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7780 DISCUSSION

A new difference based finite element method

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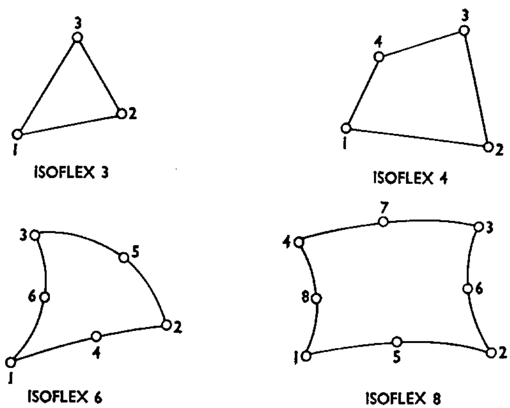
The Authors' new difference based finite element method leads to a performance in analysis which is comparable with well established finite elements. However, if one considers the design of slabs to resist bending, the preferred requirements for thin plate flexure elements may be summarized as follows.



Nodal variables

$$\delta_i = \begin{Bmatrix} w \\ \theta_n \\ \theta_y \end{Bmatrix}_i = \begin{Bmatrix} w \\ \dfrac{\partial w}{\partial y} \\ -\dfrac{\partial w}{\partial n} \end{Bmatrix}_i \text{ for corner nodes}$$

$$\delta_i = \begin{Bmatrix} \Delta\theta_T \end{Bmatrix}_i \begin{Bmatrix} \Delta\dfrac{\partial w}{\partial n} \end{Bmatrix}_i$$ the departure from linearity of the midside tangential rotation

Fig. 9. The ISOFLEX family and nodal configurations

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(*a*) The elements should be capable of being used in triangular and quadrilateral form, although a triangle is only occasionally required for mesh refinement and irregular boundaries. The elements should also be capable of representing tapering thickness and curved boundaries when necessary.

(*b*) The nodal configuration should be simple and allow the standard grillage beam element to be incorporated into an idealization.

(*c*) As a mesh of arbitrarily shaped elements is refined, convergence to the correct solution should be ensured. With certain provisos, the existence of fine mesh convergence for an element can be established by the patch test[24]

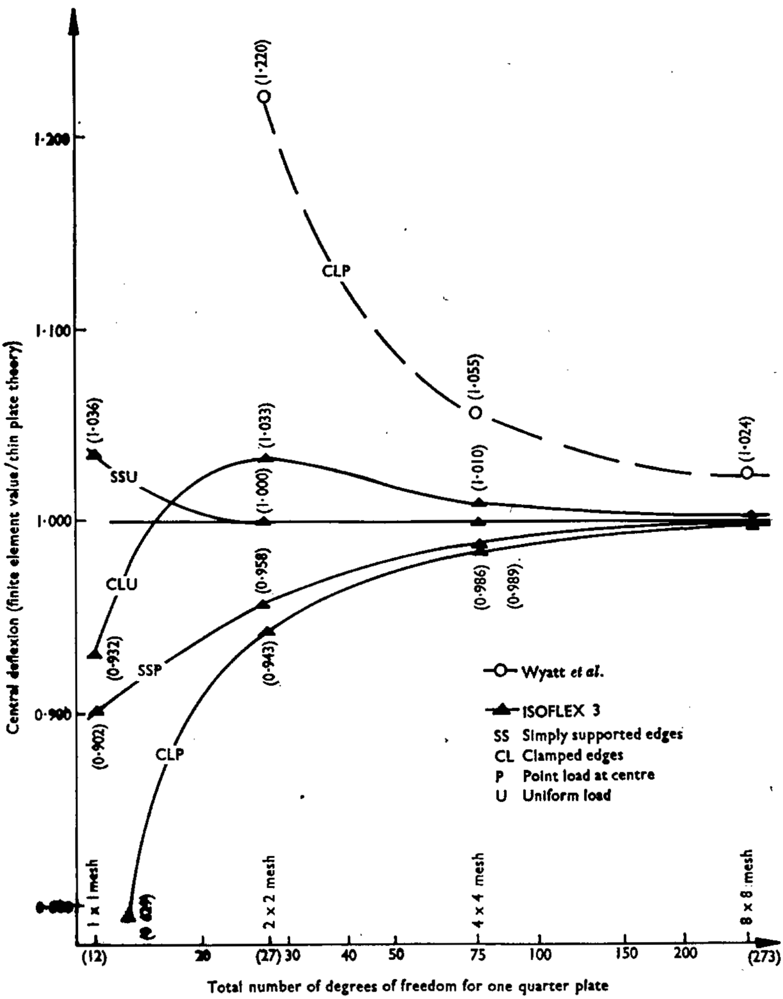

Fig. 10. Convergence of triangular elements for a square plate

(although as yet there are relatively few simple plate flexure elements which pass this test).

