

and all possible ratios were shown on the chart given in the first Paper which could be referred to for the effect of a different ratio (b/d).

Comparison of designs for the same stress but different depths, and for the same depth but different stresses had been made in the first Paper; that might be the type of cost analysis requested by Mr Cochrane: in view of that the two assumptions made there for the cost comparison might be amplified.

First, the same basic cost of concrete was used with different stresses since a high-grade concrete was required to avoid creep loss irrespective of the stress used; that resulted in a lower total cost when allowing a higher stress.

Secondly, the ratio of the cost of the cable to the cost of the completed beam was assumed to be 70% in the basic design used for comparison but altering that to 50% made no difference to the conclusions.

Those conclusions were different in the case of beams forming a deck and the Author was glad that Mr Cochrane had raised the matter which would be attended to on the lines he suggested in the proposed Paper on composite construction.

Paper No. 6085

The effect of uplift on gravity-dam profiles †

by

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Dr O. C. Zienkiewicz (Lecturer, Department of Engineering, University of Edinburgh) stated that the validity of the conclusions and the profile formulae derived, useful as they would doubtless be for economical design, hinged on the validity of one basic assumption. That was the linear distribution of the vertical component of "material" stress on horizontal sections under all conditions of uplift pressures.

That linearity, approximately true with zero uplift, appeared, if Brahtz's conclusions were generally true, to be realized only if the uplift pressures were also linear. It thus followed that if Brahtz's conclusions could be extrapolated beyond the limits of their strict applicability, the results were of little use; furthermore, any benefits of internal drainage were nullified in all gravity dams.

That vexing problem had intrigued Mr Zienkiewicz for some time and had prompted some detailed analytical work which gave an answer of practical value. A Paper on the theme had recently been published.⁷ It was proposed to quote some of the results and the approach used.

To make a full stress analysis with uplift amenable to a mathematical approach, a simplified dam (a triangular section of large height) was considered and the uplift variation was assumed to have the same shape at all similar sections.

It was well known that the profile chosen exhibited an exactly linear stress distribution with no uplift and approximated elastically to the portions of the dam not affected by the foundation contact.

† Proc. Instn Civ. Engrs, Pt III, vol. 5, p. 196 (Apr. 1956).

⁷ O. C. Zienkiewicz, "The effect of pore pressures on stresses in gravity dams". J. Pow. Divn, No. Po. 4, Proc. Am. Soc. C.E. (Aug. 1956).

The basis for the analysis was provided by the general equations of plane stress elasticity in which the effect of uplift or internal pore pressure was introduced by body forces proportional to the pressure gradients.

Defining the three stress components in terms of a stress function ϕ as followed:

$$\left. \begin{aligned} \text{Stress in} \\ \text{X direction:} \quad \sigma_x &= \frac{\partial^2 \phi}{\partial y^2} + p\beta - w_s y \\ \text{Stress in} \\ \text{Y direction:} \quad \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} + p\beta - w_s y \\ \text{Shear stress:} \quad \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\} \dots \dots \dots (31)$$

where y was measured downwards, w_s was the unit weight of the saturated material, β an area coefficient, and p the uplift pressure, it followed from elastic compatibility, that:

$$\nabla^2(\nabla^2 \phi + Kp) = 0 \dots \dots \dots (32)^*$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

and $K = \frac{1 - 2\nu}{1 - \nu} \beta$ ($\nu =$ Poisson's ratio)

Brahtz's conclusions, mentioned on p. 198 of the Author's Paper, were based on the particular distribution of p where $\nabla^2 p = 0$ reducing the equations to a simple form. Solutions of cases where that relation was not satisfied were obviously of prime interest

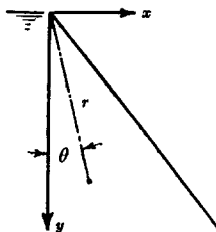


FIG. 18

For a very high triangular profile shown in Fig. 18, the uplift pressure in general could be assumed to be of the type:

$$p = r\Psi \dots \dots \dots (33)$$

where Ψ was a function of angle θ only. With that stipulation it was found that the stress function was of the form:

$$\phi = r^3 f \dots \dots \dots (34)$$

where f again was a function of the angle only. Substitution into equation (2) resulted in a simple ordinary differential equation:

$$\frac{d^4 f}{d\theta^4} + 10 \frac{d^2 f}{d\theta^2} + 9f = -K \left(\Psi + \frac{d^2 \Psi}{d\theta^2} \right) \dots \dots \dots (35)$$

which could be solved for any assumed distribution of pressure, i.e., a known function Ψ .

