourne

The method of geometric programming is a systematic procedure for solving a class of non-linear optimization problems, and the Author has presented a succinct account of the method applied to structural design problems. However, I feel that the general presentation, introducing the signum function, is, perhaps, not so readily appreciated as the presentation given by Duffin, Peterson and Zener.[3] They restrict their discussion to 'posynomials', which may be defined as 'polynomials' with positive coefficients but with the exponents not restricted to the positive integers.

61. The Author has necessarily been brief in his presentation of the geometric programming method and has therefore not explicitly mentioned features that I consider to be noteworthy. One feature is that the dual formulation provides constraints that are linear, even though the constraints in the primal problem may be non-linear. In the special case when the degree of difficulty is zero a unique solution is then obtained in a straightforward manner. Another point is that when a constraint is active in the primal problem the inequality constraints on the dual variables are inactive.

62. When the degree of difficulty is greater than zero, the dual problem can be reformulated in terms of basic variables, where the number of basic variables equals the number of degrees of difficulty. Reference was made to this in § 34, but I feel that the Author has passed over this aspect rather lightly. In small problems considered to date I have usually had a positive degree of difficulty, so that the effort involved in obtaining a solution has increased considerably. In real-life problems it can be expected that this condition will more likely be normal. The Author's example 2 illustrates this point, as it would be usual practice to include stress constraints initially and not make the assumption (*e*) of § 20. It is not obvious at the start of a problem which constraints will be active for an optimal design. Methods of treating problems with positive degrees of difficulty are in need of further investigation and development.

63. A simple example will be considered to illustrate a problem with a positive degree of difficulty, but the simpler presentation, using posynomials, will be used.[3]

64. A timber beam with fixed span and loading is to be designed, where the cost function C is given by:

$$C = ab+0{\cdot}2b^2 \qquad \ldots \ldots \ldots \quad (42)$$

In the expression a is the width and b is the depth of the cross section. It is noted that there is a penalty on the depth of the section. The following constraints may be formed for this problem:

$$\begin{aligned} &\text{flexural stress} && \frac{270}{ab^2} \leqslant 1 \\ &\text{deflexion} && \frac{2020}{ab^3} \leqslant 1 \\ &\text{depth/breadth} && \frac{b}{3a} \leqslant 1 \qquad \ldots \ldots \ldots \quad (43) \end{aligned}$$

65. As there are five terms and two variables, the degree of difficulty is two. The normality and orthogonality conditions for the dual variables become:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 2 & -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{21} \\ w_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\ldots \quad (44)$$

66. A unique solution is not possible, but a trial and error approach can be adopted since the following relationship holds between the primal and dual problems:

$$y_0 \geqslant y_0{}^* = d^* \geqslant d \quad \ldots\ldots\ldots\ldots \quad (45)$$

which is a more general form of the Author's equation (4).

67. After diagonalization, equation (44) can be rewritten as follows, with the left hand matrix augmented by the right hand vector and the vector of weights omitted.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1/3 & -1 & 2/3 \\ 0 & 1 & 0 & -1/3 & 1 & 1/3 \\ 0 & 0 & 1 & 4/3 & 0 & 2/3 \end{array}\right] \quad \ldots\ldots \quad (46)$$

68. This form allows easy calculation of the weights for assumed values of w_{21} and w_{31}. For the case when these latter are zero:

$$\begin{bmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{21} \\ w_{31} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (47)$$

The value of the dual function then becomes $d=46{\cdot}1$. From equation (45) this provides a lower bound to the optimal value d^*.

69. Determining, now, the value of the primal problem using values of the primal variables corresponding to the dual variables w, an estimate can be made of the usefulness of the results given in equation (47).

70. From equation (7a), it follows that:

$$\begin{aligned} ab &= \frac{2}{3}\times 46{\cdot}1 = 30{\cdot}8 \\ 0{\cdot}2b^2 &= \frac{1}{3}\times 46{\cdot}1 = 15{\cdot}4 \\ \frac{270}{ab^2} &= \frac{2}{3}\times\frac{3}{2} = 1 \\ \frac{2020}{ab^3} &= \frac{0}{0} \\ \frac{b}{3a} &= \frac{0}{0} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

The last two conditions are indeterminate and therefore are of no value in finding a and b. This means that these constraints are inactive in the primal problem for this particular solution.