(*d*) The equations produced should not be ill-conditioned and fail for certain geometries.

(*e*) The coarse mesh performance should be such that if an idealization is in error the results are not unreasonable.

(*f*) The element(s) should be easy to implement and computationally efficient.

37. To illustrate the extent to which these requirements may be satisfied, reference is made to the ISOFLEX family[25] of thin plate flexure elements. These elements belong

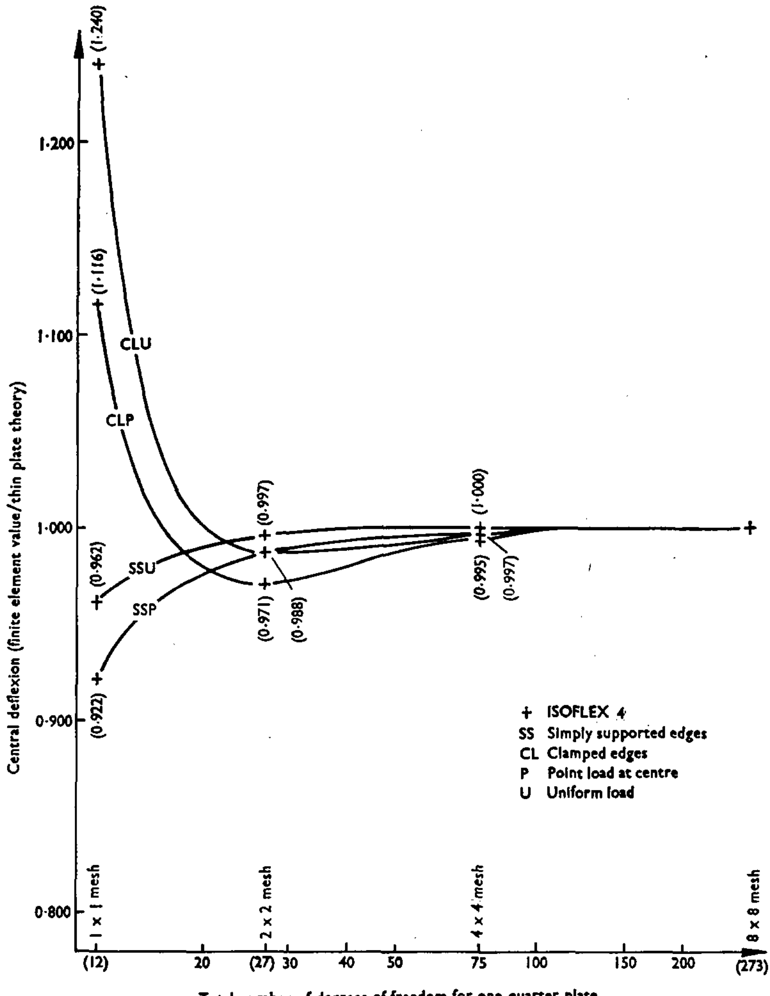

Fig. 11. Convergence of a quadrilateral element for a square plate

to the second generation of isoparametric elements and have an extensive ancestry. Their formulation is based on the three-dimensional shell element given by Ahmad *et al.*,[26] uses reduced integration techniques presented by Zienkiewicz *et al.*,[27] and employs an essential modification to invoke the Kirchhoff hypothesis at discrete points within the element, similar to that developed by Irons.[28, 29]

38. The ISOFLEX elements may have tapering thickness, and have the simplest nodal configurations (Fig. 9) which allow the standard grillage beam element to be incorporated in an idealization. These elements fulfil the requirement for convergence, because even a mixed mesh of triangular and quadrilateral elements of arbitrary geometry passes the patch test. Further, no limitations, such as low rank and spurious mechanisms, have been noted.

39. The performance of the straight sided ISOFLEX elements is indicated by the analysis of a square plate with various boundary and loading conditions. In each case (see Figs 10 and 11) the central deflexion rapidly converges to the exact theoretical value as the mesh is refined. For comparison the only case presented by the Authors is also included in Fig. 10. The convergence of the ISOFLEX elements is unbounded and fluctuates about the theoretical value in a manner associated with hybrid and mixed formulations. For a coarse 2×2 mesh of one quarter of the plate, the maximum error