One example followed, shown in Fig. 19. In that case, a line of drainage was assumed to exist along a radial plane reducing the pore pressure from the full hydrostatic head at the upstream face to zero at that line. Extension of Brahtz's hypothesis on similar lines

to those used in deriving Fig. 5 resulted in a "material" stress distribution shown in Fig. 19b. The correct elastic solution derived from equation (35) resulted in the distributions of material stress shown in Fig. 19c. Those results were interesting because they depended appreciably on the values of Poisson's ratio assumed.

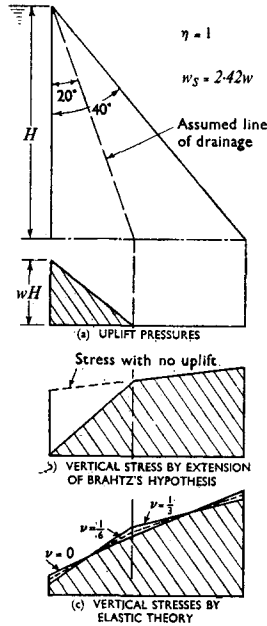


FIG. 19

It was striking, however, that the departure of the material stresses distribution from linearity, in the example chosen, was fairly small for usual values of the Poisson's ratio. Numerous other cases of uplift distribution had been investigated on those lines, with the same general result. That was indeed fortunate validating as it did, at least approximately, the basic assumption on which the results of the Author's work were based. It was hoped that it would renew the confidence many engineers placed in suitable drainage devices which reduced the resultant uplift force without necessarily decreasing the maximum value of the pore pressure.

In conclusion, it should be mentioned that Brahtz's hypothesis was already extended beyond the strict limits of its applicability even in the apparently simple case given in Figs 3-5. The Author's conclusion that the drainage provided by horizontal galleries was of no avail in reducing the stresses was, therefore, founded on false premises and required careful examination.

Mr James Park (Research Engineer, W. S. Atkins and Partners) observed that when employing Brahtz's conclusions on the effect of pore pressure on stresses in porous bodies, it was always necessary to examine carefully whether the problem fell strictly within the bounds of applicability of Brahtz's results. In so doing it was useful to examine the precise terms in which Brahtz defined his main result*:

* Up to that point, the analysis followed closely the method outlined in Brahtz's original work. (See reference 5 of Paper, p. 206.)

“In order to obtain the contact stresses [pore pressure acting] the mean stresses determined by the boundary and body forces in the absence of pore pressure must be decreased by an amount βp if internal liquid pressure is also present”,

where β denoted the area factor and p the pore pressure.

The proof of that statement was fairly straightforward. The stress components, σ_x , σ_y , and τ_{xy} were expressed in terms of a stress function ϕ thus:

$$\left. \begin{aligned} \sigma_x &= \frac{\partial^2 \phi}{\partial x^2} - w_s y + \beta p \\ \sigma_y &= \frac{\partial^2 \phi}{\partial y^2} - w_s y + \beta p \\ \tau_{xy} &= - \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\} \dots \dots \dots (36)$$

where w_s was the saturated weight of the material. The term βp was omitted for a non-porous body.

By considering the condition for compatibility of strain, and the boundary conditions, it was easily shown that:—

- (1) The general form of the stress function ϕ was identical for porous and non-porous bodies.
- (2) The arbitrary constants in the general expression for ϕ were identical for porous and non-porous bodies.

Thus, since ϕ was not dependent in any way on the porosity of the material, Brahtz’s conclusion follows from equations (36), from which it was seen that the normal stresses in porous and non-porous bodies differed only by an amount βp . The shear stress τ_{xy} was not affected by pore pressure.

It was an essential feature of Brahtz’s conclusion that the pressure function p should obey the Laplace equation throughout the region considered.