71. Using the second and third expressions of equation (48) values of $a=3{\cdot}51$ and $b=8{\cdot}77$ are found. The primal objective function then gives $y=46{\cdot}2$. Equation (47) indicates that this value must be an upper bound to the optimal value of the objective

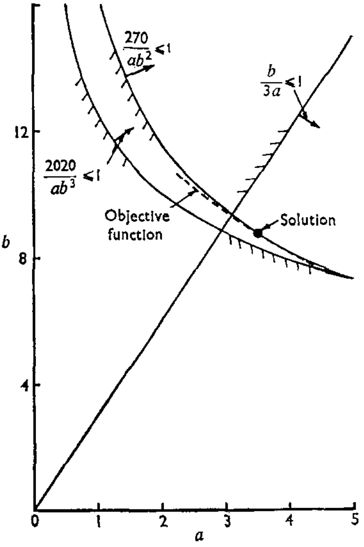

Fig. 6.

function. As the values of the upper and lower bounds are so close the problem can be considered solved. A unique solution is obtained since the two primal constraints on deflexion and depth/breadth ratio are inactive.

72. For illustrative purposes, a graphical plot of the solution is shown in Fig. 6.

73. In a larger problem it may not be obvious how to alter the unknown dual variables in order to maximize *d*; an automatic procedure is required if a computer is to be used. At present I favour the approach presented by Powell[6] for this purpose.

Mr C. Osgerby, Heriott Watt University

The Author has raised several points, which are of interest to all design engineers, and geometric programming may well be of use where *all* the constraints are considered. In both examples I was surprised to find no initial constraint linking the strength of the structure to the applied loads.

75. In the first example, there will be pressure on the walls dependent on the density and height of the enclosed material and tension in the walls will also be dependent on the depth from the top and the diameter. If the construction is of reinforced concrete, the steel area to resist this tension is dependent on the grade of steel used (at whatever extra cost for the higher grade) and the size of the bars since cost/kg increases as diameter decreases. The wall thickness may be determined solely on cover requirements or a cracking criterion may apply but here again the cost per unit volume of wall increases as the thickness decreases. I think that a force condition should have been applied and this should have been optimized for minimum cost. The same result as that obtained by the Author could have been found by differentiation of equation (8).

76. In the second example the Author obtains the dual objective function in equation (27) and containing the 'constants' K_F, K_B, K_1 and K_2. Values for K_1 and K_2 are related to the cross-sectional area and the second moment of area and Figs 3 and 4 give the *average* values for these.

77. What is important, however, is the ratio K_1^2/K_2 which is used in equation (27) and this ratio must be that for a specific joist (actually the joist which is finally chosen).

Substituting for K_1 and K_2 from equation (22) the ratio can be shown to be equal to a'/k^2 where k is the radius of gyration of the section about the axis of bending. Thus the cost varies as a function of the area, and for a given area the cost is less as k increases (i.e. the beam is deeper). For the whole range of Universal Beams the ratio varies from 0·51 to 0·195 compared with that assumed by the Author of 0·312 and a value of 0·275 for the joist finally selected.

78. The fact that the depth of the beam now comes into consideration means that where headroom is important, the cost of the extra height of the supporting structure must be allowed for. The cost of the plate K_F varies little with thickness, but that for the beam can be 10–15% less as the beam depth increases and surely this is important.

79. Using the Author's prices and in equation (27) a value of $K_1{}^2/K_2$ for the joist which is finally used (using a feedback principle) I estimate that a further reduction in cost of about 10% can be made, though this would not adhere to the Author's value of minimum plate thickness. I am not suggesting that this is a practical solution, only making the point that values assumed to be constant must in fact be so, or else a feedback loop must be provided to check the initial factors. It is for this reason that I feel it would be preferable to program a computer to produce a series of designs and then to feed in cost values which are in discrete steps, so that a realistic cost comparison can be made.

Mr A. S. Watson and Mr G. M. Mills, University of Bradford

The technique of geometric programming appears to have great potential in structural design, particularly when the primal problem can be formulated in terms of continuous functions and constraints as in the first example.

81. However, if any constraints are required to be discontinuous (e.g. CP 114 shear reinforcement requirements), their formulation in terms of polynomial inequalities is difficult. One solution might be to use a conservative replacement constraint of continuous nature and accept that the resulting solution is not an absolute optimum. Alternatively, the required effect may be produced (exactly or approximately) by replacing the appropriate primal variables by two or more components which can then be given continuous constraints.[7] The latter approach will significantly increase the magnitude of the problem.

82. Another difficulty occurs in dealing with relatively small scale problems in which the magnitudes of the structural components vary in discrete steps. Examples are Universal Beam sections in steelwork and bar sizes in reinforced concrete. Thus, referring to the Author's second example, the value of the dual objective function in both equations (27) and (33b) depends on a positive power function of $K_1{}^2/K_2$, and the minimum cost is therefore likely to occur if a Universal Beam section is chosen with the lowest value for this quantity. Typical values for selected beams are given in Table 1 from which it is clear that the value for K_1 of 0·78 used by the Author in the primal equation is not representative of the most economic section.