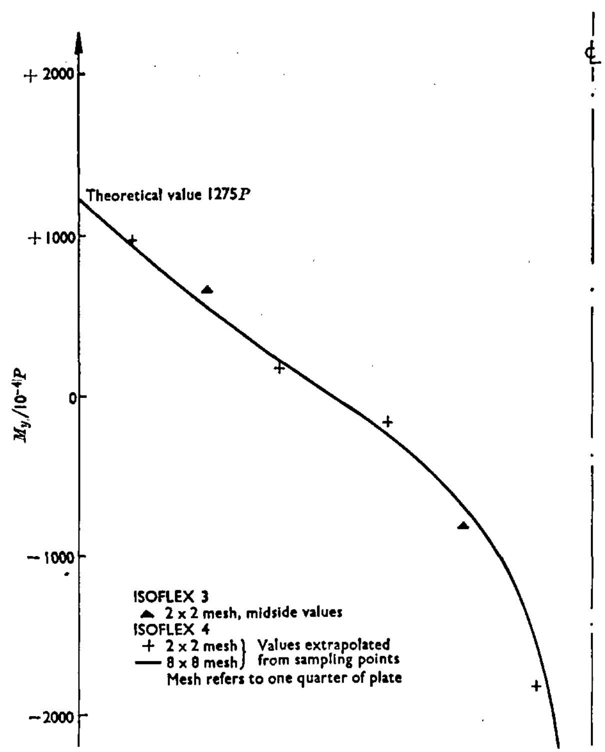

Fig. 12. Distribution of M_y moments along centre line of clamped square plate with a point load at centre

for the ISOFLEX 3 element is 5·7% and for the ISOFLEX 4 element only 2·9%. The distribution of M_y moments along the centre line of the plate, also given by the Authors, is acceptable even for a 2×2 mesh as shown in Fig. 12.

40. The ISOFLEX family of elements is a unified formulation which can be easily implemented from a single compact shape function subroutine. Compared with other well-known elements they are also computationally efficient.

41. Elements based on the Authors' approach satisfy some of the requirements of § 36 for thin plate flexure elements; quadrilateral and triangular elements are available, the standard grillage beam element can be incorporated in an idealization, and the equations produced are not ill-conditioned. Is it possible to satisfy the other requirements?

42. The results of the analysis of a square clamped plate under a central point load, given by the Authors, show rapid convergence of the central deflexion. Could the Authors present results for the other boundary and load conditions analysed here? Can they indicate whether or not they would expect an improved coarse mesh performance to be obtained from their approach in due course, and if so, how would they expect their approach to be developed?

Dr Wyatt, Dr Davies and Mr Snell

The comments by Mr Lyons and Dr Cassell appear in the main to invite comparison between the proposed new method and the ISOFLEX elements. The results presented for the ISOFLEX elements are clearly encouraging. The questions raised regarding the finite difference approach may with one exception be answered briefly. The system allows a nodal pattern of extreme flexibility. Within our inevitably limited experience, the equations are not ill-conditioned. Computation within the elements is not difficult. The solutions derived from a limited number of points can have errors of reasonable size only where the nodal points have been chosen with full understanding of the physical problem, and particularly of the important boundary conditions. This is true of all methods of approximation. The extension of the algebra to include variation of thickness and curved boundaries can be seen to be straightforward, although we have not investigated such cases. Numerical integration will, of course, become necessary for non-triangular elements.

44. Convergence to the true solution and the satisfaction of a patch test are questions which deserve more serious study. There has been considerable interest in proofs of convergence 'in the limit' but in practical terms the cost of establishing the limit with progressive reduction in element size is prohibitive except only for the most simple cases, so that the extensive literature is often of no practical help.

45. It is more rewarding, in engineering terms, to examine the error of solutions which are a long way from the limiting case of a very large number of very small elements. The fully conforming element moves to the true solution monotonically. The assumption within the element represents a constrained solution with a higher minimum total energy (the so-called Rayleigh principles). For non-conforming elements, where the work done at the non-conforming boundaries is ignored in the energy minimizing process, the solution will move towards a lower minimum total energy, which in our experience is appreciably nearer the true minimum.

46. In some cases, and in particular for the uniform strain condition used in patch tests, where the elements remain non-conforming, the minimum energy can be below the true value, and this happens with the method described in the Paper. The elements can be made to satisfy the true minimum total energy by including in the derivation of the stiffness matrix that work done on the sides of the elements, but this must make all approximations to non-uniform strain conditions further from the true minimum total energy, and would lose the advantage which non-conformity appears to confer on the least squares based finite difference approach.

47. Any non-conforming element which for simple strain becomes conforming (in polynomial terms this implies zero coefficients of the higher order terms) will satisfy

the patch test. The least squares method is non-conforming in the fullest possible sense, without conformity even of the nodal variables. Work is at present proceeding to examine the effect of including the work done due to the mismatch in the total energy summation.

48. In view of the limited space available the Authors concentrated on the presentation of the theory and were therefore inevitably limited in the number of examples that could be presented. However, considerably greater detail has been provided by Wyatt,[18] who also discusses the significance of various weighting conditions and the inclusion of in-plane rotations in the analysis of thin plate box systems.

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