Proc. Instn Civ. Engrs, Part 2, 1977, **63**, Sept., 753–756

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DISCUSSION

The elasto-plastic analysis of imperfect square plates under in-plane loading

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It is significant that the Authors favour the unmodified dynamic relaxation method for solving their simultaneous equations, derived in this case from finite difference equations. Their statement in § 2 that 'Rushton had applied a modern finite difference technique—dynamic relaxation—to obtain large deflexion solutions' would probably be taken by most readers to imply that the dynamic relaxation method of solution is restricted to equations based on finite differences. To my knowledge, this has never been seriously suggested; dynamic relaxation can be used for any simultaneous equations which converge, including those from finite elements, for example.

66. Steps towards a fully self-adaptive method of iterative solution were made in 1971.[21] Both beam bending and the biharmonic equation, involving symmetrical fourth order partial differential equations which may be very slow to converge, give rise to particularly simple damping expressions which are phase sensitive. It is not necessary to use trial cycles to find suitable damping constants, as favoured by Rushton.

67. There seems to be a strong possibility that useful advance could be made in rapid solution of simultaneous equations by using a general self-adaptive iterative technique, which changes the damping constants during solution. The restrictions implied by the mechanical analogy in dynamic relaxation, which provided such a useful beginning, do not seem to be necessary. Indeed one development suggests that the actual damping factors can be made to be remarkably unimportant providing that they are fairly low—even zero damping seems to work quite well for beam bending.

68. Day[22] originated the dynamic relaxation method and the late Mr J. R. H. Otter did much to promote its early use and development.

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The collapse analysis of thin steel plates when subjected to in-plane compressive (or shear) loading is not an easy problem to solve accurately. The nature of the problem involves the interaction between large deflexions and the spread of plasticity through the volume of the plate, and therefore necessitates numerical solution using a computer. The quantity of data available on this subject has been growing since the initial work of Graves Smith,[23] and the Paper is a useful addition.

70. The method of analysis is based on a finite difference dynamic relaxation technique which is used to find successive equilibrium deflected shapes for the plate. Large deflexion effects are correctly allowed for and the stress–strain behaviour is calculated

Paper published: *Proc. Instn Civ. Engrs*, Part 2, 1977, **63**, Mar., 137–158.

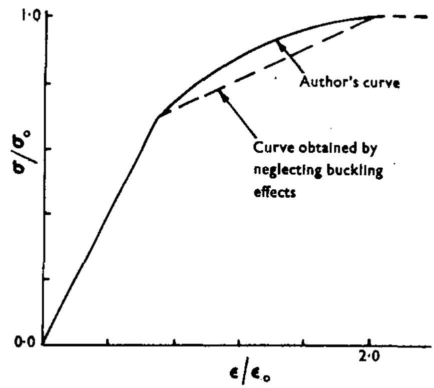

Fig. 18. Effect of residual stress on the load shortening curve of a stocky plate in uniaxial compression

from plastic flow theory. Dynamic relaxation has been used extensively for the large deflexion analysis of elastic plates.[3-6] The Authors' approach is novel in extending this elastic analysis capacity to allow for plastic flow. As I understand their method, it is assumed that, at any given point in the plate, the effective stiffness (E_{ij} or E_{ij}^*) during the current shortening increment can be calculated solely from a knowledge of the stress state of that material at the end of the previous increment. Consequently, even though the incremental stress–strain response may vary throughout the plate, at any given point it will be given one constant value during any particular sequence of iterations. This will considerably simplify and accelerate the analysis as it avoids consideration, during the iterations, of the two discontinuities associated with plastic behaviour: being brought to yield and unloading from yield. For certain cases, the plate will be assumed to respond with an unrealistic stiffness which would result in convergence to an incorrect shape. An example which is easy to explain is that of an initially perfect plate, so proportioned that its buckling stress is equal to σ_0, subjected to uniaxial compression. Up to a shortening strain value of σ_0/E, the analysis will predict the plate to remain flat, and at this value the whole plate will yield. On further increase of shortening, every point in the plate will be assumed to remain on the yield surface, even though it is clear that the plate will buckle with a substantial region unloading elastically from yield. However, if the previous shortening strain value were slightly less than σ_0/E, the plate would be assumed to be completely elastic during the iterations for a further shortening increase, even though, on buckling, a substantial region would yield. Anomalies of this nature reduce confidence in the overall applicability of the method, even though sufficient accuracy may be obtained for many cases of a single square plate. Such errors (as with several other types) can be reduced by using very small shortening increments. Unfortunately, this is a costly procedure, but it seems essential with the Authors' approach.

71. Once the new equilibrium shape has been found, it is sometimes necessary, presumably, to adjust the stresses at certain points to ensure that $f \ngtr 1$ (for the no strain hardening case). Any such adjustment will destroy, to some extent, the state of equilibrium. Did the Authors allow for this?

72. The Authors are correct in maintaining that the inclusion of strain hardening requires only a simple modification to their elastic–perfectly plastic program. Indeed, the classical work on plasticity theory usually includes strain hardening automatically. In my opinion it is doubtful whether strain hardening is an important factor in the collapse of steel plates, except possibly for steel of very high yield strength. Materials tests have been performed at Cambridge on steels, the approximate range of observed σ_0 values being 220–490 N/mm². In most cases there was a horizontal yield plateau

of several times the yield strain. Such large strains would not be present in most practical plates until beyond collapse. However, for aluminium structures, strain hardening must be allowed for.