Table 1

Beam	K_1	K_2	$K_1{}^2/K_2$
14 × 6¾ UB at 51 kg/m .	0·753	1·92	0·295
15 × 6 UB at 52 kg/m .	0·744	2·02	0·274
15 × 6 UB at 60 kg/m .	0·776	1·94	0·310
16 × 6 UB at 60 kg/m .	0·753	2·03	0·279
16 × 7 UB at 54 kg/m .	0·722	2·07	0·251
16 × 7 UB at 60 kg/m .	0·732	2·00	0·269

83. The final design selected by the Author shows 92 beams at 570 mm centres. This appears to be incorrect as 90 beams of section 15 × 6 Universal Beam at 52 kg/m will satisfy the deflexion criteria at a spacing of 555 mm and the corrected total cost y_0* becomes £3466. However, reference to Table 1 would suggest that the more economic section is 16 × 7 Universal Beam at 54 kg/m for which it can be shown that the optimum design requires 85 beams at 590 mm spacing, giving $y_0^* = £3400$. This seems to be the lowest cost solution. It may be noted that if $K_1 = 0{\cdot}72$ is substituted in equation (27), a similar cost is obtained.

84. The importance of the factor K_1^2/K_2 only becomes apparent from a study of the dual objective function and it may be that similar analysis for other specific design problems with discretely varying components will also indicate the optimum solution. In such circumstances, we agree with the Author that the use of a computer is not essential. However, the majority of design problems tend to be highly constrained. For these, the use of a computer may be necessary, although, in the present stage of development of geometric programming, the techniques of solution for optimum designs are likely to be unwieldy.

85. A typical problem with many degrees of difficulty is a doubly reinforced rectangular concrete beam element. This can be represented with reasonable accuracy by 10 variables. The corresponding objective function contains 11 terms and, even using one or two approximations, a comprehensive range of constraints requires 27 equations giving an additional 36 terms. Thus, the degree of difficulty $= (11+36) - 10 - 1 = 36$. If such a beam forms part of a larger structure, the final degree of difficulty could be much greater. As the Author states, a solution may nevertheless be found but there is no formal mathematical solution which can be applied for the general case. The only practicable method is to use a computer routine to maximize the dual objective function subject to dual constraints.

86. If the use of a computer is necessary, one may perhaps question the value of attempting to solve the dual problem rather than the primal problem. If the latter is solved directly, there is a saving in the work of conversion from one problem to the other. However, in most cases, this work is relatively small when compared with the effort required to solve a primal problem consisting of a non-linear objective function subject to non-linear constraints. Several computer methods of solution for the latter problem have been developed but their effectiveness tends to depend on the characteristics of the particular problem.

87. Alternatively, by applying geometric programming to the primal problem, the constraints are reduced to linear form but at the expense of an increased number of variables in the dual problem. Also, unless indicated by previous experience with a particular problem, the likely range of the dual variables is unknown. Fortunately, the dual objective function is always of the same form despite the infinite range of possible primal objective functions and this indicates the possibility of a standard computer solution to all dual problems with a non-zero degree of difficulty. In practice, such a standard solution may not be the most efficient method of solving a given problem as the characteristics of the multi-dimensional feasibility surface depend on the primal problem.

88. To solve the dual problem requires the location of the peak of the feasibility surface. This is normally a difficult problem although, hopefully, an easier one than the solution of the primal problem. There are three basic general techniques of solution:

(*a*) one at a time methods, when the objective function is maximized with respect to each variable in turn

(*b*) gradient assessment methods, when all variables are changed simultaneously to give the greatest rate of increase in the objective function

(*c*) grid search methods, when the possible locations of the optimum point are reduced in number by a systematic multi-dimensional search.

These are often used in combination and all seek to improve an existing feasible solution. Computerized solution techniques based on one or more of these general methods have been developed for problems of the same type as the dual problem. However, to obtain a reasonable efficiency with specific classes of problems, the basic methods have frequently been augmented by special routines. In such cases, the computer programs tend to be effective for only a relatively small range of problems. The effectiveness of a proposed standard solution to the dual problem has yet to be investigated.[8]

89. The main difficulty with any of the methods described above is that they do not necessarily yield the global optimum. Thus, any solution must be checked to ensure its correctness and one convenient approach would be to repeat the optimization process from different feasible initial solutions and compare the results. Experience with different problems may indicate a pattern for the optimum dual variables (weightings) which may then be used to predict good initial solutions.