In the example considered by the Author under the heading “The effect of drains on the stress distribution”, the presence of the longitudinal gallery created a problem beyond the scope of Brahtz’s result, for the transverse section of the dam was, in terms of the theory of elasticity, a region of multiple connexion; at least it might be considered such if the gallery was sufficiently large compared with the area of the section. Otherwise, if the gallery was treated as a point drain, the pressure function was not harmonic at that point and Brahtz’s conclusion was therefore immediately inapplicable. Assuming, however, that the gallery was of reasonable size, the section must be considered as a region of multiple connexion, in the “solid” part of which the pressure function p obeyed the Laplace equation. It was a general rule that in such a region the boundary and body forces were insufficient to determine the stress distribution, and displacements must also be considered, usually imposing the condition that they must be single-valued functions only. It was debatable whether Brahtz was aware of that limitation when he worded his conclusion specifying stresses “determined by the boundary and body forces”, but the necessity for imposing additional conditions made Brahtz’s conclusion inapplicable in a region of multiple connexion, for the proof of the conclusion was based on the calculation of stress for a simply-connected region only.

Cylinder problem

A simple example illustrated the type of discrepancy involved, without going into the elaborate calculations required to analyse the section of a gravity dam. In the two-dimensional problem of a section of a porous cylinder, of internal radius a and external radius b , subjected to an internal water pressure p_0 with a resulting radial percolation of water, if the cylinder were non-porous, the radial and tangential stress components would be, respectively:

$$\left. \begin{aligned} \sigma_r &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 - \frac{b^2}{r^2} \right] \\ \sigma_\theta &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 + \frac{b^2}{r^2} \right] \end{aligned} \right\} \dots \dots \dots (37)^8$$

In a porous cylinder, assuming that radial flow obeyed Darcy's law so that $\nabla^2 p = 0$, the pressure distribution was given by:

$$p = p_0 \frac{\log b/r}{\log b/a}$$

where log denoted logarithms to the base e .

The stress components obtained by applying Brahtz's result were then:

$$\left. \begin{aligned} \sigma_r &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 - \frac{b^2}{r^2} \right] + \beta p_0 \frac{\log b/r}{\log b/a} \\ \sigma_\theta &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 + \frac{b^2}{r^2} \right] + \beta p_0 \frac{\log b/r}{\log b/a} \end{aligned} \right\} \dots \dots \dots (38)$$

If, however, the problem was worked out from first principles, the equilibrium equation contained the body-force term $-\beta \frac{dp}{dr}$ and the true stress components were found to be:

$$\left. \begin{aligned} \sigma_r &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 - \frac{b^2}{r^2} \right] \left[\frac{\beta}{2(1 - \nu)} + 1 - \beta \right] + \frac{\beta p_0}{2(1 - \nu)} \frac{\log \frac{b}{r}}{\log \frac{b}{a}} \\ \sigma_\theta &= \frac{p_0 a^2}{(b^2 - a^2)} \left[1 + \frac{b^2}{r^2} \right] \left[\frac{\beta}{2(1 - \nu)} + 1 - \beta \right] + \frac{\beta p_0}{2(1 - \nu)} \frac{\log \frac{b}{r}}{\log \frac{b}{a}} + \frac{\beta p_0}{2 \log \frac{b}{a}} \frac{(1 - 2\nu)}{(1 - \nu)} \end{aligned} \right\} (39)$$

where ν denoted Poisson's ratio.

To determine how the discrepancy arose between equations (38) and (39), Brahtz's proof must be re-examined with regard to multiply-connected bodies.

For a non-porous section with axial symmetry, the stress components could be expressed in terms of a stress function ϕ such that:

$$\left. \begin{aligned} \sigma_r &= \frac{1}{r} \cdot \frac{d\phi}{dr} \\ \sigma_\theta &= \frac{d^2\phi}{d\theta^2} \end{aligned} \right\} \dots \dots \dots (40)$$

Expressions (40) automatically satisfied the equation of equilibrium:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

When there was no body force the condition of compatibility was:

$$\nabla^2(\sigma_r + \sigma_\theta) = 0$$

and substituting expressions (40) resulted in:

$$\nabla^4\phi = 0 \dots \dots \dots (41)$$

For a porous section the stress components became:

$$\left. \begin{aligned} \sigma_r &= \frac{1}{r} \frac{d\phi}{dr} + \beta p \\ \sigma_\theta &= \frac{d^2\phi}{dr^2} + \beta p \end{aligned} \right\} \dots \dots \dots (42)$$

⁸ S. Timoshenko and J. N. Goodier, "Theory of elasticity." 2nd edn. McGraw-Hill, New York, 1951.