73. With regard to the treatment of residual stresses, the Authors' curve for $b/t=20$, $w_0/t=0{\cdot}08$, longitudinal $\sigma_R=0{\cdot}3\sigma_0$ and transverse $\sigma_R=0$ (Fig. 8) is compared with the curve obtained by ignoring buckling effects in Fig. 18. The latter curve corresponds to the well-known rectangular residual stress distribution used at Cambridge.[24] Clearly the Authors' analysis does not use the same distribution, as their results lie significantly higher for much of the range of ϵ/ϵ_0. I suspect that the discrepancy results from an adjustment of the Cambridge distribution in order to fit in with the finite difference mesh. It would seem advisable to clarify what the actual distribution is, especially as the apparent adjustment is on the unsafe side. The Authors' correct statement that at low b/t 'residual stress merely affects the shape of the stress–strain curve in the region before the plateau' should not be allowed to obscure the fact that the effect is one of a premature drop in plate stiffness which can severely weaken a non-stocky compression member of which the plate forms part (e.g. a stiffened panel).[25]

74. The Authors explain that their analysis technique will find the new equilibrium shape provided that the previous shape was also one of equilibrium (equations (11)). The initial state of a plate with both out of flatness and residual stress blocks is not one of equilibrium. How did the Authors deal with this? Is it necessary to include in their program a formulation based on total equilibrium (equations (6)) as well as incremental equilibrium (equations 11))?

75. What degree of precision did the Authors decide was necessary to gain sufficient accuracy? Of what magnitude were the computational requirements of the Authors' program? In any method which involves, as does the Authors', the numerical evaluation of derivatives, there is a danger that round off errors may be large. The conflict is that a fine mesh is required in order to reduce errors inherent in the truncated Taylor series approximation to the derivative, whereas the finer the mesh, the larger is the effect of round off error. This factor would be an argument against using the Authors' program on a computer of low precision.

Dr Harding, Dr Hobbs and Professor Neal

We did not intend to suggest that dynamic relaxation used as an iterative technique is applicable only to the solution of finite difference equations. It is, as Dr Chaplin says, a general solution technique capable of solving any simultaneous equations and has already been used in conjunction with the solution of finite element formulations.

77. Nevertheless, it has become clear that dynamic relaxation is best used in the solution of the equations obtained by applying finite difference approximations to the equations of elasticity in a separated form, as this leads to a simple computer implementation which does not require the storage of large coefficient arrays, e.g. an explicit stiffness matrix. It is generally recognized that iterative equation solving routines are not competitive with direct solution techniques when the storage of a stiffness matrix cannot be avoided, particularly in linear structural problems.

78. For the non-linear analyses described in the Paper dynamic relaxation proved easy and economic to use, although we were aware of alternative strategies for achieving convergence and were interested to learn of Dr Chaplin's technique for modifying the damping node by node depending on the current velocities. Selection of damping factors was simple and indeed the same factors were used for all the calculations shown in the Paper, as well as for most of the more extensive study,[20] so that their selection did not affect the cost of the solutions significantly.

79. Although the Paper was not intended to review the historical development of dynamic relaxation, it is correct that Otter and Day were the originators of the technique in its present form, which has very close links with a technique due to Richardson.[26]

80. The example Dr Little chooses to illustrate his doubts about the satisfactory

convergence of our program is disingenuous. It may be easy to understand but in the study of imperfect plates a perfect plate, free of residual stresses and proportioned so that the yield stress and the conventional buckling stress coincide, is an extreme rather than a typical case. Nevertheless, by suitable selection of load increments near the critical value, as accurate a model of the behaviour as is desired could be achieved, without using large amounts of expensive computer time. For practical problems comparison of runs with a varying number of increments has shown that no problem exists in this area. The process, in fact, seems to be to some degree self-corrective as an overestimate of stiffness for one load increment will tend to produce an underestimate for the next.

81. The problem of the possible existence of multiple equilibrium paths is more difficult and there is no proof at present that any of the numerical solution techniques in all cases produces the lowest possible solution. Comparison with other analytical solutions and experiments suggests that present results correspond to the correct failure paths for the type of problem illustrated.

82. At no time, with this approach, was it found necessary to adjust the stress state and hence create an out of equilibrium situation. It was found that with the gradual multi-layer process of yielding, stresses were maintained very close to their yield values and a check at the end of a number of the runs showed that this problem was not serious.

83. It is difficult to comment on the difference in the curves of Fig. 18 without looking in detail at the results. The suggestion made by Dr Little is one possible reason, especially as it would seem that the curves eventually converge. However, it may be due to other differences in the formulation of the yielding process. None of the available solution techniques gives exact correspondence in overall stress–strain response.

84. Dr Little is correct in saying that variations in tangent stiffness throughout the loading history may be relevant to the behaviour of the overall stiffened panel.

85. Studies of solution convergence with mesh and increment size showed that, in general, the result could be expected to be within 2–3% of the true solution. This was economically acceptable, and the problems were run with the mesh described in the Paper using 40k words of store on a readily available CDC 6400 (50k word store, 60 bit word length), although much larger facilities were available. Dr Little is correct in saying that low precision machines can produce significant round off errors, but this has not been a problem with the facilities available to us. Indeed, experience with dynamic relaxation on a variety of machines suggests that it is no more prone to round off errors than, for example, the direct solution of a set of linear equations, in spite of the well-known theoretical limitations of finite difference techniques which caused so many problems in the days of hand computations.

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