90. Finally, we believe that geometric programming is potentially a very useful technique but it is still very much in its infancy and requires considerable research into its possibilities. This is particularly so in the case of problems with non-zero degrees of difficulty. Comparisons are required between the geometric programming approach and the alternative technique of direct solution of the primal problem. If this should prove very favourable to the former, then the investigation of minimum cost designs will become feasible for many problems which are presently too complex. This should result in economic benefits both to the construction industry and to the community as a whole.

Dr J. I. Nicholls, University of Washington

The Author has presented in a simple format, an application of a relatively new addition to the field of optimization in structural design. The problems themselves will not be discussed here, but rather, some additional comments will be made.

92. The geometric programming technique offers perhaps a significant advance over existing techniques such as grid search or gradient methods. A major difficulty however, appears to be that of defining the problem from the engineering standpoint. If the problem being solved has a zero degree of difficulty then an immediate solution can be derived by inverting the coefficient matrix, or by using some other decomposition algorithm. This is the type of problem solved by the Author. If however, the coefficient matrix is not square, such that the feasible solution region contains more than one feasible solution, then the problem of determining the optimal solution becomes more difficult. Since in this case the objective function is not linear (for the dual problem) linear programming may not be directly applied. By linearizing the objective function, a solution may possibly be obtained by repeated use of the linear programming algorithm. Engineering problems are in general relatively complex, and thus obtaining a zero degree of difficulty problem cannot always be assured.

93. One specific area that may produce a significant contribution is one in which the Author has made certain advances. To date, the grid search and/or gradient techniques have suffered from the fact that only a specific region could be searched due perhaps in part to the enormity of the initial problem. Expanding the area of search in general is not justified since no specific regional allocation can be attributed to the global optimal solution. With the application of some form of pre-analysis, such as geometric programming, the region to be searched can be much more clearly determined, and specifically, integer values or constrained values may be adhered to without the necessity of searching extremely large regions as has been previously carried out.

94. Since the dual problem involves linear constraints, this immediately infers that we have a convex region. Thus the global optimal solution can be obtained for this problem (provided one exists) in a similar fashion to that in which one obtains the global optimal solution to a linear programming problem. Further, the global

optimal solution to the primal problem can be derived from a knowledge of the solution to the dual problem. This step has been carried out by the Author. Given then, a set of variables which define the solution to the primal problem, one would suspect that a further solution which has a lower objective function value must be infeasible, or, there is an error in one of the solutions. With this in mind, it is hard to realize that the optimal cost for the second example can actually decrease as further restrictions are placed on the variables. The feasible solution region cannot be expanded when restrictions are placed on any of the individual variables, and thus a reduction in optimal cost does seem rather strange from a practical viewpoint.

A. B. Templeman

I was gratified by the interest shown in the Paper and would like to thank the contributors to the discussion for the opportunity of enlarging upon some points dealt with rather rudimentally in the Paper.

96. **Mr Chee** demonstrates a discrepancy of some 2% between values of Z calculated in the second example. This difference is caused by lack of precision in performing the calculations for which I apologize. In fact optimum values of all three primal variables, t, n and Z may be found exactly from three of the four terms containing them. The fourth term may then be used to check the results; a process which I omitted to perform. The section modulus Z may be shown by these means to be given by the formula:

$$Z = 2^{3/4}A^3\left(\frac{K_B{}^3K_1{}^3W^6}{K_F{}^3K_2{}^7E^6}\right)^{3/22}$$

97. The fact that the final rounded optimum cost of £3519 is less than the initial optimum cost, as noted by Mr Chee and **Dr Nicholls,** is due to the substitution of discrete section properties into the cost function as soon as they are known. Section tables give the cross-sectional area and section modulus of a 15 × 6 UB at 52 kg/m as 6640 mm² and 842 000 mm³ respectively. The cost function is:

$$y_0 = (K_F\gamma AB)t+(K_B\gamma AK_1)nZ^{2/3} \quad . \; . \; . \; . \; . \quad (23)$$

Substituting $t=4{\cdot}76$ mm, $n=92$ and $z=842\,000$ mm³, into equation (23) gives a cost of £3633 for the constants used in this example. However, the factor $K_1Z^{2/3}$ in the second term represents the beam area a' which is now known. If $a'=6640$ mm² is substituted into the second term a total cost of £3519 results as quoted in the Paper.