The equilibrium equation, which again was automatically satisfied was:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} - \beta r \frac{dp}{dr} = 0$$

The condition of compatibility became:

$$\nabla^2(\sigma_r + \sigma_\theta) = \frac{\beta}{(1 - \nu)} \nabla^2 p,$$

and since $\nabla^2 p = 0$, substituting expressions (42):

$$\nabla^4 \phi = 0 \quad (43)$$

Thus the general form of the stress function, as seen from equations (41) and (43), was the same for porous and non-porous bodies, as was found in part (1) above of Brahtz's proof for simply-connected bodies. In polar co-ordinates the solution of (43) had the form :

$$\phi = A \log r + Br^2 \log r + Cr^2 + D$$

and the stress components, from (40) and (42) were:

$$\left. \begin{aligned} \sigma_r &= \frac{A}{r^2} + B(3 + 2 \log r) + 2C + \beta p \\ \sigma_\theta &= -\frac{A}{r^2} + B(3 + 2 \log r) + 2C + \beta p \end{aligned} \right\} . . . (44)$$

the last term in each expression being omitted in the case of non-porous bodies.

Consider now the arbitrary constants *A*, *B*, and *C*. The value of *D* did not need to be considered since it did not appear in the stress equations. Only two boundary conditions existed, namely the conditions of normal stress at the inner and outer boundaries. Therefore to evaluate the three constants it was necessary to consider displacements, imposing the condition that they must be single-valued functions of *r* only in an axially symmetrical system.

Displacements

The components of radial, tangential, and shear strain (ϵ_r , ϵ_θ , and $\gamma_{r\theta}$ respectively) could be expressed in terms of the radial and tangential displacements, *u* and *v*, as followed:

$$\epsilon_r = \frac{\partial u}{\partial r} \quad (45a)$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \quad (45b)$$

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (45c)$$

The values of *u* and *v* were now required.

Using Hooke's Law, the strain components could also be expressed in terms of the stress components:

$$\epsilon_r = \frac{1}{E} [(1 - \nu^2)\sigma_\theta - \nu(1 + \nu)\sigma_r] \quad (46a)$$

$$\epsilon_\theta = \frac{1}{E} [(1 - \nu^2)\sigma_\theta - \nu(1 + \nu)\sigma_r] \quad (46b)$$

$$\gamma_{r\theta} = \frac{2(1 + \nu)}{E} \tau_{r\theta} \quad (46c)$$

From equations (45a) and (46a), substituting the expression (44):

$$\frac{\partial u}{\partial r} = \frac{1}{E} \left[\frac{A}{r^2}(1 + \nu) + B(1 - 3\nu - 4\nu^2) + 2B \log r(1 - \nu - 2\nu^2) + 2C(1 - \nu - 2\nu^2) + \beta(1 - \nu - 2\nu^2)p \right]$$

and integrating with respect to r :

$$u = \frac{1}{E} \left[-\frac{A}{r}(1 + \nu) - Br(1 + \nu) + 2Br \log r(1 - \nu - 2\nu^2) + 2Cr(1 - \nu - 2\nu^2) + \beta(1 - \nu - 2\nu^2) \int p dr \right] + f(\theta) \quad (47)$$

where $f(\theta)$ was a function of θ only, and $\int p dr$ represented the general integral only, the arbitrary constant of integration being included in $f(\theta)$.

From (45b) and (46b):

$$\begin{aligned} \frac{\partial v}{\partial \theta} &= r\epsilon_{\theta} - u \\ &= \frac{r}{E} \left[(1 - \nu^2)\sigma_{\theta} - \nu(1 + \nu)\sigma_r \right] - u \end{aligned}$$

and substituting from equations (44) and (47):

$$= \frac{1}{E} [4Br(1 - \nu^2) + \beta(1 - \nu - 2\nu^2)(pr - \int p dr)] - f(\theta)$$

Integrating:

$$v = \frac{1}{E} [4Br(1 - \nu^2)\theta + \beta(1 - \nu - 2\nu^2)(pr - \int p dr)\theta] - \int f(\theta)d\theta + f(r) \quad (48)$$

where $f(r)$ was a function of r only.

For a system of symmetrical stress distribution the shear stress ($\tau_{r\theta}$) was zero, therefore $\gamma_{r\theta}$ was also zero. Thus, substituting (47) and (48) in equation (45c):

$$\frac{1}{r} \frac{df(\theta)}{d\theta} + \frac{df(r)}{dr} + \frac{1}{r} \int f(\theta)d\theta - \frac{1}{r} f(r) = 0$$

from which, by separating functions of r and θ :

$$f(r) = F_1 r + F_2; f(\theta) = F_3 \sin \theta + F_4 \cos \theta \quad (49)$$

F_1, F_2, F_3, F_4 , being constants.

Substituting expressions (49) in (47) and (48):

$$u = \left[-\frac{A}{r}(1 + \nu) - Br(1 + \nu) + 2Br \log r(1 - \nu - 2\nu^2) + 2Cr(1 - \nu - 2\nu^2) + \beta(1 - \nu - 2\nu^2) \int p dr \right] + F_3 \sin \theta + F_4 \cos \theta \quad (50)$$

$$v = \frac{4Br(1 - \nu^2)}{E} \theta + \frac{\beta}{E} (1 - \nu - 2\nu^2) [pr - \int p dr] + F_3 \cos \theta - F_4 \sin \theta + F_1 r + F_2 \quad (51)$$

which were the required results.

Considering now the section of a cylinder, from symmetry, displacements could occur only in radial planes. v must therefore be zero for all values of θ .

It followed from equation (51) that:

$$\begin{aligned} \frac{4Br(1 - \nu^2)}{E} + \frac{\beta}{E} (1 - \nu - 2\nu^2) [pr - \int p dr] &= 0 \\ F_1 &= 0 \\ F_2 &= 0 \\ F_3 &= 0 \\ F_4 &= 0 \end{aligned}$$

For a non-porous body, the pore-pressure terms involving p must vanish and so $B = 0$.

However, where pore pressure existed:

$$\begin{aligned}
 B &= -\frac{\beta}{4} \frac{(1-2\nu)}{(1-\nu)} \frac{1}{r} [pr - \int p dr] \\
 &= -\frac{\beta}{4} \frac{(1-2\nu)}{(1-\nu)} \frac{1}{r} \left[pr - p_0 \int \frac{\log \frac{b}{r}}{\log \frac{b}{a}} dr \right] \\
 &= \beta \frac{p_0(1-2\nu)}{4 \log \frac{b}{a} (1-\nu)}
 \end{aligned}$$

The two remaining constants A and C could be determined from the boundary conditions. Without going farther it had been shown that the arbitrary constants in the general expression for the stress function were not identical for porous and non-porous multiply-connected bodies. The difference arose when imposing the condition for single-valued displacements, when an additional term, caused by the body force, appeared in the expressions for displacements (50 and 51). Such a term would always appear. In some particular case its value might be zero, depending on the pressure distribution: and only in such a case would Brahtz's result be applicable.

In general, however, the result was inapplicable in multiply-connected regions.

That conclusion did not seriously affect the main conclusions of the Author's Paper, but it was one which was not generally recognized and which sometimes led to an erroneous interpretation of Brahtz's results.

Mr J. L. Serafim (Research Engineer, Head of Dams Section, Laboratório Nacional de Engenharia Civil, Lisbon) stated that probably, in design of dams, there was no problem subject to such extensive debate as that of uplift. In recent discussions^{9, 10} the main results of the experimental findings of the area factor carried out at the Laboratório Nacional de Engenharia Civil in Lisbon were given. Such results proved that the value of the area factor for deformations in the elastic range differed from the one in rupture. It also depended on the dryness of the concrete, the percolating fluid, the characteristics of the concrete, the value of the pore pressure, etc. Values from about 0.4 to 1.0 had been obtained.