98. The above discrepancy is caused by the fact that the approximations (22) are not always precise. Messrs Chee, **Osgerby, Watson** and **Mills** all draw attention to variations in the values of the so-called constants K_1 and K_2. That K_1 and K_2 are not constants is evident from the small amount of scatter on the graphs of Figs 3 and 4. However, in my opinion the approximations (22) are necessary if any rational approach is to be made to the optimum design of steel structures. Trouble is invariably encountered when a set of discrete values is represented by a continuous function. This is the case here with Universal sections, but nevertheless the linearity of Figs 3 and 4 is so obvious that the advantage of making these particular approximations greatly outweighs the disadvantages.

99. Geometrical programming has produced in example 2 an optimum design (equations (29)–(32)) using equations (22). It is now the engineer's job to turn this mathematical design into a practical one. One way of doing this using geometric programming is presented in the Paper and the design (39) results. Messrs Chee, Osgerby, Watson and Mills are quite correct in making the point that there may be other possibilities. By all means check the constants, re-examine the design, cast around for alternatives, but it must be acknowledged that, as Dr Nicholls has pointed out, geometric programming has produced a very close estimate of the area in which this optimum practical design will be found.

100. I am well aware, as Mr Osgerby points out ,that initial constraints linking the strength of the structure to the applied loads were absent from both examples. This

was done in the first example simply to produce a mathematically suitable problem for the geometric programming technique to solve as a demonstration. In the second example such a constraint would be inactive as is shown in the Paper and furthermore would have produced a problem having at least two degrees of difficulty which would have been soluble but inappropriate. As Mr Osgerby states and **Professor Sawko** demonstrates, the first example is almost trivial and may be solved by straightforward substitution and differentiation. It was included to show the geometric programming approach and to demonstrate the order and form in which the results are obtained.

101. The reinforced concrete slab design quoted by Professor Sawko is interesting although it does not involve geometric programming. I wish that all optimization problems were as simple as this one, since the process of substituting an equality constraint into the cost function as carried out here is rarely possible. In the general case if there is more than one constraint the designer has *a priori* no way of knowing which constraints are active and which are not and so can make no substitution. In the reinforced concrete deck problem quoted there are at least three constraints implied: maximum allowable steel tension, maximum allowable concrete compression and maximum allowable span/depth ratio. Professor Sawko apparently has decided initially that the maximum allowable steel tensile stress is the only active constraint at the optimum. This may be so specifically but not generally, and Professor Sawko's equation for d_1 should be joined by at least two other equations for d_1 corresponding to the other implied constraints.

102. **Dr Schmidt's** preference for the more restrictive posynomial formulation of geometric programming is understandable but was discarded in favour of the completely general approach given in the Paper. Dr Schmidt answers the query of Messrs Watson and Mills, who question the value of solving a dual problem rather than a primal problem. The primal problem usually comprises an objective function and several constraints, all displaying a high degree of non-linearity, and is difficult to solve as such. The work of transforming to the dual form is quite minimal and the dual problem has the great advantages of linear constraints and an objective function whose form is always the same (equation (5)) whatever the form of the primal problem. Solution of the dual problem is generally considerably easier.

103. The treatment of problems involving non-zero degrees of difficulty was deliberately omitted from the Paper. Dr Schmidt shows one way of handling these problems in his timber beam example, but I feel that this method, although suitable for one or two degrees of difficulty, proves impossibly lengthy if the number of degrees of difficulty is high. Generally, the more constraints which are applied to the primal problem the higher the degree of difficulty of the dual problem becomes. It therefore became apparent, as Drs Schmidt and Nicholls and Messrs Watson and Mills suggest, that a comprehensive computer method of solving problems involving n degrees of difficulty was required. It is perhaps significant that a program to do this was developed in the USA in 1969 by the Westinghouse Electric Corporation, who were reluctant to release details to me. Over the past year I have, therefore, developed my own program for the geometric programming problem which inputs the primal problem, performs the necessary transformations, maximizes the dual function and outputs the optimum values of the primal variables and the minimum cost.

104. Maximization of the dual objective function by means of linearizing and repeatedly using a linear programming algorithm as suggested by Dr Nicholls is unsuitable since equation (5) is highly non-linear, its gradients varying continuously between $-\infty$ and $+\infty$, thus rendering linearization a very suspect process. Instead, use is made of the fact that the dual function is always of the form of equation (5) no matter what the form of the primal problem.

105. The program is now completed and is undergoing trials. Test examples have been processed involving 3, 8 and 21 degrees of difficulty and successful results have been obtained very quickly. In the near future comparative trials are to be

made against other non-linear optimization programs using test problems. Development of this program has precluded much work being done on engineering applications of the method. Since the publication of the Paper, however, I have been informed of many varied problems which appear to be in a suitable form for geometric programming and it is intended to make a systematic investigation into the possibilities of the method as soon as the program is fully developed.

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