The Author did not discuss that problem and referred to Levy's classical treatment in which nothing was stated about the "proportion of area over which uplift was assumed to act" since Levy postulated the existence of a crack extending from upstream to downstream in the dam. Surely that assumption was equivalent to an area factor of 100%, which was now generally accepted, but it did not conform to the treatment of material stresses, i.e., effective stresses in a porous body, as made in the Paper.

The Author wondered "if Brahtz's conclusion for a two-dimensional system is more generally valid . . .", but it has already been proved that it was not.¹¹ That meant, in a three-dimensional state of stress (and of percolation) the uplift stresses could not be obtained by deducting the value of the pore pressure from the total stresses assuming that the body was impermeable. In fact, even if the percolation of the water through the dam was in a steady state ($\nabla^2 p = 0$), a state of hydrostatic (pantatonic) stresses equal to p was not possible. That meant it did not satisfy the compatibility conditions of the theory of elasticity.

The Author also stated that there was no published guidance on the effect of uplift on the distribution of the "material-contact stresses" in the case of vertical drains and of a region of denser concrete at the upstream face. However, both cases could be treated and were presented in a publication on the various aspects of the uplift problem¹¹.

In view of the ideas following the statement that "Uplift pressure would thus be a

⁹ References 9-11 are given on p. 875.

welcome phenomenon . . .', it was suggested that the Author should study in detail the time aspects and other physical conditions of the development of the uplift in concrete.

The Author, in reply, expressed his appreciation of the various comments and agreed with Mr Serafim that the difficulty in reaching a generally acceptable account of uplift lay in the number of aspects of the physical problem which were uncertain.

It had been the Author's intention to present a clear-cut and, it was hoped, useful solution to what he had termed the *structural* problem, on certain stated assumptions, and from the solution to see the effect of certain variables. The solution involved no difficulties; it was as valid as the assumptions on which it was based. The most questionable of those assumptions was the customary one of linearity of vertical stress in the concrete, and the Author had deliberately drawn attention to it, for that one assumption evaded many of the controversial difficulties of the uplift problem. He was grateful to Dr Zienkiewicz and Mr Park for their arguments in support of linearity, which encouraged the hope that when all was understood the simple linear theory might be found to give a good approximation.

One of the more disturbing doubts about the assumption had arisen from the work of Brahtz, referred to in the Paper, and Mr Park had illustrated a subtle point which limited the application of Brahtz's argument to the very simplest cases. Dr Zienkiewicz, in his interesting recent Paper,⁷ had shown that linearity was approximately compatible with other, more or less reasonable, assumptions: that if the theory of homogeneous elastic plane strain was applicable, if the dam was not self-strained in any way, if the uplift obeyed assumed distributions, and if the flow of pore water was strictly two-dimensional, then the variation of the material stress would be almost linear when Poisson's ratio = 0, although it would approach Brahtz's prediction for values of Poisson's ratio higher than were likely to be applicable to concrete. It was a reflection of the complexity of the problem that such assumptions were needed before a solution could be found.

Against that background Mr Serafim's careful experiments in Lisbon had been very welcome. Really valuable experiments into those questions were very difficult to make, and published data had been apt to provoke controversies as to the significance of the results. Nevertheless, the linearity assumption was likely to be widely used for many years to come, and both design engineers and research workers would be grateful for reliable experimental results which would enable them to use it with more knowledge of its approximation to the truth.

FURTHER REFERENCES

9. R. W. Carlson, "Permeability, pore pressure and uplift in gravity dams". Proc. Am. Soc. C.E., Paper No. 700. See discussion by J. L. Serafim, J. Power Divn, Proc. Am. Soc. C.E., No. PO1, p. 21 (Feb. 1956).
10. T. C. Powers, "Hydraulic pressure in concrete". Proc. Am. Soc. C.E., Paper No. 742. See discussion by J. L. Serafim, J. Power Divn, Proc. Am. Soc. C.E., No. PO1, p. 55 (Feb. 1956).
11. J. L. Serafim, "A supressão nas barragens". Publ. No. 55, Lab. Nac. Engg Civ., Lisbon, 1954.